THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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AMERICAN MATHEMATICAL MONTHLY

BENJAMIN PEIRCE.

I. REMINISCENCES OF PEIRCE.

By President Emeritus CHARLES W. ELIOT,[1] Harvard University.

Benjamin Peirce graduated at Harvard College with the degree of A.B. in 1829. Two years later he was appointed Tutor and in 1833 University Professor of Mathematics and Natural Philosophy. This was an unendowed professorship; and its creation was one of President Quincy's enterprising adventures in the enlargement of Harvard's teaching staff. The President was doubtless

[1] The authors of the "Reminiscences" of Benjamin Peirce, presented herewith, were all his former students and each has done something notable in mathematics. President Emeritus ELIOT was a student during 1849–53. He was a tutor of mathematics in Harvard College 1854–58, and assistant professor of mathematics 1858–61; he was also assistant professor of chemistry 1858–63. James Mills Peirce, son of Benjamin, and classmate of Eliot, was appointed tutor of mathematics at the same time. In his *Analytic Geometry*, published in 1857, Tutor Peirce acknowledged that "whatever merit the book may have is owing, in a great degree, to the assistance of Mr. C. W. Eliot." President Eliot has described these early years as follows (*Report of the Harvard Class of 1853*, Cambridge, 1913, p. 98): "Tutor Peirce chose the Freshman class, leaving me the Sophomore class in that year [1854–55]. After a year's experience, we applied some new recitation-room methods which made the mathematical instruction more effective. Finding the existing method of conducting oral examinations twice a year in the presence of visiting committees of the Board of Overseers very unsatisfactory as a test of the students' knowledge and capacity, we asked leave of the Faculty to conduct the mathematical examinations of the Freshmen and Sophomores in writing. After a good deal of hesitation the Faculty granted us leave to make the experiment; and these examinations were the first examinations in writing ever conducted for entire classes in Harvard College. The innovation was gradually adopted in other departments, and ultimately spread to the whole University.

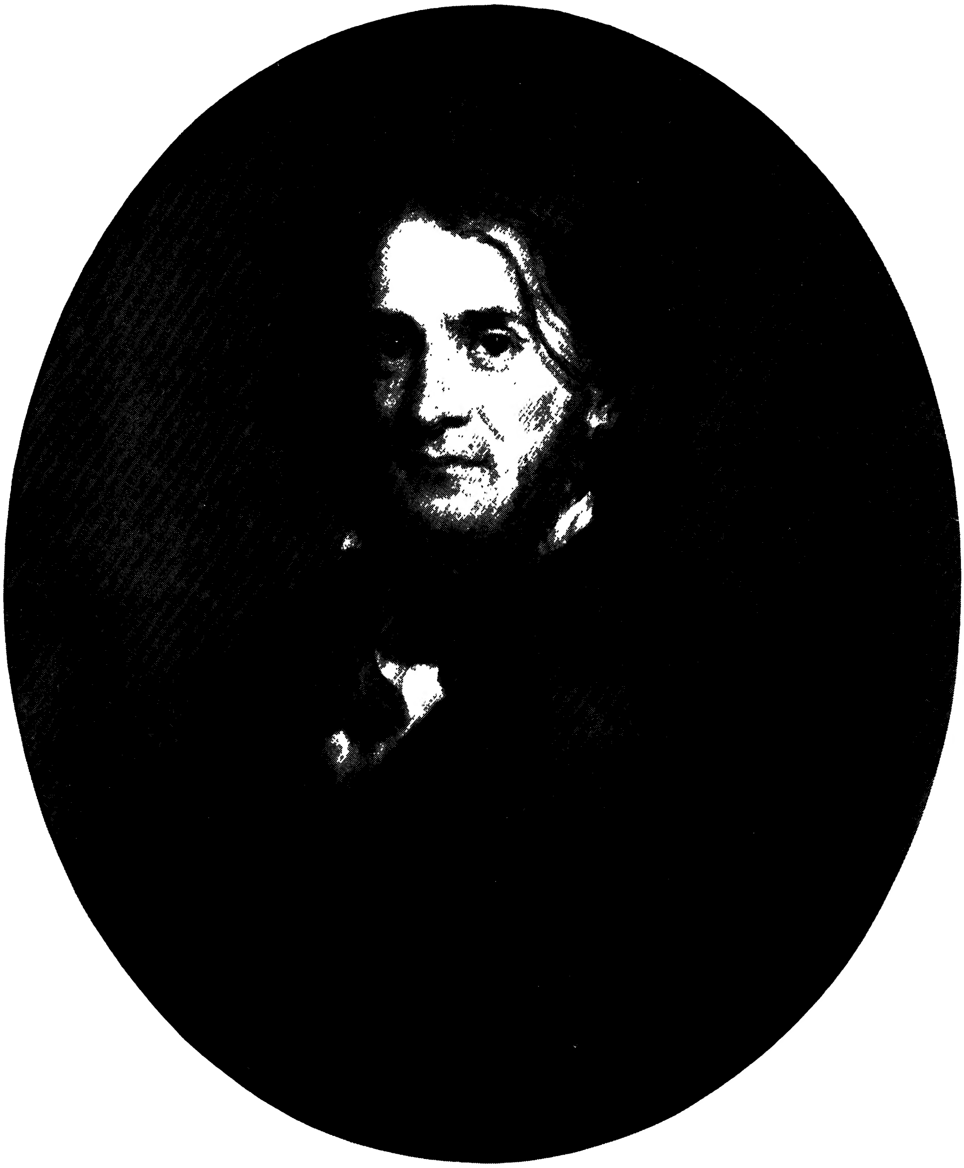
"I tried to make the teaching of mathematics to the Freshmen and Sophomores as concrete as possible, and to illustrate its principles with practical applications. For example, while the class was studying trigonometry, I taught simple surveying to a group of volunteers, and with their help made a survey of the streets and open spaces of that part of Cambridge which lies within a mile and a half of University Hall. These volunteers made under my direction a careful map of what was then the College Yard, with every building, path, and tree delineated thereon—a map which is preserved in the college library."

President LOWELL was a student under Peirce 1873–77, and his paper on "Surfaces of the second order as treated by quaternions," read before the American Academy of Arts and Sciences, was published in its *Proceedings* (vol. 13, 1878, pp. 222–250).

Professor BYERLY was a student under Peirce 1867–71. He was assistant professor of mathematics at Cornell University 1873–76; assistant professor of mathematics at Harvard College 1876–81; professor 1881–1906; Perkins professor of mathematics 1906–1913. Since 1913 he has been Perkins professor emeritus. He was the first one (in 1873) to receive the degree of Doctor of Philosophy, in mathematics, at Harvard University; his thesis was entitled "The heat of the sun." He is the author of mathematical articles, pamphlets, and textbooks.

Chancellor CHACE studied with Peirce in 1878–79, and his paper on "A certain class of cubic surfaces treated by quaternions," published in the *American Journal of Mathematics* (vol. 2, 1879, pp. 315–323), was a result. The Chancellor has recently prepared a translation, with commentary and notes, of the Rhind mathematical papyrus, which is about to be sent to the press.

R. C. ARCHIBALD.



BENJAMIN PEIRCE, 1853 (?)

From a painting by J. A. Ames in the possession of Harvard University.

supported in this adventure by Nathaniel Bowditch, who was then at the height of his influence as a Fellow of the Corporation (the President and Fellows of Harvard College). As soon as the endowed Perkins Professorship of Astronomy and Mathematics was established (1842), Benjamin Peirce was transferred to that chair, which he held till his death in 1880.

Benjamin Peirce was never a professor of Mathematics only. In the title of the University professorship he held, the broad subject of Natural Philosophy appeared, and in the title of the Perkins professorship Astronomy was the first subject named. These titles represented the real breadth of Benjamin Peirce's mental interests and imaginative powers, and this breadth characterized his teaching in Harvard College from beginning to end.

He was no teacher in the ordinary sense of that word. His method was that of the lecture or monologue, his students never being invited to become active themselves in the lecture room. He would stand on a platform raised two steps above the floor of the room, and chalk in hand cover the slates which filled the whole side of the room with figures, as he slowly passed along the platform; but his scanty talk was hardly addressed to the students who sat below trying to take notes of what he said and wrote on the slates. No question ever went out to the class, the majority of whom apprehended imperfectly what Professor Peirce was saying.

When I entered College in 1849 Professor Peirce had ceased to have to do with the elementary courses in Mathematics. He taught only students who had been through the two years of prescribed Mathematics and had elected to attend his courses, which were given three times a week throughout the junior and senior years. Two or three times in the course of the hour, Professor Peirce would stop for a moment or two to give opportunity for the members of the class to ask questions or seek explanations; and these opportunities were utilized by all the members who really wanted to learn. If a question interested him, he would praise the questioner, and answer it in a way, giving his own interpretation to the question. If he did not like the form of the student's question, or the manner in which it was asked, he would not answer it at all, but sometimes would address an admonition to the student himself which went home.

One day in my senior year, when Professor Peirce had already acquired the habit of giving me the highest possible marks on all my notes of his lectures and on every other exercise for which marks could be given, to the great concern of my competitor for the first place in the class, a concern which he liked to communicate to me his next door neighbor in Hollis—I graduated second—I ventured to say that what he had just been saying to us about functions and infinitesimal variables seemed to me to be theories or imaginations rather than facts or realities. Professor Peirce looked at me gravely, and remarked gently, "Eliot, your trouble is that your mind has a skeptical turn. Be on your guard against that tendency or it will hurt your career." That was new light to me; for I had never thought at all about my own turns of mind. The diagnosis was correct.

In spite of the defects of his method of teaching, Benjamin Peirce was a very

inspiring and stimulating teacher. He dealt with great subjects and pursued abstract themes before his students in a way they could not grasp or follow, but nevertheless filled them with admiration and reverence. His example was much more than his word. I remember that this great master began one day with unusual promptness to put on the slates a series of calculations and formulæ in which he seemed to be much interested. He said but little; but wrote diligently with the chalk, stopping now and then to examine his work and to rub out some of it, but only to resume it, and go on eagerly. The class before him said not a word, took notes as well as they could of what he wrote on the slates, and watched him. Suddenly near the end of the hour the worker looked despairingly at the contents of the last slate he had filled, turned to the class, and remarked, "there is an error somewhere in this work, but I cannot see where it is. This last line—the conclusion—is obviously wrong." Whereupon he seized the rubber and rapidly rubbed out everything he had put on the slates. Professor Peirce sat down in his armchair visibly fatigued. The class slowly folded their notebooks and departed without a word, even to each other. I, for one, have always remembered vividly that hour's spectacle.

In 1862, Thomas Hill was elected President of Harvard University. One of his first measures was the institution of courses of lectures open to graduate and other advanced students, and called University Lectures. President Hill's idea was to give advanced students living in Cambridge and the interested Cambridge public opportunities to hear the best scholars and scientists of the country speaking on their favorite subjects, the subjects in which they had won distinction or renown. Benjamin Peirce was one of the first persons to be appointed University lecturer, and he served gladly in this capacity in five different years. The University Lectures were not to be technical, though advanced. They were to be stimulating as well as informing, and women were encouraged to attend them as well as men. Benjamin Peirce's lectures dealt, to be sure, with the higher mathematics, but also with theories of the universe and the infinities in nature, and with man's power to deal with infinities and infinitesimals alike. His University Lectures were many a time way over the heads of his audience, but his aspect, his manner, and his whole personality held and delighted them. An intelligent Cambridge matron who had just come home from one of Professor Peirce's lectures was asked by her wondering family what she had got out of the lecture. "I could not understand much that he said; but it was *splendid.* The only thing I now remember in the whole lecture is this—'Incline the mind to an angle of 45°, and periodicity becomes non-periodicity and the ideal becomes real.'"

When Professor Bache retired from the superintendency of the U. S. Coast Survey, he procured the appointment of his intimate friend Benjamin Peirce as his successor in the superintendency. Those of us who had long known Professor Peirce heard of this action with amazement. We had never supposed that he had any business faculty whatever, or any liking for administrative work. A very important part of the Superintendent's function was to procure from

Committees of Congress appropriations adequate to support the varied activities of the Survey on sea and land. Within a few months it appeared that Benjamin Peirce persuaded Congressmen and Congressional Committees to vote much more money to the Coast Survey than they had ever voted before. This was a legitimate effect of Benjamin Peirce's personality, of his aspect, his speech, his obvious disinterestedness, and his conviction that the true greatness of nations grew out of their fostering of education, science, and art.

In his younger days Benjamin Peirce enjoyed taking part in private theatricals. As an actor he was apt to be too violent and impetuous; but he was always interesting. He had, indeed, a gift for dramatic expression which served him well in many incidents, both comical and tragical, of his maturer life. For this reason, among others, only persons who saw and heard him can fully appreciate the influence of his life and work.

II. Reminiscences.

By President A. Lawrence Lowell, Harvard University.

Looking back over the space of fifty years since I entered Harvard College, Benjamin Peirce still impresses me as having the most massive intellect with which I have ever come into close contact, and as being the most profoundly inspiring teacher that I ever had. His personal appearance, his powerful frame, and his majestic head seemed in harmony with his brain.

The amount of instruction in mathematics then given was small compared with what is offered in any large university at the present day. The teaching of the calculus and everything beyond was done by Benjamin Peirce and his son, the father at this period giving only the more advanced courses for the few upper classmen who elected them. He expected and received close and rapid attention in class, and hard, though not extensive, work outside. We read his *Analytic Mechanics*, Briot and Bouquet on *Elliptic Functions*, Tait and Hamilton on *Quaternions;* while his direct instruction consisted mainly, but not wholly, in solving problems by writing on the blackboard that covered the end of the room a series of equations which we copied into our notebooks.

As soon as he had finished the problem or filled the blackboard he would rub everything out and begin again. He was impatient of detail, and sometimes the result would not come out right; but instead of going over his work to find the error, he would rub it out, saying that he had made a mistake in a sign somewhere, and that we should find it when we went over our notes.

Described in this way it may seem strange that such a method of teaching should be inspiring; yet to us it was so in the highest degree. We were carried along by the rush of his thought, by the ease and grasp of his intellectual movement. The inspiration came, I think, partly from his treating us as highly competent pupils, capable of following his line of thought even through errors in transformations; partly from his rapid and graceful methods of proof, which reached a result with the least number of steps in the process, attaining thereby an artistic or literary character; and partly from the quality of his mind which

tended to regard any mathematical theorem as a particular case of some more comprehensive one, so that we were led onward to constantly enlarging truths. To those of us who have not pursued the study of mathematics since college days the substance of what he taught us has faded away, but the methods of thought, the attitude of mind and the mode of approach have remained precious possessions.

III. Reminiscences.

By Professor Emeritus W. E. Byerly, Harvard University.

When I entered Harvard in 1867, a particularly unsophisticated freshman from New Jersey knowing absolutely no one in the college, Cambridge was a small straggling town. The inhabitants still spoke of visiting Harvard Square as going down to the village.

The Square itself was occupied by the hay scales and the town pump. The portion of the college yard east of University Hall was a hayfield, from which the University drew a modest annual profit. The dormitories, Massachusetts, Hollis, Stoughton, and Holworthy, were grouped in the neighborhood of the college pump, the water supply of all the students in the yard. Steam heating and plumbing were unknown.

The College was small; the Faculty was small, but distinguished and picturesque. "There were giants in those days," bearded giants mainly, though Agassiz and Child were beardless, Sophocles, Longfellow, Lowell, Asa Gray, Benjamin Peirce. There are giants in the faculty now, but they are more or less lost in the crowd. Then, poets, discoverers, philosophers, seers, in soft hats and long cloaks, looked their parts, and we newly-fledged freshmen gazed at them with admiration and awe.

The appearance of Professor Benjamin Peirce, whose long gray hair, straggling grizzled beard and unusually bright eyes sparkling under a soft felt hat, as he walked briskly but rather ungracefully across the college yard, fitted very well with the opinion current among us that we were looking upon a real live genius, who had a touch of the prophet in his make-up.

When I knew him later in the class-room, I will not say as a teacher, for he inspired rather than taught, and one's lecture notes on his courses were apt to be chaotic, I always had the feeling that his attitude toward his loved science was that of a devoted worshipper, rather than of a clear expounder. Although we could rarely follow him, we certainly sat up and took notice.

I can see him now at the blackboard, chalk in one hand and rubber in the other, writing rapidly and erasing recklessly, pausing every few minutes to face the class and comment earnestly, perhaps on the results of an elaborate calculation, perhaps on the greatness of the Creator, perhaps on the beauty and grandeur of Mathematics, always with a capital M. To him mathematics was not the handmaid of philosophy. It was not a humanly devised instrument of investigation, it was Philosophy itself, the divine revealer of TRUTH.

I remember his turning to us in the middle of a lecture on celestial mechanics and saying very impressively, "Gentlemen, as we study the universe we see every-

where the most tremendous manifestations of force. In our own experience we know of but one source of force, namely will. How then can we help regarding the forces we see in nature as due to the will of some omnipresent, omnipotent being? Gentlemen, there must be a GOD."

At another time he was lecturing on his favorite subject, which was then beginning to attract the attention of mathematicians and philosophers, Hamilton's new calculus of quaternions, which he believed was going to be developed into a most powerful instrument of research. He must have been working recently on his "Linear Algebras" for he said that "of possible quadruple algebras the one that had seemed to him by far the most beautiful and remarkable was practically identical with quaternions,[1] and that he thought it most interesting that a calculus which so strongly appealed to the human mind by its intrinsic beauty and symmetry should prove to be especially adapted to the study of natural phenomena. The mind of man and that of Nature's God must work in the same channels."

In one of his lectures on the theory of functions he established the relation connecting π, e, and i, $e^{\pi/2} = \sqrt[i]{i}$, which evidently had a strong hold on his imagination. He dropped his chalk and rubber, put his hands in his pockets, and after contemplating the formula a few minutes turned to his class and said very slowly and impressively, "Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

I have hinted that his lectures were not easy to follow. They were never carefully prepared. The work with which he rapidly covered the blackboard was very illegible, marred with frequent erasures, and not infrequent mistakes (he worked too fast for accuracy). He was always ready to digress from the straight path and explore some sidetrack that had suddenly attracted his attention, but which was likely to have led nowhere when the college bell announced the close of the hour and we filed out, leaving him abstractedly staring at his work, still with chalk and eraser in his hands, entirely oblivious of his departing class.

Outside of the class-room I used to see him at meetings of a little informal mathematical club, attended by the more advanced students, where he frequently took part in the discussions and was always alert and suggestive; and at meetings of the American Academy where he frequently took an active part in the informal debate on the paper of the evening, usually to the enlightenment or the discomfiture of the author.

The first meeting of the Academy I ever attended gave him an opportunity to show his remarkable ability to think clearly and quickly. The paper of the evening was a very elaborate one, describing the lecturer's investigations into the tides of The Gulf of Maine. An important member of the Coast Survey, he had been engaged all summer in hydrographic work at the mouth of the Bay of Fundy, but he confessed himself completely staggered by the phenomena he had

[1] Compare pages 15-16, 28 of this monograph.—R. C. A.

observed and had just described to us, which seemed to him absolutely inexplicable. At the close of the address Professor Peirce rose from his seat and began to ask leading questions. The lecturer, rather puzzled at first, began to answer them hesitatingly but soon discovered that step by step he was being led up to a theory that met all his difficulties and dissolved all his paradoxes. It was as pretty a piece of work as ever I saw done, and was manifestly entirely unrehearsed.

Benjamin Peirce, mathematician and mystic, was not always on the heights. Calling at his house one day to consult him on some abstruse problem I found him on all fours in the parlor playing bear with one of his grandchildren, and I was invited to take part in the game.[1]

In his personal relations with his students he was always courteous, kind, and helpful, if rather prone to overrate their ability and promise, and they reverenced and loved him.

IV. Reminiscences.

By Chancellor Arnold B. Chace, Brown University.

I have very pleasant memories of Professor Benjamin Peirce. In the later seventies being desirous of taking up the study of quaternions, which were then beginning to be talked about, and having worked at them myself for a while, I decided that I needed some help—and, going one day to Cambridge, after making some inquiries, I called on Professor Peirce, introduced myself, and asked him if he would assist me. He received me very pleasantly and seemed much interested in my request. He had at that time retired from active service in the college, but, as he told me with reference to a recent request from the head of Radcliffe, which was just starting, he was glad to assist anyone who deserved it.

I went to his house one afternoon a week for nearly a year and, sitting in his pleasant study before an open fire, I would show him the work that I had done in the previous week, and he, an old man, and I, a young man, discussed quaternions and many other matters in a most friendly way.

He was one of the most stimulating men that I have ever known. I can picture him now with his large noble brow, his beautiful white hair, his flashing eyes, his animated but kindly face, and most inspiring personality.

[1] A quotation from Henry Cabot Lodge's *Early Memories* (New York, 1913, pp. 55–56) may be recalled in this connection: "Altogether he had a fascination which even a child felt, and all the more because he was full of humor, with an abounding love of nonsense, one of the best of human possessions in this vale of tears. I know that I was always delighted to see him, because he was so gentle, so kind, so full of jokes with me, and so 'funny.' As time went on I came as a man to know him well and to value him more justly, but the love of the child, and the sense of fascination which the child felt, only grew with the years."

Another germane quotation may be made from E. W. Emerson's *The Early Years of the Saturday Club* (Boston and New York, 1918, p. 102): "A pleasant reminiscence of the family life is given by his daughter, another instance of Leasts and Mosts in this remarkable man. Before breakfast he always went to walk with his younger children, now a delightful memory to them. This man who could divine and see remotest suns in space, amused his little ones by allowing no pin to hide from his eyes in the dust of the sidewalk;—'although he never seemed to be looking for them, he would suddenly stoop to pick up a pin. He had various 'pincushions'; one was the trunk of an elm tree near our gate, others on Harvard and Brattle streets. Those on Quincy and Kirkland streets are still standing.' "—R. C. A.

He was very enthusiastic in his belief that Hamilton's *Lectures on Quaternions* and *Elements of Quaternions* marked a most important step in the progress of mathematical science, a belief which I think has been fully justified in all our modern vector analysis. A recent writer has placed [1] these books of Hamilton's among the ten most important works in the development of mathematics.

It was at just about this date, in the winter 1878–79, that Professor Peirce delivered a course of lectures at the Lowell Institute entitled "Ideality of the Physical Sciences." In the sixth and last of these lectures he made the statement that "Ideality is preëminently the foundation of mathematics." It was in this lecture also, after stating that two geometers had computed independently of each other the elements of the orbit of a planet which would reconcile the apparent discrepancies in the orbit of the recently discovered planet Uranus, and had determined that this new planet was situated at a certain point in the heavens, he told how on December 23, 1846, Dr. Galle of Berlin directed his telescope at the designated spot and discovered the new planet Neptune. Professor Peirce then made the surprising statement that the planet so discovered by Dr. Galle was not in the place that had been figured out, but that there were two possible positions of the planet and that by a remarkable coincidence on the given night the two positions were in a straight line from the earth. I remember very well as I sat in his study that he repeated this story to me with much animation, and when I questioned him further about it he said he was sure of his reasoning, but the calculations were so long and laborious that he had never had the courage to go through them a second time.[2]

V. Biographical Sketch.[3]

By R. C. Archibald, Brown University.

Mathematical research in American Universities began with Benjamin Peirce. His influence on students and contemporaries was extraordinary; this is borne out by the "Reminiscences" given above. In September, 1924, President Lowell wrote also: "I have never admired the intellect of any man as much as that of Benjamin Peirce. I took every course that he gave when I was in College, and whatever I have been able to do intellectually has been due to his teaching more than to anything else."

Hence no apology is required for taking the greater part of an issue of the Monthly to exhibit the life and work of such a man, if information of this kind is not already easily available. An appreciative sketch appeared in this Monthly nearly thirty years ago,[4] but it seemed evident that something of a more com-

[1] In this Monthly, *1923*, 320.—R. C. A.

[2] Compare page 14 of this monograph.—R. C. A.

[3] The gist of this section and of the next was given in a paper on Benjamin Peirce read at a joint session of the History of Science Society, Section L of the American Association for the Advancement of Science, and of the Mathematical Association of America, Washington, D. C., January 1, 1925.

[4] "Benjamin Peirce" by F. B. Matz, in this Monthly, *1895*, 173–179; also in *A Mathematical Solution Book* by B. F. Finkel, fourth ed., Springfield, Mo., 1902, pp. 524–528.

prehensive nature should be attempted, and that sources of further information should be indicated for the present generation and for the future historian of American mathematics. Previously [1] there has been no adequate indication of the extent of Peirce's publications; even the lists of his periodical articles in the *Royal Society Catalogue of Scientific Papers*, and in "Poggendorff," [2] are by no means complete. Furthermore, critical estimates of Peirce's notable work in linear associative algebra, in connection with the problem of the discovery of Neptune, and in other fields, are not readily to be found by the average inquirer.

Benjamin Peirce was born at Salem, Mass., April 4, 1809, and died at Cambridge, Mass., October 6, 1880. He was descended from John Pers, a weaver of Norwich, Norfolk Co., England, who emigrated to this country in 1637. His father was Benjamin Peirce (1778–1831), who graduated from Harvard University in 1801, represented Salem in the lower branch of the legislature for several years and was later sent to the state senate. For the last five years of his life he was librarian at Harvard University; his history of the University was published after his death.

Benjamin Peirce entered Harvard University in 1825 and graduated in the class of 1829, with membership in the Phi Beta Kappa Society. Oliver Wendell Holmes, James Freeman Clarke, educator and prolific author, and George T. Bigelow and Benjamin R. Curtis, eminent jurists, were classmates. For the two years immediately after graduation, Peirce was associated with George Bancroft as teacher at the famous Round Hill School, Northampton, Mass. In 1831 he was appointed tutor in mathematics at Harvard College and

[1] Some of the best printed sources of information concerning Peirce's ancestry, life and work are: (1) F. C. Peirce, *Peirce Genealogy*, Worcester, 1880; (2) R. S. Rantoul, *Historical Colls. Essex Institute*, vol. 18, 1881, pp. 161–176; (3) *Proc. Amer. Acad. Arts and Sciences*, vol. 16, 1881, pp. 443–454, by [H. A. Newton]. This appeared in slightly different form in *Amer. Jl. Sci.*, vol. 122, 1881, pp. 167–178; (4) *Benjamin Peirce . . . A Memorial Collection*, edited by M. King, Cambridge, Mass., 1881, 64 pp. [includes sketch by Thomas Hill from *The Harvard Register*, May, 1880; editorials and sketches from various newspapers and periodicals; sermons by A. P. Peabody, T. Hill and C. A. Bartol; and poems by Oliver Wendell Holmes, G. Thwing and T. W. Parsons. The poem by Holmes appeared originally in the *Atlantic Monthly*, vol. 46, 1880, p. 823; also in *Kansas City Review*, vol. 4, p. 510]; (5) *Mo. Notices Royal Astr. Soc.*, vol. 41, 1881, pp. 191–193; (6) *Proc. Royal Soc. Edinb.*, vol. 22, 1882, pp. 739–743, by Simon Newcomb; (7) "The Services of Benjamin Peirce to American mathematics and astronomy" by J. J. See, *Popular Astronomy*, vol. 3, 1895, pp. 49–57; (8) *Encyclopædia Britannica*, eleventh edition, vol. 11, 1911; (9) F. Cajori, *A History of Mathematics*, second ed., 1919, pp. 338–339, etc.; and (10) E. W. Emerson, *The Early Years of the Saturday Club, 1855–1870*, Boston and New York, 1918, pp. 96–109. Other references will be given in the following pages. Sketches of minor importance occur in: (1) F. S. Drake, *Dictionary of American Biography*, Boston, 1872; (2) *The Harvard Book, a series of historical and biographical and descriptive sketches* by various authors collected and publ. by F. O. Vaille and H. A. Clark, vol. 1, Cambridge, 1875, pp. 104, 172–173; (3) Appleton's *Cyclopædia of American Biography*, New York, vol. 3, 1888; and (4) *National Cyclopædia of American Biography*, New York, vols. 8, 9, 10, 1898, 1907, 1909.

A considerable quantity of Benjamin Peirce's manuscripts and correspondence was presented to the American Academy of Arts and Sciences in 1913. This collection is soon to be augmented by many other letters of great value which have been in the possession of Peirce's grandson, Benjamin P. Ellis of Cambridge, Mass. I am much indebted to Mr. Ellis for allowing me free access to this material.

[2] J. C. Poggendorff, *Biographisch-Literarisches Handwörterbuch zur Geschichte der exacten Wissenschaften*, Leipzig, vols. 2, 3, 1863, 1898.

was in full charge of the mathematical work. In 1833 he received the A.M. degree from Harvard, was appointed professor of mathematics and natural philosophy, and was married to Sarah H. Mills of Northampton. His academic title was changed to that of professor of astronomy and mathematics in 1842.

Professor Peirce had four sons and a daughter. One son, Benjamin Mills (1844–70), was a mining engineer; another, Herbert Henry Davis [1] (1849–1916), was a diplomat.[2] The eldest, James Mills (1834–1906), was assistant professor of mathematics at Harvard 1861–69, and professor from 1869 till his death; he was also dean of the graduate school of Arts and Sciences 1890–95, and of the faculty of Arts and Sciences 1895–98. But the son who seemed largely to inherit his father's intellectual powers [3] was Charles Santiago Saunders (1839–1914) who was lecturer at Harvard, in philosophy and logic, 1869–71. Of these brothers, the brilliant Benjamin Osgood Peirce (1854–1914) was a second cousin once removed.[4]

Professor Benjamin Peirce was honored in various ways both in this and in other countries. He was elected a Member of the American Philosophical Society in 1842; an Associate of the Royal Astronomical Society, London, 1850; a Foreign Member (limited to 50) of the Royal Society of London in 1852; an Honorary Member of the State Historical Society of Wisconsin, 1854; a Fellow of the American Academy of Arts and Sciences in 1858; an Honorary Fellow of the University of St. Vladimir, at Kiev, Russia (now Ukrainia), 1860; a Corresponding Member of the British Association for the Advancement of Sciences in 1861; an Honorary Fellow (limited to 36) of the Royal Society of Edinburgh in 1867; and a Correspondent in the mathematical class of the Royal Society of Sciences at Göttingen in 1867. He received the degree of LL.D. from the University of North Carolina in 1847, and from Harvard [5] in 1867. He was: One of a committee of five appointed by the American Academy of Arts and Sciences to draw up a "Program for the Organization of the Smithsonian Institution," 1847; Consulting Astronomer for the Nautical Almanac, 1849–67; Director of the

[1] Probably named after Admiral Charles Henry Davis (1807–77), who married Benjamin Peirce's wife's sister.

[2] See *National Cyclopædia of American Biography*, vol. 10, p. 449, and vol. 9, p. 539.

[3] Papers of C. S. S. Peirce are referred to in such works as: B. Russell, *The Principles of Mathematics*, vol. 1, Cambridge, 1903; C. I. Lewis, *A Survey of Symbolic Logic*, Berkeley, 1918; and F. Enriques, *Per la Storia della Logica*, Bologna, 1922. See, also, E. W. Davis, "Charles Peirce at Johns Hopkins," *The Mid-West Quarterly*, New York, vol. 2, pp. 48–56; it is here stated that Sylvester considered C. S. S. Peirce "a far greater mathematician than his father."

[4] At Harvard he was instructor in mathematics, 1881–84, then assistant professor of mathematics and physics, 1884–88, and finally professor of mathematics and natural philosophy, 1888–1914.

[5] In a letter dated July 29, 1867, the president of Harvard College, who was also somewhat of a mathematician, wrote as follows:

"I have the honor of informing you that the University, on Commencement Day, conferred upon you the degree of Doctor of Laws in recognition of the transcendent ability with which you have pursued mathematical physical investigations, and in particular for the luster which she has herself for so many years borrowed from your genius.

"With the sincerest regard,

"Very truly and gratefully yours,

"THOMAS HILL."

longitude determinations of the United States Coast Survey, 1852–67; Member of the Scientific Council (J. Henry, A. D. Bache, B. Peirce) of the Dudley Observatory, Albany, 1855–58; Superintendent of the Coast Survey, February 26, 1867, to February 16, 1874, while continuing to serve as professor at Harvard; consulting geometer of the Survey,[1] 1874–80; President of the American Association for the Advancement of Science, 1853, and elected Fellow, 1875; Chairman of the Department of Education of the American Social Science Association, 1869–72, acting president[2] in 1878, and vice-president in 1880; One of the fifty incorporators of the National Academy of Sciences, one of the nine members of the committee of organization, and chairman of the mathematics and physics class,[3] 1863; Director of the expedition to Sicily to observe the eclipse of the sun, December, 1870; Coöperating Editor of the *American Journal of Mathematics*, volume 1, 1878; Special Lecturer on physical philosophy at the Concord Summer School of Philosophy and Literature, 1879; Lecturer at the Lowell Institute, 1879, and at the Peabody Institute, 1880.

In a recently published article, Professor Coolidge pointed out[4] that before the time of Benjamin Peirce it never occurred to anyone that mathematical research "was one of the things for which a mathematical department existed. Today it is a commonplace in all the leading universities. Peirce stood alone—a mountain peak whose absolute height might be hard to measure, but which towered above all the surrounding country." In his publications[5] and papers before scientific bodies, Peirce touched on a wide range of topics.

Of his eleven works, in twelve volumes, six were elementary texts, some of which went through several editions. The first, on plane trigonometry, appeared in 1835, and the second and third, on spherical trigonometry and sound, in 1836. The seventh work, in two volumes (1841–46), dealt with analytic geometry,

[1] This post was held while retaining his professorship. He was appointed "consulting geometer" with compensation at the rate of $4,000 per annum, and subsistence at the same rate per diem as was allowed the late Hydrographic Inspector." For an account of Peirce's connection with the Survey, see *Centennial Celebration of the United States Coast and Geodetic Survey, April 5 and 6, 1916*, Washington, 1916, p. 137. T. C. Mendenhall here remarks, "As a genius in mathematics and astronomy he is easily a star of first magnitude in the Coast Survey galaxy." In an account of the centennial celebration in the *Scientific Monthly*, vol. 3, 1916, p. 616, the story is told that at a meeting of the National Academy of Sciences he spent an hour filling the blackboard with equations, and then remarked, "There is only one member of the Academy who can understand my work and he is in South America." Was this B. A. Gould? Hilgard was the managing head of the Survey during Peirce's administration; see anonymous note by S. Newcomb, *Nation*, March 5, 1874, vol. 13, p. 157.

[2] He declined to take the titular office of president offered to him in this year although performing all the duties of the office. For a sketch of Professor Peirce including an account, by F. B. Sanborn, of his relations to the American Social Science Association, see *Journal of Social Science*, no. 12, 1880, pp. ix–xi. For the title of his address delivered before the Association in 1878, see the List of Peirce's Writings in the next Section.

[3] Compare *A History of the First Half-Century of the National Academy of Sciences, 1863–1913*, Washington, 1913, pp. 9, 10, 20, 21, 23, 27, 168–171, 215, 223, and 256. Peirce was one of the first sixteen to read papers before the Academy, January, 1864.

[4] "The Story of Mathematics at Harvard" by J. L. Coolidge, *Harvard Alumni Bulletin*, January 3, 1924, vol. 26, p. 376.

[5] A list of these, which probably closely approximates to completeness, is given in Section VI of this monograph.

differential and integral caculus, and differential equations. His other volumes were: *Tables of the Moon* (1853–56); a notable work on *Analytic Mechanics* (1855), the remarkably original *Linear Associative Algebra* (1870), and the posthumous volume of lectures, *Ideality in the Physical Sciences* (1881).

About one quarter of the titles of Peirce's publications relate to topics of pure mathematics and three quarters to questions mainly in the fields of astronomy, geodesy and mechanics. His first publications, when only sixteen years of age, were solutions of problems in algebra and mechanics. Very early in life, possibly through having Ingersoll Bowditch as schoolmate, Peirce had the good fortune to become acquainted with Ingersoll's father, Dr. Nathaniel Bowditch [1] (1773–1838), author of *The New American Practical Navigator* (which has gone through so many editions in the last hundred years), and translator of Laplace's celestial mechanics. During the ten years before he was thirty, Peirce revised and corrected the proof sheets of this translation. Among other works, Peirce contributed an original notable result regarding perfect numbers; gave certain methods of determining the number of real roots of equations applicable to transcendental as well as to algebraic equations; made an important advance in the treatment of Kirkman's school-girl problem;[2] discussed a new binary system of arithmetic; wrote on probabilities at the three-ball game of billiards, on the extension of Lagrange's theorem for development of functions, and on transformation of curves; and supplemented his volume on associative algebra by a memoir on the uses and transformations of linear algebras. Apart from volumes already referred to, his publications on applied mathematics included: various papers on the perturbations of Neptune and Uranus; a mathematical treatment (also translated into German) of the problem of Saturn's rings, leading to the result that the rings were fluid; [3] a note upon the conical pendulum; papers on the relation between the elastic curve and the motion of the pendulum, on a criterion for the rejection of doubtful observations, on the catenary on a vertical right cone, on the internal constitution of the earth, and on a mathematical investigation of the fractions which occur in phyllotaxis.

The concluding characteristic paragraph of this last paper is as follows:

"May I close with the remark, that the object of geometry in all its measuring and computing, is to ascertain with exactness the plan of the great Geometer, to penetrate the veil of material forms, and disclose the thoughts which lie beneath them? When our researches are successful, and when a generous and heaven-eyed inspiration has elevated us above humanity, and raised us triumphantly into the very presence, as it were, of the divine intellect, how instantly and entirely are human pride and vanity repressed, and, by a single glance at the glories of the infinite mind, are we humbled to the dust." [4]

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[1] The dedication of Peirce's *Analytic Mechanics* is as follows: "To the cherished and revered memory of my master in science, Nathaniel Bowditch, the father of American geometry, this volume is inscribed."

[2] Sylvester referred to this treatment as "the latest and probably the best" (*Philosophical Magazine*, 1861, vol. 21, p. 520; also *Collected Mathematical Papers of . . . Sylvester*, vol. 2, 1908, p. 276). See also my notes on this title in the next Section.

[3] Compare S. Newcomb, *Popular Astronomy*, 1879, p. 358; also Newcomb-Engelmann, *Populäre Astronomie*, 5th ed., 1914, p. 429.

[4] Another quotation may be given to illustrate Peirce's manner of thought to which reference has been made above in the "Reminiscences." The following are concluding sentences from a

While Peirce read before scientific societies many papers concerning his investigations, the printed reports of them are unfortunately often mere abstracts. "His mind moved with great rapidity, and it was with difficulty that he brought himself to write out even the briefest record of its excursions."

"His elementary books were remarkable for their condensation. In the geometry, especially, the short and terse and comprehensive forms of mathematical thought and expression, natural to the mathematician, were substituted for the minute demonstrations of Euclid. Free use was also made of infinitesimals." [1]

In order to bring out more clearly the place Peirce occupies in the development of American mathematics it seems desirable to comment further on four of the subjects which he discussed in a notable manner: 1. criterion for the rejection of doubtful observations; 2. perturbations of Uranus and the discovery of Neptune; 3. analytic mechanics; 4. linear associate algebra.

1. "Peirce's criterion," as the term has appeared in scientific literature, had as its object the solution of a delicate and practically important problem of probability; this problem is: "Being given certain observations of which the greater part is to be regarded as normal and subject to the ordinary law of error adopted in the method of least squares, while a smaller unknown portion is abnormal, and subject to some obscure source of error, to ascertain the most probable hypothesis as to the partition of the observations into normal and abnormal." This criterion has been regarded as one of Peirce's best contributions to science. In volume 46 (St. Petersburg, 1898) of the great Russian encyclopædia (based on "Brockhaus") it is especially referred to in a ten-line biographical notice of Peirce.

The excessive brevity of Peirce's statement concerning the criterion, when it appeared in 1852, resulted in frequent misunderstandings. The tables which Gould published three years later [2] facilitated its application. But it was not till 1878 that Peirce somewhat remedied his original statement by giving fuller explanations. In 1920, however, R. M. Stewart proved the statement fallacious.[3]

The criterion and its application are set forth at length in W. Chauvenet's *Manual of Spherical and Practical Astronomy* [4] (1868), and a paragraph is devoted to it in W. S. Jevons's *Principles of Science* [5] (1877). An illustration of a recent work where application of the criterion is suggested is H. M. Wilson's *Topographic, Trigonometric and Geodetic Surveying* [6] (1912).

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paper on Saturn's rings: "But in approaching the forbidden limits of human knowledge, it is becoming to tread with caution and circumspection. Man's speculations should be subdued from all rashness and extravagance in the immediate presence of the Creator. And a wise philosophy will beware lest it strengthen the arms of atheism, by venturing too boldly into so remote and obscure a field of speculation as that of the mode of creation which was adopted by the Divine Geometer."

[1] S. Newcomb, *Royal Soc. Edinb., Proc.*, vol. 22, p. 739.

[2] B. A. Gould, "On Peirce's criterion for the rejection of doubtful observations with tables for facilitating its application," *Astron. Jl.*, vol. 4, pp. 81–87. G. B. Airy expressed himself as believing Peirce's criterion defective in its foundation and illusory in its results (*Astron. Jl.*, vol. 4, pp. 137–138, 1856); Joseph Winlock showed (pp. 145–146) that his argument was wholly unsound. Compare articles by Stone and Glaisher in *Mo. Notices R. Astr. Soc.*, vols. 28, 33–35.

[3] *Popular Astronomy*, vol. 28, pp. 2–3. See also J. L. Coolidge, *Probability*, Oxford, 1925, pp. 126-127.

[4] Volume 2, fourth edition, 1868, pp. 558–566, 596–599.

[5] London, second edition, p. 391. [6] New York, third edition, pp. 604–606.

2. The computation of the general perturbations of Uranus and Neptune was the first work to extend Peirce's reputation. Simon Newcomb's compact statement[1] in this connection is here reproduced with two added footnotes:

"The formulæ to which he was led were published in the first volume of the *Proceedings of the American Academy*,[2] but were accompanied by no description of his process. Subsequent investigations, however, showed them to have been remarkably accurate. In his views of the discrepancy between the mean distance of Neptune as predicted by Leverrier, and as deduced from observations, he was less fortunate, although when due consideration is given to Leverrier's conclusions, there was much plausibility in the position taken by Peirce. As the subject has frequently been discussed without a due comprehension of all the circumstances, a brief review of them may be appropriate.[3]

"Leverrier, from his researches, found for the mean distance of the disturbing planet, 36.1539, and a consequent period of 217 years. He also announced that the limits of the mean distance which would satisfy the observed perturbation of Uranus were 35.04 and 37.90. He founded this conclusion on a supposed inadmissible increase of the outstanding differences between theory and observation, as the mean distance was diminished below 35. But when the planet was discovered, its mean distance was found to be only 30; and yet the observations of Uranus were as well satisfied as by Leverrier's hypothetical planet. It was, therefore, an expression of Peirce's high confidence in the accuracy of Leverrier's conclusions that led him to announce that there were two solutions to the problem; the one being that found by Leverrier, and the other that corresponding to the actual case. He also sought to show a cause for the two solutions in a supposed discontinuity in the form of the perturbations, when the period was brought to the point at which five revolutions of Uranus would be equal to two of Neptune. As a matter of fact, however, it has been shown by Professor Adams that there is no such discontinuity in the actual perturbations during the limited period; from which it would follow that Leverrier must have made a mistake in tracing out the conclusions which would follow when the mean distance of the disturbing planet was diminished."

H. H. Turner has also published a very readable account of this matter in his *Astronomical Discovery* (London, 1904).

3. As to Peirce's *Analytic Mechanics*, Simon Newcomb referred to it[4] as "the most characteristic as well as the most extensive of his works." Then he continues: "The exposition of dynamical concepts in the first forty pages is pleasant reading for one already acquainted with the subject, but that a student beginning the subject could understand it without clearer distinction of definitions, axioms, and theorems seems hardly possible." In his later years Peirce often said he wanted to rewrite his *Mechanics* and introduce quaternions into it. Sir

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[1] *Roy. Soc. Edinb., Proc.*, vol. 22, pp. 740–741.

[2] Published in 1848. It is related that when, in 1846, Peirce announced in the American Academy that Galle's discovery of Neptune in the place predicted by Leverrier was a happy accident, the President, Edward Everett, "hoped the announcement would not be made public: nothing could be more improbable than such a coincidence."—"Yes," replied Peirce, "but it would be still more strange if there was an error in my calculations,"—a confident assertion which the lapse of time has vindicated. In this connection, it is noteworthy that Peirce was not made a fellow of the American Academy till many years after he had been honored by the American Philosophical Society and two foreign bodies. In 1878 Peirce sent in his resignation as a Fellow of the Academy, but this was never accepted.

[3] A full statement of Peirce's views, which he maintained to the last, is given by J. M. Peirce, on pages 200–211 of B. Peirce's *Ideality in the Physical Sciences*. It was 28 years after Peirce's criticisms of the work of Leverrier and Adams were published (1848), that is, 1876, that J. C. Adams made a reply, *Journal de Mathématiques* (Liouville), vol. 41, 1876; see also *The Scientific Papers of John Couch Adams*, vol. 1, 1896, pp. xxxiii, 57, 64.

[4] *Royal Society Edinb., Proc.*, vol. 22, p. 742.

Thomas Muir has given [1] an analysis of the section on determinants and functional determinants. The analysis is as follows:

"At the outset of his Tenth Chapter, which deals with the integration of the differential equations of motion, Peirce feels the need for making his reader acquainted with the properties of functional determinants. He accordingly gives as a preparation a brief account (§§ 327–348, pp. 172–183) of determinants in general, and then expounds within the space of sixteen broad-margined pages the main theorems of Jacobi's 'De determinantibus functionalibus.' The treatment of the original is free and masterly, the order being altered with good effect. For example, Jacobi's incorrectly stated proposition is brought forward to occupy the second place, the enunciation being *If either* (i.e., *any one*) *of the given functions contains any of the other functions, these* (*latter*) *functions may be regarded as constant in finding the functional determinant.* There is thence deduced Jacobi's last proposition of all, namely, that expressing the determinant as a single product: and this in turn is used to discuss the connection between the vanishing of the determinant and interdependence of the functions.

"Had Peirce's exposition been less condensed and been published as part of an ordinary text-book of determinants, its value at this time to English-speaking students would have been considerable."

4. There seems to be no question that his *Linear Associative Algebra* [2] was the most original and able mathematical contribution which Peirce made. He himself held the work in high esteem; on April 4, 1870, he wrote in the introduction, "This work has been the pleasantest mathematical effort of my life. In no other have I seemed to myself to have received so full a reward for my mental labor in the novelty and breadth of the results." In his *Synopsis of Linear Associative Algebra* (published by the Carnegie Institution in 1907) J. B. Shaw characterized the work as "really epoch-making," and devoted a number of pages (52–55, 101–106) to formulating the main results. While the monograph attracted wide and favorable comment in England and America,[3] continental investigators on the subject (1889–1902) did not give Peirce the credit which his results and methods deserved. Adverse criticism had been "due in part to a misunderstanding of Peirce's definitions, in part to the fact that certain of Peirce's principles of classification are entirely arbitrary and quite distinct in statement from those used by Study and Scheffers,[4] in part to Peirce's vague and in some cases unsatisfactory proofs, and finally to the extreme generality of the point of view from which his memoir sprang, namely a 'philosophic study of the laws of algebraic operation.'" In order that Peirce's work should receive due recognition, H. E. Hawkes discussed and answered the following questions: [5]

[1] T. Muir, *The Theory of Determinants*, London, vol. 2, 1911, p. 251.

[2] The first sentence of the work is often quoted. It is: "Mathematics is the science which draws necessary conclusions." On page 5 we find a reference to the "mysterious formula" $i^{-i} = e^{\pi/2} = 4.810477381$. Compare this MONTHLY, *1921*, 115–121.

[3] The substance of the work was reviewed by William Spottiswoode in his retiring presidential address delivered before the London Mathematical Society in 1872 ("Remarks on Some Recent Generalizations in Algebra," *London, Math. Soc., Proc.*, vol. 4, 1873, pp. 147–164). Peirce refers to this (1875) as a "fine, generous, and complete analysis." See also Cayley, *Collected Mathematical Papers*, vol. 11, pp. 457–8; vol. 12, pp. 60–71, 106, 303, 459, 465. The first reference is to Cayley's address before the British Association in 1883, when he spoke of "the valuable memoir by the late Benjamin Peirce." At the last reference Cayley writes (1887): "the general theory of associative linear forms is treated in a very satisfactory manner in Peirce's memoir."

[4] A complete list of references may be found in Shaw's work. So also for references to other works developed from Peirce's ideas.

[5] "Estimate of Peirce's Linear Associative Algebra," *Amer. Jl. Math.*, vol. 24, 1902, pp. 87–95; "On Hypercomplex Number Systems," *Amer. Math. Soc., Trans.*, vol. 3, 1902, pp. 312–330.

(1) What problem did Peirce attack, and to what extent did he solve it? (2) What relation does this problem bear to that treated by Study and Scheffers? (3) To what extent do Peirce's methods assist in the solution of that problem? In part Hawkes summed up his conclusions as follows:

"We can now state precisely the problem that Peirce set for himself. He aimed to develop so much of the theory of hyper-complex numbers as would enable him to enumerate all inequivalent, pure, non-reciprocal number systems in less than seven units. The relation to the problem treated by Scheffers is plain if we remember that the first two of Peirce's principles of classification are identical with those of Scheffers, and the other three are only slightly modified. Peirce solved this problem completely. The theorems stated by him are in every case true, though in some cases his proofs are invalid."

Hawkes showed also that by using Peirce's principles as a foundation, we can deduce a "method more powerful than those hitherto given," by such writers as Study and Scheffers, for enumerating all number systems of the type considered by Scheffers. Since Study is the author of the article on "Theorie der gemeinen und höheren complexen Grössen" in the *Encyklopädie der math. Wissenschaften,* one is not surprised to find his references to Peirce so wholly inadequate.[1]

Benjamin Peirce died in 1880 in the seventy-second year of his life and in the fiftieth of his service in the University. The teaching of mathematics at Harvard during this half century has been described by Florian Cajori [2] and J. K. Whittemore.[3] Other information of interest, supplementing the "Reminiscences" given above, may be found in the printed Reports of the presidents of the University during those years, in the 1829 Class Records (Harvard Univ. Library), and in the first volume of the *Annals of the Harvard Observatory.*

In the account of "How I was Educated," Edward Everett Hale, '39, wrote: [4] "The classical men made us hate Latin and Greek; but the mathematical men (such men! Pierce [*sic*] and Lovering) made us love mathematics, and we shall always be grateful to them." Lovering taught mathematics and natural philosophy at Harvard, 1838–88. In another place Hale wrote: [5] "I had but four teachers in college,—Channing, Longfellow, Peirce and Bachi. The rest heard me recite but taught me nothing."

Colonel Henry Lee has written of Peirce as follows in *The Harvard Book*:[6]

"Why we should have given him the affectionate diminutive name of 'Benny' I cannot say, unless as a mark of endearment because he could fling the iron bar upon the Delta farther than any undergraduate, or perhaps because he always thought the bonfire or disturbance outside the college

[1] Volume 1, pp. 159 and 167. In Cartan's form of the article, *Encyclopédie des Sciences Mathématiques*, tome 1, vol. 1, a very different presentation is found; see, for example, pp. 369, 401–2, 417, 422–5. While scores of trivialities are reviewed in *Jahrbuch über die Fortschritte der Mathematik* for 1881, absolutely nothing is given concerning Peirce's notable work.

[2] F. Cajori, *The Teaching and History of Mathematics in the United States*, Washington, 1890, pp. 133–148, 278, 397. See also J. L. Coolidge, Story of math. at Harvard, *l. c.*, pp. 372–376.

[3] In a sympathetic sketch of "James Mills Peirce," *Science*, n.s., vol. 24, 1906, pp. 40–48. Rather curiously J. M. Peirce, who died in 1906, was, as his father also, in the seventy-second year of his life and in the fiftieth of his service in the University.

[4] *Forum*, vol. 1, 1886, p. 61.

[5] *Outlook*, June 4, 1889, vol. 59, p. 316; also in E. E. Hale, *James Russell Lowell and his Friends*, Boston, 1899, p. 128.

[6] Volume 1, Cambridge, 1875, p. 104.

grounds, and not inside, and conducted himself accordingly. His softly lisped *sufficient* brought the blunderer down from the blackboard with a consciousness of failure as overwhelming as the severest reprimand. There was a delightful abstraction about this absorbed mathematician which endeared him to the students, who hate and torment an instructor always on the watch for offences, and which confirmed the belief in his peculiar genius."

Peirce's force and judgment in a great emergency are shown in the following anecdote by one who was present:[1]

Jenny Lind's last concert of the original series, given under the auspices of Phineas T. Barnum, was given at the hall over the Fitchburg Railroad Station. Tickets were sold without limit,—many more than the hall could hold,—and there was every prospect of a riot. Barnum had taken the precaution to leave for New York. I got about one-third up the main aisle, but could get no farther. Just ahead of me was Professor Peirce. The alarm was increasing. The floor seemed to have no support underneath, but to hang over the railroad track by steel braces from the rafters above. Would it hold? The air was stifling and windows were broken, with much noisy crashing of glass, in order to get breath. Women were getting uneasy. And there was no possibility of escape from a mass of human beings so packed together. We knew, from the conductor's baton, that the orchestra was playing, but no musical sound reached us. Professor Peirce mounted a chair. Perfect silence ensued as soon as he made himself seen. He stated, very calmly, certain views at which he had arrived after a careful study of the situation. The trouble was at once allayed. Jenny Lind recovered her voice and the concert went on to its conclusion."

Another contemporary has made this record:[2]

"He was among the first to read any new and noteworthy poem [3] or tale, to hear a new opera or oratorio; and his judgment and criticism upon such matters was keen and original. His interest in religious themes was deep, but it was in the fundamental doctrines rather than in the debates of sectarians; he was a devout believer in Christianity, but held to no established creed."

Among Peirce's students who afterwards became eminent were Simon Newcomb and George W. Hill. In his *Reminiscences of an Astronomer*,[4] Newcomb has happily hit off some of Peirce's most striking characteristics as follows:

"Professor Peirce was much more than a mathematician. Like many men of the time, he was a warm lover and a cordial hater. It could not always be guessed which side of a disputed question he would take; but one might be fairly sure that he would be at one extreme or the other. As a speaker and lecturer he was very pleasing, neither impressive nor eloquent, and yet interesting from his earnestness and vivacity. For this reason it is said that he was once chosen to enforce the views of the university professors at a town meeting, where some subject of interest to them was coming up for discussion. Several of the professors attended the meeting, and Peirce made his speech. Then a townsman rose and took the opposite side, expressing the hope that the meeting would not allow itself to be dictated to by these nabobs of Harvard College. When he sat down, Peirce remained in placid silence, making no reply. When the meeting broke up, some one asked Peirce why he had not replied to the man. 'Why! did you not hear what he called us? He said we were nabobs! I so enjoyed sitting up there and seeing all the crowd look up to me as a nabob that I could not say one word against the fellow.'"

An estimate by one who was not a scientist may be added. In a centennial address Wendell Phillips referred [5] to Peirce as "the largest natural genius, the man of the deepest reach and firmest grasp and widest sympathy, that God has

[1] E. W. Emerson, *The Early Years of the Saturday Club*, Boston and New York, 1918, pp. 100–101.

[2] *Nation*, New York, October 14, 1880.

[3] Judge Addison Brown wrote that "Professor Peirce seemed a poet in a mathematical dream, his mind so preoccupied, as it were gazing at the stars" (*Annals of the Harvard Class of 1852*, by G. W. Edes, Cambridge, 1922, p. 322; there are references to Peirce on several other pages).

[4] Boston, 1903, pp. 77–78; see also pp. 276–277.

[5] *The Scholar in a Republic. Address at the Centennial Anniversary of the Phi Beta Kappa Society of Harvard College*, Boston, 1881, p. 13.

given to Harvard in our day,—whose presence made you the loftiest peak and farthest outpost of more than mere scientific thought,—the magnet who, with his twin Agassiz, made Harvard for forty years the intellectual Mecca of forty States."

Andrew P. Peabody, '26, who was preacher to the University and professor of Christian morals at Harvard for the last two decades of Peirce's service, has devoted several pages of his *Harvard Reminiscences* [1] to Peirce. In referring to the last few years of Peirce's life he tells how he

"had for his pupils only young men who were prepared for profounder study than ever entered into a required course, or a regularly planned curriculum; but he never before taught so efficiently, or with results so worthy of the mind and heart and soul, which he always put into his work. His students were inflamed by his fervor, and started by him on the eager pursuit of the eternal truth of God, of which mathematical signs and quantities are the symbols."

It was not alone "young men" whom Peirce was willing to direct, as the following extract from a letter [2] written in 1879 shows: "I will do the same for the young women that I do for the young men. I shall take pleasure in giving gratuitous instruction to any person whom I find competent to receive it. I give no elementary instruction, but only in the higher mathematics."

The pall-bearers at Peirce's funeral were President Eliot, Ex-President Thomas Hill, C. P. Patterson (Superintendent of the Coast Survey), J. J. Sylvester,[3] J. Ingersoll Bowditch, Simon Newcomb, Joseph Lovering, Andrew P. Peabody, and his classmates, James F. Clarke and Oliver Wendell Holmes.

At a meeting of the President and Fellows of Harvard College, October 11, 1880, the entry made upon the records regarding Peirce stated that

"The University must long lament the loss of an intelligence so rare, an experience so rich, and a personal influence so strong, as his.

"As a teacher, he inspired young minds with a love of truth, and touched them with his own enthusiasm; as a man of science, his attainments and achievements and his public services have reflected honor upon the University and the country."

What has been indicated above, coupled with the bibliography which follows, provides material necessary for forming an intelligent opinion as to the activities, personality, brilliant and powerful mind, and "wonderfully stimulating influence" of one of the most eminent and original scientists that America produced in the last century. In this country Peirce was the leading mathematician of his time, and a pioneer in achieving notable mathematical research, some of it anticipating work by well-known Europeans of later date. It is interesting to speculate as to the possible publication harvest if Peirce had been able throughout his career constantly to meet his mathematical equals or peers, and if he had

[1] Boston, 1888, pp. 180–186.

[2] From a letter in the possession of D. E. Smith of Columbia University.

[3] "When Professor Sylvester was called [1876] to the chair of mathematics in the Johns Hopkins University, Professor Peirce of Harvard, being asked what he thought would be the opinion of American mathematicians respecting the new appointment, replied that no American mathematician had a right to have any opinion on the subject, except himself, and one of his old pupils, a distinguished professor of mathematics in one of our leading colleges." L. A. Wait, "Advanced Instruction in American Colleges," *The Harvard Register*, vol. 3, p. 127, 1880; see also page 119.

had a capable disciple always at hand to put his ideas on paper in a form suitable for publication.

In a passport dated May 14, 1860, Peirce is described as aged 51, of height 5 feet $7\frac{3}{4}$ inches, and with high forehead, hazel eyes, straight nose, regular mouth, round chin, brown hair, light complexion and oval face. A ticket dated Crystal Palace, London, October 5, 1873, gives his weight as 190 pounds.

Portraits of Peirce may be seen in the following sources:

Daguerreotype in possession of Peirce's grandson, Mr. Benjamin P. Ellis. This is the earliest known portrait of Peirce. Probably it dates from about 1845.

Painting in the Faculty Room, University Hall, Harvard University, executed by J. A. Ames (1816–1872) about 1853(?). It was Mrs. Peirce's favorite portrait, and was presented to Harvard after her death in 1888. Reproduced herewith; also in *Harvard Alumni Bulletin,* January 3, 1924, vol. 26, p. 375.

Painting executed in 1857 by Daniel Huntington (1816–1906) purchased at auction by George A. Plimpton of New York City, in 1924, and presented to Harvard University.

Photograph by Whipple and Black, Boston, in 1858, in the class of 1829 album (in ms.), Harvard University Library.

Steel engraving, *The Mathematical Monthly,* ed. by J. D. Runkle, vol. 2, July 1860, facing page 329. Engraved by H. W. Smith from a daguerreotype by Southworth and Hawes in 1860, a "most accurate likeness." The same engraving appeared as frontispiece to *Annual of Scientific Discovery . . . for 1870,* Boston, 1870.

Very interesting reproduction of a photograph, full-length, of Peirce working at the blackboard and apparently taken about 1865; *Centennial Celebration of the United States Coast and Geodetic Survey,* Washington, 1916, p. 153. Copied in *The Scientific Monthly,* vol. 3, December, 1916, p. 618. Also on a plate opposite page 96 of *The Early Years of the Saturday Club 1855–1870,* by E. W. Emerson, Boston, 1918.

Photograph taken about 1872, nearly full-length, and reproduced in *The Harvard Book* by F. O. Vaille and H. A. Clarke, vol. 1, Cambridge, 1875, opposite page 172. Copied in *The Outlook,* New York, vol. 59, 1898, p. 323. Also in *Universities and Their Sons,* vol. 2, Boston, 1899, p. 228.

Reproduction of photograph, taken about 1872, F. C. Peirce, *Peirce Genealogy,* Worcester, 1880, oppo. p. 118.

Reproduction of photograph by G. W. Pach, New York, 1879, in *Harper's New Monthly Magazine,* March 1879, vol. 58, p. 508. Also in this MONTHLY, *1895,* oppo. p. 173.

Photograph, taken about 1879, reproduced as a steel engraving frontispiece in B. Peirce, *Ideality in the Physical Sciences,* 1881. Also in *Amer. Jl. Mathematics,* vol. 24, 1902, frontispiece. Woodcut of the same, apparently, in

The Harvard Register, May 1880, vol. 1, p. 91; in *Popular Science Mo.,* vol. 18, 1881, oppo. p. 578; and in M. King, *Benjamin Peirce . . . A Memorial Collection,* 1881.

Many other unpublished photographs of Benjamin Peirce are in Mr. Ellis's possession. There are also at least two other paintings of Peirce owned by grandchildren, one an inferior portrait by Wite and the other by a Miss Whitney from a photograph.

VI. THE WRITINGS OF PEIRCE.

By R. C. ARCHIBALD.

[Solutions of problems.]
The Mathematical Diary, New York, vol. 1, 1825, ed. by R. Adrian, pp. 281, 286.

[Solutions of problems, and a problem for solution.]
The Mathematical Diary, New York, vol. 2, ed. by J. Ryan; no. X, 1828, p. 89; no. XI, 1830, pp. 116, 118; no. XIII, 1832, pp. 211–212, 216, 237, 244, 246, 310.

Laplace, *Mécanique Céleste translated with a commentary* by Nathaniel Bowditch.
Boston, 4 volumes, 1829–1839.
B. Peirce revised the entire work and detected many errata (compare the memoir in volume 4, pp. 61 and 140).

The American Almanac and Repository of Useful Knowledge. [For the years 1830–1851, the astronomical department was under the direction of B. Peirce.]
Boston, 1829–1850.

On perfect numbers.
The Mathematical Diary, vol. 2, no. XIII, 1832, pp. 267–277.
Euler showed that every even perfect number is expressible in the Euclidean form $2^{n-1}\cdot p$ where $p = 2^n - 1$ is a prime. In Peirce's paper it is shown that there can be no odd perfect number "included in the form a^r, a^rb^s, $a^rb^sc^t$, where a, b, and c are prime numbers and greater than unity." In his *History of the Theory of Numbers,* vols. 1–3, 1919–1923, L. E. Dickson does not mention this paper. He does record that in 1844 "V. A. Lebesgue stated that he had a proof that there is no odd perfect number with fewer than four distinct prime factors." We now see that an American mathematician published a proof of this theorem twelve years earlier.

An elementary Treatise on Plane Trigonometry with its applications to heights and distances, navigation and surveying.
Cambridge and Boston, Munroe, 1835, 7 + 90 pp. + 1 fold. pl.

First part of an elementary Treatise on Spherical Trigonometry.
Boston, J. Munroe & Co., 1836, 4 + 71 pp. + 1 fold. pl.

An elementary Treatise on Sound: being the second volume of a course of natural philosophy designed for the use of high schools and colleges. Compiled by Benjamin Peirce.
Boston, J. Munroe & Co., 1836, 56 + 220 pp.; diagrs. on 10 fold. pl.
"The Catalogue of works relating to sound," pp. v–lvi, is interesting, and the list of titles connected with musical matters, pp. xvii–xlviii, valuable. The work is based on J. F. W. Herschel's treatise on sound, in the *Encycl. Metropolitana.* Reviewed in *The New York Review,* vol. 4, 1839, pp. 164–176.

[Problems and solutions.]
The Mathematical Miscellany, ed. by C. Gill, New York, no. II: 1836, pp. 81–94, 94–97, 101–107; III: 1837, pp. 160–163 (194); IV: 1837, pp. 210–211, 233–234, 251–255, 258; V: 1838, pp. 289–290, 296, 309–311, 316–318, 327; VI: 1838, 359, 362–363, 383–387, 392–395, 397, 399; vol. 2, no. VII: 1839, pp. 16, 33–34, 42–47, 61, 63; VIII: 1839, 91, 92, 97–98, 110–113, 114, 117.
For the question discussed vol. 2, pp. 97–98, compare *Sphinx Œdipe,* vol. 8, 1913, pp. 93–94. It seems probable that many problems proposed by "P." in the *Mathematical Miscellany* were due to Peirce. See, for example, vol. 1, pp. 53, 55, 109, 110, 257–258, etc.

An elementary Treatise on Algebra: to which are added exponential equations and logarithms.
Boston, J. Munroe & Co., 1837, 10 + 276 pp.
Another edition, 1842, 4 + 284 pp.; 1843; fifth ed., 1845; sixth ed., 1846; other editions or reprints, 1850, 1851, 1855, 1858, 1860; and published by W. H. Dennett in 1864, 1865, 1870.

An elementary Treatise on Plane and Solid Geometry.
Boston, J. Munroe & Co., 1837, 20 + 159 pp. + 6 fold. pls.
Other editions or reprints, 20 + 3–150 p. + 6 fold. pls., 1841, 1847, 1851, 1853, 1855, 1857, 1860; and publ. by W. H. Dennett in 1863, 1865, 1866, 1867, 1868, 1869, 1870, 1871, 1872.

An account of Mr. Talbot's "Researches in the integral calculus" [*Philosophical Transactions*, vols. 54–55, 1836–37].
The Mathematical Miscellany, ed. by C. Gill, vol. 1, no. VI, 1838, pp. 404–411.

[Anonymous review of Laplace's *Mécanique Céleste*, vols. 1, 3–5; N. Bowditch's translation, vols. 1–4; and A. Young, D. A. White, and J. Pickering on Bowditch.]
N. Amer. Rev., vol. 44, 1839, pp. 143–180.

[Anonymous review of J. Pickering, D. A. White, and A. Young on N. Bowditch.]
The New York Review, vol. 4, 1839, pp. 308–323.

An elementary Treatise on Plane Geometry. . . . Printed for the use of the blind.
Boston, At the Press of the Perkins Institution and Massachusetts Asylum, 1840. 71 pages of definitions and demonstrations + 19 pages embossed diagrams.
In Boston line type; size 11 x 10 inches.

An elementary Treatise on Plane and Spherical Trigonometry, with their application to navigation, surveying, heights and distances and spherical astronomy, and particularly adapted to explaining the construction of Bowditch's Navigator and the Nautical Almanac.
Boston, J. Munroe & Co., 1840, 4 + 428 pp. + 5 fold. pls.
Third edition with additions, 1845, 4 + 449 pp. + 4 fold. pls.
New edition revised, with additions, 1852, 6 + 360 pp. + 5 fold. pls.
Another edition or reprint, 1861.
Pages 317–357 + plates 4–5 were reprinted in 1852 with the following title page: "*The chapter on Eclipses extracted from Peirce's Spherical Astronomy. For the Use of the Nautical Almanac.*

An elementary Treatise on Curves, Functions, and Forces. Volume first; containing analytical geometry and the differential calculus, 1841: *Volume second; containing calculus of imaginary quantities, residual calculus, and integral calculus*, 1846.
Boston, James Munroe & Co., 8 + 304 pp. + 14 pls. + 8 + 290 pp. + fold. pl.
New edition of volume 1, 1852. 7 + 301 pp. + 14 pls.
A third volume of this work, dealing with the applications of analytic mechanics, was projected, but not published. Probably the treatise issued in 1855 took its place. The following text by J. M. Peirce is based on this treatise: *A Text-book of Analytic Geometry; on the basis of Professor Peirce's Treatise.* Cambridge, 1857, 7 + 228 pp. + 6 fold. pl. Extracts from the preface: "I would acknowledge, in closing, my obligations for the aid and encouragement which I have received from others. Professor Peirce has given me the benefit of his advice in repeated instances. Whatever merit the book may have is owing, in a great degree, to the assistance of Mr. C. W. Eliot, who, besides many less definite, but important services, has read and criticised a considerable part of the manuscript before it was sent to the press."
In 1845 the following work was published in Boston by Thomas Hill, afterwards president of Harvard College: *An Elementary treatise on Arithmetic, designed as an Introduction to Peirce's Course of Pure Mathematics* [that is the series of works published 1835–1846] *and as a sequel to the Arithmetics used in the High Schools of New England.*

The Cambridge Miscellany of Mathematics, Physics, and Astronomy, no. I (April, 1842), edited by B. Peirce; no. II (July, 1842), no. III (October), no. IV (last, January, 1843), edited by B. Peirce and J. Lovering.
Boston, James Munroe & Co., 1–48 + 49–96 + 97–144 + 145–192 pp. + 3 plates.

Contributions by B. Peirce: Problems and Solutions, pp. 23–24, 58, 60–61, 66–72, 97, 102, 119, 145, 149, 155, 156, 159, 168; "American astronomical and magnetic observers," pp. 25–28; "Distances of the fixed stars," pp. 28–31; "Meteors," pp. 44–46; "Varieties of climate," pp. 46–47; "The barometer," p. 48; "On Espy's theory of storms," pp. 141–144. Joseph Lovering (1813–92) was teacher of mathematics and natural philosophy at Harvard for the fifty years prior to 1888.

[Anonymous review of S. C. Walker's work on Meteors.]
N. Amer. Rev., vol. 56, 1843, pp. 409–435.

On the perturbation of meteors approaching the earth.
Amer. Philos. Soc., *Trans.* n.s., vol. 8, 1843, pp. 83–86.
A letter dated Dec. 24, 1840, to S. C. Walker. Presented to the Society, January 15, 1841.

Bowditch's Useful Tables. [Preface by J. I. Bowditch, pages iii–v; "Remarks of Professor Pierce" (*sic*), pp. vii–viii.]
New York, E. & G. Blunt, 1844. Other editions were published in 1856, 1859, 1863, and 1866. An edition was issued by the U. S. Bureau of Navigation in 1885. Recent editions (such as the one for 1911) do not contain Peirce's remarks.
The tables in question are taken from Nathaniel Bowditch's *Practical Navigator*.

[Elements of the] third comet of 1845.
Amer. Jl. Sci., vol. 49, 1845, pp. 220–221.

The latitude of Cambridge Observatory, in Massachusetts, determined, from transits of stars over the prime vertical, observed during the months of December, 1844, and January, 1845, by William C. Bond, James D. Graham, and George P. Bond.
Amer. Acad., *Memoirs*, Boston, n.s., vol. 2, 1846, pp. 183–203.

[Orbits for Bond's comet.]
Boston Courier, March 27, 1846, p. 2, col. 2.
Also in *Amer. Jl. Sci.*, n.s., vol. 1, 1846, p. 348.
Letter dated March 26, 1846.

The Perturbations of Uranus.
Boston Courier, April 30, 1847, p. 2, col. 3.
Also in *Amer. Acad.*, *Proc.*, vol. 1 (1846–48), 1848, pp. 144–145.
Also in *Amer. Jl. Sci.*, n.s., vol. 4, 1847, pp. 132–133.
Communication dated April 29, 1847.

Mass of Neptune.
Boston Courier, October 25, 1847, p. 2, cols. 2–3.
Also in *National Intelligencer*, Washington, October 26, 1847, p. 2, col. 5.
Leverrier's reply is in the *Intelligencer*, March 10, 1848, p. 2, cols. 5–6, and in the *Boston Daily Advertiser*, March 20, p. 1, cols. 5–6. To this Peirce replied in a letter dated March 13, 1848; this is in the *Intelligencer*, March 23, p. 1, col. 2; and in the *Advertiser*, March 24, p. 2, col. 2, "Leverrier and Mr. Peirce." The following letter from Asa Gray, the distinguished botanist then professor of natural history at Harvard, is dated March 26:

Dear Peirce

When I read, in the Daily, your letter, I was on the point of sitting down to write you a line to tell you that I think it a *perfect gem*, and the most beautiful contrast to the *Johnny Crapeau* vociferation of Le Verrier. I am perfectly charmed with its spirit, and all that I have heard speak of it have taken the same view. I am not alone, therefore, in the opinion that it does you the highest credit and it is just the style of reply calculated to place you at the greatest advantage. As one zealous for the highest interests and character of American Science and American *Savans*, I thank you most sincerely, and am

Faithfully yours,

A. Gray

Also in *The Siderial Messenger*, Cincinnati, vol. 2, pp. 28–29, 1847.
Letter dated October 22, 1847.

[Elements of an elliptic orbit of De Vico's fourth comet.]
Amer. Acad., *Proc.*, vol. 1 (1846–48), 1848, pp. 39–42.

Notice of the computations of Mr. Sears C. Walker, who found that a star was missing in the *Histoire Céleste Française*, observed by Lalande on the 10th of May, 1795, near the path of the planet Neptune at that date, which may possibly have been this planet.
Amer. Acad., Proc., vol. 1 (1846–48), 1848, pp. 57–68 See also pp. 41–42.
Pages 57–65 appeared also in *The Siderial Messenger*, Cincinnati, vol. 1, pp. 125–128, 1847, under the title "The planet Neptune"; some parts of the other pages are given, in substance, on pages 85–86 of the same volume in an article entitled "Le Verrier's planet."
Pages 64–67 appeared as a quotation in *Amer. Jl. Sci.*, n.s., vol. 3, 1847, pp. 441–443.

[Review of Nichol's *Contemplations on the Solar System.*]
N. Amer. Rev., vol. 66, 1848, pp. 253–255.

[Formulæ for the perturbations of Neptune's longitude and radius vector.]
Amer. Acad., Proc., vol. 1 (1846–48), 1848, pp. 285–295.

[Investigations into the action of Neptune upon Uranus.]
Amer. Acad., Proc.,[1] vol. 1 (1846–48), 1848, pp. 332–342.

(*a*) Perturbations [of Neptune]; (*b*) Elements of Neptune; (*c*) Satellite of Neptune; (*d*) [Perturbations of Uranus].
Mo. Notices R. Astr. Soc., vol. 8 (1847–48), 1848, pp. 38–40, 128, 202–203.

Ueber die Störungen des Neptuns.
Astron. Nachrichten, vol. 27, cols. 215–218, 1848.

[Calculations on the perturbations of Uranus.]
Amer. Jl. Sci., n.s., vol. 5, 1848, pp. 435–436.

Development of the perturbative function of planetary motion.
Astr. Jl., vol. 1, pp. 1–8, 1849; pp. 31–32, 33–36, 1850.

Certain methods of determining the number of real roots of equations applicable to transcendental as well as to algebraic equations [abstract].
Amer. Assoc. Adv. Sci., Proc., vol. 1 (1848), 1849, pp. 38–39.

A. Guyot, *The Earth and Man.* Translated from the French by C. C. Felton. Boston, 1849.
Quotation from the preface: "Besides Prof. Felton, who read all the proof sheets, the author returns his acknowledgments to Professors Agassiz, Peirce and Gray who have had the goodness to revise portions of them."

On Fresnel's dioptric apparatus for lighthouses, by B. Peirce and J. Lovering.
Franklin Inst., Jl., s. 3, vol. 18, 1849, pp. 249–252.

"Poor Richard." Poor Richard's Almanac for 1850(–52) as written by Benjamin Franklin for the years 1733, 1734, 1735, (1736–1741). The astronomical calculations by . . . Benj. Peirce. New York, John Doggett, Jr., 1849(–51).

(*a*) On the connection of comets with the solar system; (*b*) On the relation between the elastic curve and the motion of the pendulum; (*c*) Mathematical investigation of the fractions which occur in phyllotaxis.
Amer. Assoc. Adv. Sci., Proc., vol. 2 (1849), 1850, pp. 118–122; 128–130; 444–447.
As to the topic discussed in (*c*) compare my articles on Golden section and the Fibonacci series in this MONTHLY, *1918*, 232–238; or better in Jay Hambidge, *Dynamic Symmetry*, New Haven, 1920, pp. 152–157. See also C. Wright, "On the phyllotaxis," *Astron. Jl.*, vol. 5, pp. 22–24, 1856.

(*a*) Note of Professor Peirce to the editor [demonstrating the parallelogram of forces]; (*b*) On the orbit of α Virginis regarded as a double star.
Astr. Jl., vol. 1, pp. 23, 138–139, 1850.

On the constitution of Saturn's rings.
Amer. Jl. Sci., vol. 12, 1851, pp. 106–108.
Also in *Astr. Jl.*, vol. 2, pp. 17–19, 1851.
German translation: Ueber die Beschaffenheit des Saturnringes, *Annalen der Physik und Chemie*, vol. 160, 1851, pp. 313–319.

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[1] There are numerous references in the *Proceedings* to papers read by Peirce but not published. See vol. 1 (1846–48), 1848, p. 185; vol. 2 (1848–52), 1852, pp. 111–2, 147, 235, 240, 250, 256, 258, 282, 289–90, 298, 310; vol. 3 (1852–57), 1857, pp. 8, 9, 28, 31, 67, 83.

Report of the Committee upon Prof. Mitchell's system of astronomical observations by Benjamin Peirce, chairman.
Amer. Assoc. Adv. Sci., Proc., vol. 5, 1851, pp. 69–71.

(*a*) [Report on the results of the U. S. Coast Survey, by B. Peirce for the Committee: B. Peirce, D. Treadwell, J. I. Bowditch, and J. Lovering]; (*b*) [Report by B. Peirce, E. N. Horford, and J. Lovering on a paper, entitled, "Description of the causes of the explosion of steam boilers, and of some newly-discovered properties of heat and other matters; for the purpose of showing that the application of steam for the production of motive force is susceptible both of immense improvement and economy," by James Frost]; (*c*) [On a new method of computing the constants of the perturbation function of planetary motion]; (*d*) Report by B. Peirce (in behalf of the committee D. Treadwell, B. Peirce, J. Lovering, H. L. Eustis, and M. Wyman) concerning C. H. Davis's paper on the deterioration of Boston harbor.
Amer. Acad., Proc., vol. 2 (1848–52), 1852, pp. 124–128; 129–130; 197–198; 288–289.

An account of Longstreth's lunar formula.
Amer. Assoc. Adv. Sci., Proc., vol. 6 (1851), 1852, pp. 143–144.

The American Ephemeris and Nautical Almanac for the year 1855 [vol. 1] (-1861).
Washington, 1852(–1858).
From the preface: "The theoretical department of the work has been placed under the special direction of Professor Benjamin Peirce, LL.D., and most of the calculations have passed under his final revision."

(*a*) Note upon the conical pendulum; (*b*) Criterion for the rejection of doubtful observations.
Astr. Jl., vol. 2, 1852, pp. 137–149; 161–163.
An elucidation of the second of these papers is given in the paper on the Criterion, published by Peirce in 1878. See further the notes on this subject in Section V.

The semidiameters of Venus and Mars investigated. From the observations made with the mural-circle of the Naval Observatory at Washington during the years 1845 and 1846.
Astr. Jl., vol. 3, pp. 9–10, 1852.

Tables of the Moon; constructed from Plana's theory, with Airy's and Longstreth's corrections, Hansen's two inequalities of long period arising from the action of Venus, and Hansen's values of the secular variations of the mean motion and of the motion of the perigee. Arranged in form designed by . . . Benjamin Peirce.
Washington, For the use of the Nautical Almanac, 1853. 4to, 326 pp.
See also *Tables of the Moon's Parallax*, 1856.
The second edition (1865, 348 pp.) was "the same as the first, with the exception of the correction of typographical errors; the substitution of the Tables of the Moon's Parallax constructed from Walker's and Adams' formulas, in the place of the original Parallax Tables; and the addition of a Table adapted to a convenient modification of the method of computing the latitude, by Professor J. D. Runkle.
"A third edition will shortly be issued, of which the basis will still be Plana's theory, while the Tables will be corrected to conform to the new Solar Parallax, and the corrected elements of the Moon's orbit." (Preface; the third edition does not seem to have been published.)
The work was reviewed in *Mo. Notices R. Astr. Soc.*, vol. 14, pp. 26–32, 184. On page 32 it is remarked that "the arrangement . . . is the result of a plan devised by Professor Peirce. It is very clear and masterly, and is in every respect worthy of that eminent mathematician."

On longitudes from moon culminations.
Coast Survey, Report for 1853, Washington, 1853, app. 31, p. 84.

Address of Professor Benjamin Peirce, president of the American Association for the year 1853, on retiring from the duties of president. [Printed by order of the Association.]
[Cambridge], 1854, 17 p.
Also in *Amer. Assoc. Adv. Sci., Proc.*, vol. 8 (1854), 1855, pp. 1–17.
Also bound in limp cloth, gilt edges, with cover title *The Song of Geometry*, and with special title-page and dedication. The copy of the title page is as follows: *Ben Yamen's Song of Geometry, sung by the Florentine Academy, at the accession of Her Majesty the Queen, degraded into prose by Benjamin the Florentine*, Cloverden, 1854.
Also most of pages 2–5 appears on pages 105–108 of *The Early Years of the Saturday Club, 1855–1870*, by E. W. Emerson, Boston, 1918.

Residual differences between the theoretical and observed longitudes of Uranus, from the theories of Peirce, LeVerrier and Adams.
Amer. Phil. Soc., Proc., vol. 5, 1854, p. 16.

(*a*) Elements of the comet, 1854, III; (*b*) Quantities to be added to the solar ephemeris of the American Nautical Almanac to obtain that given by Hansen's solar tables with the obliquity of the ecliptic of the Nautical Almanac; (*c*) The investigation of the catenary upon a cone of revolution with a vertical axis.
Astr. Jl., vol. 4, pp. 7, 9, 27–29, 1854.

Report upon the determination of longitude by moon-culminations.
Coast Survey, Report for 1854, Washington, 1854, app. 36, pp. 108–120.

On the Adams prize-problem for 1856.
Astr. Jl., vol. 4, pp. 27–29, 1854.
At the conclusion of the article are the words "to be continued"; no continuation has been found.

[Report of the Committee of the American Academy of Arts and Sciences on a Program for organization of the Smithsonian Institution, December 8, 1847.]
Smithsonian Institution, Eighth Annual Report, Washington, 1854, pp. 148–155.
Also in *The Smithsonian Institution. Documents Relative to its Origin and History* (*Smithsonian Misc. Colls.*, vol. 17, Washington, 1879, pp. 964–970).
The Committee consisted of E. Everett (Chairman), Jared Sparks, Benjamin Peirce, H. W. Longfellow, and Asa Gray.

Physical and Celestial Mechanics . . . developed in four Systems of Analytic Mechanics, Celestial Mechanics, Potential Physics, and Analytic Morphology. Then the second title page: *A System of Analytic Mechanics.*
Boston, Little, Brown & Co., 1855, 39 + 496 pp. + a fold. pl.
This work on mechanics was intended as the first of a series of four volumes, the other three to be respectively on Celestial Mechanics, Potential Physics, and Analytic Morphology.
Extract from preface: "I have . . . reexamined the memoirs of the great geometers, and have striven to consolidate their latest researches and their most exalted forms of thought into a consistent and uniform treatise. If I have, hereby, succeeded in opening to the students of my country a readier access to these choice jewels of intellect, if their brilliancy is not impaired in this attempt to reset them, if in their new constellation they illustrate each other and concentrate a stronger light upon the names of their discoverers, and still more, if any gem which I may have presumed to add is not wholly lustreless in the collection, I shall feel that my work has not been in vain. The treatise is not, however, designed to be a mere compilation. The attempt has been made to carry back the fundamental principles of the science to a more profound and central origin; and thence to shorten the path to the most fruitful forms of research. See further comments in Section V of this monograph.
Reviewed in: *Christian Examiner*, Boston, vol. 64, 1858, pp. 276–293 [by Thos. Hill]; *N. Amer. Rev.*, vol. 87, 1858, pp. 1–21 [by T. Hill].
Another issue with new title-page, New York, Van Nostrand, 1865; also 1872.

On the method of determining longitudes by occultations of the Pleiades.
Coast Survey, Report for 1855, Washington, 1855, app. 42, pp. 267–274.

Six articles upon the Smithsonian Institution . . . together with the letters of Professor Peirce and Agassiz.
Boston Post, January 27, February 5, 7, 13, 21, 22, 1855.
Also as a pamphlet, Boston, Printed at the office of the *Boston Post*, 1855, 44 pp.
The letter of Professor Peirce was dated January 29, 1855. The articles were signed "N. P. D." The last three of the five paragraphs of Peirce's letter were published in *The Congressional Globe, Appendix* (33d Congress, 1853–55, House of Representatives, February 27, 1855), vol. 31, p. 285, and reprinted in *The Smithsonian Institution, Documents relative to its Origin and History* (*Smithsonian Miscellaneous Collections*, vol. 17), Washington, 1879, pp. 588–9, 619.

Letter of Professor Peirce to President Quincy, with two letters from Admiral Beaufort annexed to it, and a list of zenith stars from Professor Airy.
Annals of the Harvard Observatory, vol. 1, 1856, pp. xciv–xcv.
The letter was dated May 10, 1845.

Opening address of Professor Benjamin Peirce . . . President of the Association.
Amer. Assoc. Adv. Sci., Proc., vol. 7 (1853), 1856, pp. xvii–xx.

(*a*) Abstract of a paper on researches in analytic morphology. Transformation of curves; (*b*) Abstract of a paper upon the solution of the Adams prize problem for 1867; (*c*) Abstract of a paper on partial multipliers of differential equations; (*d*) Abstract of a paper upon the catenary on the vertical right cone; (*e*) Abstract of a paper upon the motion of a heavy body on the circumference of a circle which rotates uniformly about a vertical axis; (*f*) Abstract of a paper on the resistance to the motion of the pendulum; (*g*) Method of determining longitudes by occultations of the Pleiades.
Amer. Assoc. Adv. Sci., Proc., vol. 9 (1855), 1856, pp. 67–74, 97–102.

Tables of the Moon's Parallax, constructed from Walker's and Adams's formulæ, arranged as a supplement to the first edition of Peirce's Tables of the moon.
Washington, For the U. S. Nautical Almanac Office, 1856, pp. 303–329.

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Cambridge, Mass., 1856, 4 pp.
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On the determination of longitude by occultations of the Pleiades.
Coast Survey, Rept. for 1856, Washington, 1856, app. 24, pp. 191–197.

An investigation of the cases of complete solution by integration by quadratures of the problem of the motion of a material point acted upon by forces which emanate from a fixed axis.
Astr. Jl., vol. 5, pp. 38–39, 1857.

[Report, dated June 29, 1857, by a committee consisting of Benjamin Peirce (chairman), Louis Agassiz, B. A. Gould, and E. N. Horsford, on spiritualistic phenomena presented by a Dr. Gardner in an attempt to win a $500 prize offered in 1857 by the *Boston Courier*.]
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Determination of longitudes by occultations of the Pleiades and solar eclipses.
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(*a*) Note on the Red Hill catalogue of circumpolar stars; (*b*) Note on the extension of Lagrange's theorem for the development of functions.
Astr. Jl., vol. 5, pp. 137, 164, 1858.

On the formation of continents.
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Mathematical Monthly, ed. by J. D. Runkle, vol. 1, pp. 11, 16–18, 58, 60, 1858; 167–168, 170, 1859.

Cotidal lines of an inclosed sea, derived from the equilibrium theory.
Coast Survey, Rept. for 1858, Washington, 1858, app. 30, pp. 210–213.

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Albany, Weed, Parsons & Co., 1858, 93 pp.
Signed by Joseph Henry, A. D. Bache, Benjamin Peirce, Dudley Observatory, July 1858. See also *The Dudley Observatory and the Scientific Council, Statement of the Trustees*, Albany, 1858; letters of Peirce, pp. 54, 101, 102. Also, *A Key to the "Trustee's Statement." Letters to the Majority of the Trustees of the Dudley Observatory, showing the Misrepresentation, Garblings, Perversions of their Misstatements*, by George H. Thacher. Albany, Atlas & Argus, Oct. 1858, p. 126.
Third edition, Albany, 1858.

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Astr. Jl., vol. 5, pp. 186–188, 1858; vol. 6, pp. 50–56, 1859.

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The Atlas and Daily Bee, Boston, vol. 34, Aug. 11, 1859, p. 1, col. 7.

[W. C. Bond, director of Harvard Observatory; obituary notice.]
Amer. Acad., Proc., vol. 4 (1857–60), 1860, pp. 163–166.

Lettre addressée a M. le président de l'Académie des Sciences sur la constitution physique des comètes.
Comptes Rendus de l'Académ. d. Sc., vol. 51, 1860, pp. 174–176.

(*a*) [Abstract of a memoir on the peculiarities of astronomical observers;] (*b*) Memoir upon the tail of Donati's Comet.
Amer. Acad., Proc., vol. 4 (1857–60), 1860, pp. 197–199, 202–206.

Cyclic Solutions of the school-girl puzzle.
Astr. Jl., vol. 6, pp. 169–174, 1860.
The problem here discussed is the following: "A given number, *f*, of girls are required to walk in a given number, *g*, of ranks, of which each rank consists of a given number, *k*, of girls; subject to the condition that each girl is to walk once, and only once, in the same rank with every other girl." This important paper is apparently one of a series inspired by T. P. Kirkman's problem published in the *Lady's and Gentleman's Diary* for 1850, p. 48: "Fifteen young ladies in a school walk out thrice abreast for seven days in succession; it is required to arrange them daily, so that no two shall walk twice abreast." An account of this problem and some references to the literature are given in W. W. R. Ball, *Mathematical Recreations*, tenth edition, London, 1922, pp. 193–223; see also *Messenger of Mathematics*, vol. 41, 1911, pp. 33–56; *Jahrbuch über die Fortschritte der Mathematik*, 1911, p. 250, and W. Ahrens, *Mathematische Unterhaltungen und Spiele*, vol. 2, Leipzig, 1919, pp. 102–117.

(*a*) Report upon the determination of the longitude of America and Europe from the solar eclipse of July 28, 1851; (*b*) Report on an example for the determination of longitudes by occultations of the Pleiades.
Coast Survey, Rept. for 1861, Washington, 1861, (*a*) app. 16, pp. 182–195; (*b*) app. 17, pp. 196–221.

(*a*) Abstract of a memoir upon the attraction of Saturn's ring; (*b*) Upon the system of Saturn.
Amer. Acad., Proc., vol. 5 (1860–62), 1862, pp. (*a*) 353–354; (*b*) 379–380.

(*a*) On the computations of the occultations of the Pleiades for longitude; (*b*) Upon the tables of the moon used in the reduction of the Pleiades.
Coast Survey, Rept. for 1862, Washington, 1862; (*a*) app. 12, pp. 155, 156; (*b*) app. 13, pp. 157, 158.

Report upon the occultations of the Pleiades in 1841–42.
Coast Survey, Rept. for 1863, Washington, 1863, app. 17, pp. 146–154.

On the computations for longitudes by occultations of the Pleiades.
Coast Survey, Rept. for 1864, Washington, 1864, app. 11, p. 114.

(*a*) Report on the progress of determining longitude from occultations of the Pleiades [continued, compare *Report for 1863*]; (*b*) Method of determining the corrections of lunar semi-diameter, mean place, ellipticity of orbit, longitude of perihelion, coëfficient of annual parallax, and longitude of Europe and America from the occultation of the Pleiades.
Coast Survey, Rept. for 1865, Washington, 1865; (*a*) app. 12, pp. 138–146; (*b*) app. 13, pp. 146–149.

On the lunar bolis.
Amer. Acad., Proc., vol. 6 (1862–65), 1866, p. 36.

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Coast Survey, Report, B. Peirce, Superintendent [For the years 1867–1873].
House Executive Documents, Washington, 1867–1873.
Details concerning the exact number of these reports in the *Documents* and the number of pages in each report may be found in B. P. Poore, *A Descriptive Catalogue of the Government Publications of the United States, 1774–1881*, Washington, 1885. See also E. L. Burchard, *List and Catalogue of The Publications issued by the U. S. Coast and Geodetic Survey, 1816–1902*, Washington, 1908.

Obituary on Alexander Bache.
Coast Survey, Rept. for 1867, Washington, 1867, app. 19, p. 330.

Communication of vibration.
Amer. Assoc. Adv. Sci., Proc., vol. 16 (1867), 1868, pp. 17–18.

Report on Weights and Measures.
Washington, Coast Survey, 1869, 4 pp.
Report upon the progress made in the construction of metric standards of length, weight, and capacity, in pursuance of a joint resolution of Congress of July 27, 1866.

The solar eclipse of December 22, 1870.
Coast Survey, Rept. for 1870, Washington, 1870, app. 16, pp. 229–232.

Linear Associative Algebra (Lithographed).
Washington City, 1870, 153 pp.
Edition limited to 100 copies issued through "labors of love" by persons engaged on the Coast Survey. This work was developed from papers read before the National Academy of Sciences, 1866–1870.
New edition, with addenda and notes by C. S. Peirce, son of the author.
Amer. Jl. Math., vol. 4, 1881, pp. 97–229.
Reprinted, New York, Van Nostrand, 1882, 4 + 133 pp.
This contains pp. 120–125, a reprint of: (1) his article on the uses and transformations of linear algebra, published in 1875; and (2) C. S. Peirce's notes, pp. 125–133, which appeared at the same time.
On page 656, volume 3 (1869) of R. P. Graves's *Life of Sir William Rowan Hamilton*, occurs the following with reference to Peirce's work as it appeared in the *American Journal:* "The author of this Paper in a *note*, on p. 105, makes objection to Quaternions on the ground of the treatment of imaginaries. A reply to this objection may, I believe, be gathered from what will be found stated by Sir William Hamilton in pages 578, 579 of vol. II and pages 84, 85 of vol. III of this work, as well as *passim* in the correspondence with Professor De Morgan."
See notes in Section V, pp. 15–16.

[Problem proposed.] Given the skill of two billiard players at the three-ball game, to find the chance of the better player gaining the victory if he gives the other a *grand discount.*
Our Schoolday Visitor, Philadelphia, vol. 15, 1871, p. 220, problem 108.
Also as problem 71 in *The Mathematical Visitor*, vol. 1, p. 46, 1878; solution by the proposer on p. 69, 1879. Compare Peirce's paper, "Probabilities at the three-ball game of billiards," 1877.

Observations of the eclipse of December 22, 1870, at Catania.
Boston Daily Advertiser, , 1871.
Also in *Amer. Jl. Sci.*, s. 3, vol. 1, 1871, p. 155*.
Letter dated Dec. 22, 1870.

On the mean motions of the four outer planets.
Amer. Jl. Sci., s. 3, vol. 3, 1872, pp. 67–68.
From a letter to H. A. Newton, dated December 13, 1871.

Harbor of New York: its Condition, May, 1873. Letter . . . to the Chamber of Commerce of New York, with the report of Prof. Henry Mitchell on the Physical Survey of the Harbor.
New York, Press of Chamber of Commerce, 1873. 3–38 pp. + 8 charts and tables.
The letter occupies pages 3–5.

[On the formation of the shell of the earth by shrinkage.]
Amer. Acad., Proc., vol. 8 (1868–73), 1873, pp. 106–108.

The rotation of the planets as a result of the nebular theory.
Nature, vol. 8, 1873, pp. 392–393.
Reprint of a report of the 1873 meeting of the A. A. A. S., in the *New York Tribune*.

Ocean lanes for steamships.
Amer. Acad., Proc., vol. 9 (1873–74), 1874, pp. 228–230.

On the uses and transformations of linear algebras.
Amer. Acad., Proc., vol. 10, 1875, pp. 395–400.
Also in *Amer. Jl. Math.*, vol. 4, 1881, pp. 216–221; in this form reprinted in *Linear Associative Algebra*, 1882, pp. 120–125.

A new system of binary arithmetic.
Coast Survey, Rept. for 1876, Washington, 1876, app. 6, pp. 81–82.

The conflict between science and religion.
Unitarian Review, Boston, vol. 7, 1877, pp. 656–666.
Also reprinted with cover title, Boston, 1877, 12 pp.
A discourse delivered in the First Church, Boston, May 6, 1877.

"Qualitative algebra."
Johnson's New Universal Cyclopædia, New York, vol. 3, 1877, pp. 1487–88.

(*a*) Probabilities at the three-ball game of billiards; (*b*) on Peirce's criterion.
Amer. Acad., Proc., vol. 13 (1877–1878), 1878, pp. (*a*) 141–144; (*b*) 348–351.
The second of these papers was in elucidation of the paper on the same subject published in 1852.

The National Importance of Social Science in the United States. An address delivered by Professor Benjamin Peirce, at the opening of the session of the American Social Science Association at Cincinnati, 18 May, 1878.
Boston, Little, Brown & Co., 1878, cover-title, 16 p.
Also in *Journal of Social Science*, no. 12, 1880, pp. xii–xxi.

[Problem proposed] 5564. Find the probabilities at a game of a given number of points, which is played in such a way that there is only one person who is the actual player, and when the player is successful he counts a point, but when he is unsuccessful, he loses all the points he has made and adds one to his opponent's score.
Educational Times, London, vol. 31, 1878, p. 88; solution, pp. 135–136.
Also in *Mathematical Questions with Solutions from the Educational Times*, vol. 29, 1878, pp. 72–73.
Also as problem 66 in *The Mathematical Visitor*, Erie, Pa., vol. 1, p. 45, 1878; Peirce's solution is given on page 66, 1879.
Also in E. J. Boudin, *Leçons de Calcul des Probabilités*, edited by P. Mansion, Paris, 1916, pp. VIII and 36 ff. Several solutions of the problem are given, one by A. Claeys, another by A. Demoulin, and its connections with important theory are set forth.

[Problem proposed] 5968. If two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, prove that each will appear to the other to move in a plane, whatever be the mutual attraction.
Educational Times, vol. 32, 1879, p. 152; solutions, vol. 33, 1880, pp. 141 (by C. J. Munro) and 309 (by Asaph Hall).
Also in *Mathematical Questions . . .* , vol. 33, 1880, p. 91; vol. 34, 1881, p. 111.
Also as problem 145 in *The Mathematical Visitor*, vol. 1, p. 84, 1879; quaternion solution by the proposer, and a solution by De V. Wood, p. 146, 1880.

Internal constitution of the earth.
Coast Survey, Rept. for 1879, Washington, 1879, app. 14, p. 201.

[Problems proposed] (*a*) 174. To find by quadratic equations a triangle of which the angles are given and the distances of the vertices from a given point in the plane of the triangle. (*b*) 202. Find a curve which is similar to its own evolute.
The Mathematical Visitor, vol. 1, pp. 99, 116, 1880; solutions of 174 by W. Hoover and W. Siverly, p. 174, 1881; solution of 202 by A. S. Christie, vol. 2, pp. 21–2, 1882.

Propositions in cosmical physics.
Amer. Acad., Proc., vol. 15, 1880, p. 201.

The intellectual organization of Harvard University.
The Harvard Register, April 1880, vol. 1, p. 77.

Ideality of the Physical Sciences. Edited by J. M. Peirce.
Boston, Little, Brown & Co., 1881, 7 + 9–211 pp. Portrait frontispiece (fine steel engraving).
The editing consisted in verbal changes and the addition of footnotes and the appendix (pp. 195–211).
Reprint, 1883.
This volume contains the six lectures delivered by B. Peirce in February and March, 1879, in a Lowell Institute course, Boston, Mass. They were also given, January 20 to February 5, 1880, at the Peabody Institute, Baltimore. The lectures are entitled: 1. Ideality in Science; 2. Cosmogony; 3. From nebula to star; 4. Planet, comet, and meteor; 5. The cooling of the earth and the sun; 6. Potentiality. The dedication is as follows:

I dedicate
these lectures
To my wife
with my whole heart.
Benjamin Peirce.

Cambridge, 790320

The last of these lectures was quoted under the heading, "Prof. Peirce on the spiritual body" in *Banner of Light*, Boston, May 3, 1879.

THE MATHEMATICS OF BIOLOGY.

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If we may accept the amount of mathematical expression of a branch of science as a measure of how scientific that branch is, then biology may be said to be fast approaching its scientific stage. This is especially true for experimental biology or general physiology, though even the more descriptive phases are becoming more mathematical. Since biology uses data from other branches of science, particularly chemistry and physics, it is able to grow proportionally faster than it could if it had to solve their problems as well as its own. Unless the biologist shares his problems with the mathematician as well as with the chemist and the physicist, he cannot expect his necessary coöperation. It is the purpose of this paper to indicate briefly the extent and types of mathematical analysis found in present day biology. The value of such a summary is apparent to any adviser of students. This adviser realizes that he must not only recommend mathematics to the student but also show him the necessity of overcoming his aversion to mathematics in order to understand the experimental literature in biology and medicine.

Many people think that biological statistics represents the mathematics of biology. This is largely true if we think of *mathematical statistics* in contradistinction to the customary view that to carry on a statistical inquiry means to gather some numerical data, to pick out a convenient looking formula, and to reduce the data to a coefficient by means of that formula. The average person fails to understand that biological problems have been the cause of much pure mathematical research in statistical fields which required the extensive knowl-

edge of the research mathematician. The contributions of the Pearson school which have stimulated much further research illustrate this assertion. The publication of Pearl's (14) [1] *Medical biometry and statistics* last year shows the need for a book on statistics devoted to the special interests of the medical man. This book is most valuable since it is written to give the non-mathematically trained reader an accurate working knowledge of the subject. It is one of the very few books which present the limits of usability of the various measures without discussing their complete derivation. The study of death and its causes is an important branch of mathematical statistical biology. In this field one who wishes to understand what he reads must be familiar with the calculus and special methods of least squares, moments, and other methods of curve fitting.

These facts are not so obvious to the average intelligent reader for he sees only the final results separated from the methods and formulæ used in their calculation. Few people realize the immense amount of work involved in calculating life tables or in the localization and control of an epidemic. Yet any one who even idly turns the pages of the *Journal of General Physiology* must be forcibly reminded of his calculus text. Until recently one could read by omitting the mathematics. Now, however, it is necessary to follow the mathematics in order to understand the logic because this form of expression permits the presentation of much information in a more concise manner.

There are few treatises to aid the biology student for most mathematics teachers do not realize his particular needs. The mathematician cannot be expected to follow the special biological literature as well as that of his own exacting field. Feldman's recent book (9) *Biomathematics* will partly supply this need of the modern student. It will require some knowledge of mathematics to study it profitably. The unique position of this book warrants a short description. The author begins with a brief review of algebra and emphasizes logarithms as a convenient calculating tool. Illustrative examples are taken from growth and dietetics. The parts of trigonometry which have a biological use are presented with illustrations from animal mechanics and photospectrometry, instead of surveying. Limits of accuracy in measuring are considered. Then follows a study of series and the properties of the Napierian base e. Protein digestion furnishes examples. The compound interest law is considered with respect to the processes which do and do not follow its rate of increase. The chapter on functions, variables, and constants with its wealth of illustrative material is most interesting. Differentials, maxima and minima, successive differentiation and integral calculus follow in turn. The next chapter is headed "Biochemical applications of integration." This shows the use of calculus in analyzing hæmolysis, destruction of agglutinins as well as biological phenomena which follow the mono-, bi-, and multimolecular chemical equations. Thermodynamics with its biological applications is considered. Another chapter presents special methods of integration with further applications to animal mechanics. The nature of harmonic analysis by means of Fourier series is discussed. Differential

[1] See bibliography on p. 36.

equations are treated in the solution of problems of fermentation and in the hæmodynamics of heart valve incompetence. A chapter sums up many investigations wherein mathematical analysis was used in interrelating various problems. Biometrics is presented as well as special methods, such as the preparation of nomograms, etc. This suffices to indicate the broad aspect of the use of mathematics in experimental biology. We will now turn to more specific examples.

From the preceding discussion it is obvious that many of the problems of the biologist are of the same type as those of the chemist and physicist. His problems, however, are further complicated by that special synthesis which probably is the difference between living and nonliving material. Many of these problems are considered in Bayliss' classic *Principles of General Physiology*. The interested reader will there find problems of surfaces, interfaces and their tensions, energetics, osmosis and permeability, colloidal and crystalloidal state and ionization as well as those processes usually considered to be physiology proper, such as digestion, respiration, etc. A superficial analysis of respiration leads one immediately to the gas laws and the relations between a gas mixture (air made of N, O, H, etc.), a semipermeable membrane in the lungs and a liquid, the blood.

We may profitably examine du Nouy's recent work (6) for an excellent illustration of this thesis. While studying surface tension of blood serum with his newly-invented measuring device he found that it decreased spontaneously with time. We read about his method and see the actual decrease of the tension with respect to time on his graphs. Then we read, "the most stable arrangement of any solution must be accompanied by a minimal surface tension. This follows from the simple study of Gibbs' equation $\frac{cdy}{dc} = -\frac{udp}{dc}$ in which u is the quantity adsorbed, p the osmotic pressure, c the concentration, and y the surface tension" (p. 589). This simple problem of surface tension immediately involves the use of differential equations in its solution. The result of the mathematical analysis of this important phenomenon is the equation $y = y_0e^v$, where $v = -k\sqrt{t}$, which expresses the fall of surface tension with respect to time. If the t be changed to c, it expresses the decrease in time due to adsorption. A second paper (7) dealt with the effect of diluting the serum with sodium chloride solutions. Further experiment (8) showed that the addition of moderate amounts of substances which have great surface activity, such as sodium oleate, only lowers the surface tension temporarily. The serum by some buffer action recovers its previous tension. The recovery follows the logarithmic law $y = ae^{bx}$ provided the temperature remains constant. At present du Nouy is investigating the effect of temperature. This illustrates the use of mathematical analysis in one of the simpler biological fields. Now let us turn to a more general case, that of vision.

Selig Hecht (11) began the first experiments of his study of vision with the photosensory system of the common mud clam, *Mya arenaria*. This animal is well adapted for such experiments. If the intensity of light falling on the

extended neck of the clam is increased, after a measurable latent period, the neck will be retracted. The continuous flow of water through the syphons insures the temperature of the neck remaining constant. From his experimental data he concluded that the following expressions indicate the processes involved in the retraction of the neck due to photic stimuli. A photosensitive substance S is broken down into two substances by means of light, then either or both of these resulting substances catalyze a third substance which, when catalyzed, initiates the nerve action. With this and several other equations, if the amount of light and the temperature are given, the length of time before the animal's response can be calculated. The difference between the predicted and actual times was never greater than a few hundredths of a second. The interesting fact is that all his conclusions are based solely on the empirical data. His methodology was that of an exacting physicist. The student interested in scientific discovery will find Hecht's papers most interesting and profitable reading.

Next Hecht studied the kinetics of the bleaching of visual purple and found them to conform to the same schema. By means of the absorption spectra and portions of the spectrum giving maximum stimulation he showed that the photosensitive substance of the clam is very similar to the visual purple found in the retina. By investigating dark adaptation and vision with dim and bright light he has indicated that human vision also follows the same theory. He has further found that the difference between the rods and cones is a difference in the concentration of the visual purple. Without the continuous aid of mathematical analysis it is doubtful if Hecht would have reached these conclusions and shown so clearly these similarities in such widely differing organisms. In following his work we see the clever use of algebra, trigonometry, analytical geometry, calculus and differential equations.

Even the healing of a surface wound follows a law which may be formulated in strict mathematical terms. This was discovered by du Nouy (2–5). He covered surface wounds with sterile cellophane and outlined the wound's boundaries on it with a wax pencil. The cellophane was removed and the outline transferred to paper and the area measured with a planimeter. From these areas he was able to derive the equation representing the healing of wounds. He found the index of cicatrization to be a function of the size of the wound and the age of the person. The general equation for the healing of a wound is given in his tenth paper (5) and is

$$S_T = S_0 e^{-kv} \qquad \text{where} \qquad v = T + (T^2/2p),$$

where S_T is the wound area at time T, S_0 the initial area, and k and $2p$ are constants. This last equation permits the construction of the curve of healing for any wound from the initial area and the relations between the index of cicatrization and the age of the patient. Charts are given in du Nouy's paper to aid in constructing the curves. The actual healing of a wound follows this equation closely, so closely that it is being used to investigate the merits of different

antiseptics and dressings. A deviation from the theoretical curve often indicates infection before it is apparent to the attendants.

In 1908, Robertson (19) pointed out the similarity between growth curves and the graph of an autocatalytic monomolecular chemical reaction. Both give a characteristic S-shaped curve. Since the rate of a complex group of chemical reactions can be no greater than that of the slowest one of the group, Robertson advanced the theory that the "master reaction" of growth is a monomolecular autocatalytic reaction. He has been presenting data to prove this theory by showing how well observational data fit the equation for this type of reaction. The equation is $\log x - \log (A - x) = k(t - t_1)$, where x equals the growth at time t and t_1 is the time when $x = \frac{1}{2}A$. Or $x = \dfrac{Ace^{kt}}{1 + ce^{kt}}$. Robertson has been able to fit this equation with fair success to the growth of man and a few other organisms. Reed (17) has applied the formula to the growth of sunflowers, pear shoots, lemon shoots, and white rats. Others have shown that the growth of frogs, cucumber leaves, the cow, and the production of flowers on the cotton plant follow the same equation fairly well but not with sufficient mathematical exactness to prove the theory. Usually the curve does not fit the observations well when x is small. Reed has partially obviated this difficulty by raising the factor $k(t - t_1)$ to a power c. This, however, changes the differential equation and consequently does not uphold the theory. Other investigations have shown that the growth of certain other forms followed better other equations. Wilhelmy's equation $x = A(1 - e^{-kt})$ has been used with some success. The growth of apricot shoots during an entire summer, with three growth cycles, followed a curve for two successive monomolecular reactions where

$$x = a[1 - e^{-k_1(t-1)}] + be^{-k_2 t} \cos \alpha t.$$

A German, Mitscherlich (12), has proposed the formula

$$\log (\sqrt[n]{A} - \sqrt[n]{y}) = \log \sqrt[n]{A} - cx, \qquad \text{or} \qquad y = A(1 - e^{-cx})^n.$$

This last curve has not given such satisfactory results as the Robertson formula.

Pearl (15) has attacked the problem from the standpoint of population growth. His first curve was a logarithmic parabola. This curve was quite satisfactory though not in a generalized form. Later he found (18) that the general equation is

$$y = \frac{K}{1 + me^{a_1x + a_2x^2 + a_3x^3 \cdots}}.$$

This gives a general curve, free from the restrictions of the previous curves, that promises to be of great usefulness. However, it is so recent that numerical results of its use are not available.

These studies of growth while as yet inconclusive and scattered have given us a better picture of the changes in growth and a better method of attack than have the previous investigations. In analyzing so complex a problem as growth

it is absolutely necessary to have some aid in organizing and interpreting the data; mathematics is supplying that aid. To derive and use these equations requires a knowledge of calculus and mathematical statistics as well as the special methods of curve fitting, least squares, moments and approximations.

Mathematical analysis is also helping in the converse problem of senescence. After reviewing the results of many papers, including several of his own, Brody (1) finds that senescence may be expressed as a simple exponential law ". . . which is practically the equation used to represent the course of accumulation and disappearance of a substance as the result of two simultaneous, consecutive, monomolecular chemical reactions." He finds that the equations $y = Ae^{-kt}$ and $y = Ae^{-k_1t} - Be^{-k_2t}$ express vitality as the reciprocal of senescence when y measures such processes as the production of eggs by the hen and the production of milk by the cow. With one or the other of these equations he was able to calculate theoretical curves for the decline of vitality of man as a whole, or due to certain diseases, the mortality of the famous fruit fly Drosophila and variations of basal metabolism with age. All the observational data fit the theoretical curves remarkably well. Here again the extent that mathematical analysis may give us an insight into the conditions operating in the organism cannot be predicted. There is a noticeable increase in information due to the use of this analytical logic by the modern investigators of senescence.

One other research must conclude our sampling. That is Osterhout's (13) investigations in establishing a criterion of death. We speak of an organism as either dead or alive, but further consideration tells us that all parts of an organism do not die at the same time. A decapitated snake shows indications of life for some hours after the accident. Osterhout by using electrical methods has contributed much information with regard to death and its rate. Much of the usefulness of these studies is due to the aid of mathematics both in the analysis and statement of the resulting information. In his studies of antagonism between ions in nutrient media Osterhout has been able to express some very complicated relationships by means of ingenious solid models in a very clear and forceful manner.

The above general and special samples of researches wherein mathematics has been used indicate somewhat the extent that biology is profiting by expressing its data in mathematical terms and using that science for a more direct solution of the relations or the testing of hypotheses. While the advantages of this procedure are obvious, the biologist must ever beware of believing that the mere statement of a problem in terms of an equation is an ultimate solution. Some of the present study of growth curves seems to be a sort of mathematical contest to see who can get the best-fitting curve. Other studies are contributing to biological science by making assumptions and testing them mathematically or using the mathematics to simplify data by determining the amount and direction of its movements. The most useful branch of mathematics is the calculus, especially the infinitesimal calculus. Most biological phenomena are progressing at such rates that they cannot be directly measured accurately without sub-

stituting the measurements into the integrated form of the best differential equation. In fact, this is the only method of adequately expressing the problems of dynamic equilibria which the biologist finds involved in much of living phenomena.

From this brief survey of only a few research problems we see that the student who contemplates active research work or first class teaching, in the near future, must understand and be able to use at least college algebra, trigonometry, analytic geometry, calculus, differential equations and mathematical statistics. With the present increase of mathematical usage in experimental literature even the medical student must be familiar with the more general facts of calculus and biometrics. The student who desires only to be able to read the experimental literature, and the professor who was unable to study mathematics in his undergraduate years, might find that a year course in mathematical analysis would efficiently fulfill their needs. Some general text such as Griffin's (10) *Introduction to mathematical analysis* might be used. This book presents the elements of algebra, trigonometry, analytics and calculus together in an integrated fashion. Then this course could be followed profitably by intensive personal work, or better in conference with an instructor, in Feldman's (9) *Biomathematics*.

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THE FIRST MEETING OF THE INDIANA SECTION.

An Indiana Section of the Mathematical Association of America was reorganized, after a lapse of several years, at a meeting held in the Shortridge High School, Indianapolis, Indiana, on October 16, Professor F. H. Hodge presiding.

The attendance was twenty, including the following nineteen members of the Association:

W. C. Arnold, Gladys Banes, G. E. Carscallen, H. T. Davis, S. C. Davisson, J. E. Dotterer, E. D. Grant, H. E. H. Greenleaf, L. Hadley, D. F. Heath, Cora B. Hennel, F. H. Hodge, E. N. Johnson, June M. Lutz, T. E. Mason, A. Miller, G. E. Moore, C. K. Robbins, and H. N. Wright.

An organization was effected and the following officers elected: Professor F. H. HODGE, Purdue University, Chairman; Professor E. N. JOHNSON, Butler College, Vice-Chairman; Professor H. T. DAVIS, Indiana University, Secretary-Treasurer, these officers to constitute the executive committee of the section. The next meeting will be held early in the year at Purdue, the date to be decided later by the executive committee.

The program consisted in an address by Professor L. Hadley of Purdue on the subject: "How the state requirement in commercial arithmetic is handled by Purdue"; this was followed by a stimulating discussion.

H. T. DAVIS, *Secretary-Treasurer.*

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. NOTE ON MR. WEAVER'S PAPER "A SYSTEM OF TRIANGLES RELATED TO A PORISTIC SYSTEM" (*1924*, 337–340).

By F. D. MURNAGHAN, Johns Hopkins University.

The object of this note is to show that the interesting results in Mr. Weaver's paper may be derived in a different manner possessing a certain interest of its own.

The three vertices of any triangle may be fixed by means of a complex variable or, what is much the same thing, a polar coördinate system whose origin may be taken at the center of the circumcircle of the triangle. Denoting the radius of this circumcircle by R, the three vertices are $z_1 = Rt_1$, $z_2 = Rt_2$ and $z_3 = Rt_3$ respectively, where t_1, t_2, t_3 are *turns* or complex numbers of unit modulus. Let us consider two points y and x_1 which are images of each other in the straight line joining the points z_2 and z_3. The quotients $\frac{y - z_2}{y - z_3}$ and $\frac{x_1 - z_2}{x_1 - z_3}$ are, therefore,

conjugate imaginary numbers, *i.e.*,

$$\frac{x_1 - z_2}{x_1 - z_3} = \frac{\bar{y} - \bar{z}_2}{\bar{y} - \bar{z}_3},$$

or

$$x_1(\bar{z}_2 - \bar{z}_3) = \bar{y}(z_2 - z_3) + \bar{z}_2 z_3 - z_2 \bar{z}_3.$$

Upon writing in $\bar{z}_2 = R/t_2$, $\bar{z}_3 = R/t_3$ (since the conjugate of a turn is its reciprocal), we have

$$x_1 = R(t_2 + t_3) - \bar{y} t_2 t_3.$$

Introducing the notation

$$\sigma_1 = t_1 + t_2 + t_3; \qquad \sigma_3 = t_1 t_2 t_3,$$

this may be written in the form

$$x_1 = R(\sigma_1 - t_1) - \bar{y}\sigma_3/t_1.$$

Similarly the images x_2 and x_3, respectively, of y in the other two sides of the triangle are obtained by replacing t_1, in succession, by t_2 and t_3 in this expression.

When the point y is on the circumcircle of the triangle (z_1, z_2, z_3), so that $y = Rt$ where t is a turn, the three images obtained in this way lie on a straight line through the point $R\sigma_1$. For the ratios $x_1 - R\sigma_1 : x_2 - R\sigma_1 : x_3 - R\sigma_1$ are then real. In fact $(x_1 - R\sigma_1) \div (x_2 - R\sigma_1) = (tt_1 + t_2 t_3) \div (tt_2 + t_3 t_1)$ so that its conjugate $\left(\frac{1}{tt_1} + \frac{1}{t_2 t_3}\right) \div \left(\frac{1}{tt_2} + \frac{1}{t_3 t_1}\right)$ is equal to itself; in other words it is a real quantity.

In order to see what is the point $R\sigma_1$, we may recall a result already proved in this MONTHLY.[1] If two points ξ and η are isogonal conjugates with respect to the triangle (z_1, z_2, z_3), the quotient $(\xi - z_1) \div (z_2 - z_1)$ has the same argument as $(z_3 - z_1) \div (\eta - z_1)$ and there are similar statements for the other two vertices. Hence $(\xi - z_1)(\eta - z_1) \div (z_2 - z_1)(z_3 - z_1)$ is real. Equating it to its conjugate and writing $z_1 = Rt_1$, $\bar{z}_1 = \frac{R}{t_1}$, etc., we find

$$R^3 t_1^3 - R^2 t_1^2 \left\{ \xi + \eta + \frac{\sigma_3 \bar{\xi}\bar{\eta}}{R} \right\} + Rt_1\{\xi\eta + R\sigma_3(\bar{\xi} + \bar{\eta})\} - R^3\sigma_3 = 0,$$

with two similar expressions obtained on writing in succession t_2 and t_3 for t_1 in this equation. Denoting $t_2 t_3 + t_3 t_1 + t_1 t_2$ by σ_2 so that t_1, t_2, and t_3 are the zeros of the cubic polynomial $t^3 - \sigma_1 t^2 + \sigma_2 t - \sigma_3 = 0$, we see that this must be identical with the cubic

$$t^3 - t^2 \left\{ \frac{\xi}{R} + \frac{\eta}{R} + \frac{\sigma_3 \bar{\xi}\bar{\eta}}{R^2} \right\} + t \left\{ \frac{\xi\eta}{R^2} + \frac{\sigma_3(\bar{\xi} + \bar{\eta})}{R} \right\} - \sigma_3 = 0,$$

so that

$$\xi + \eta + \frac{\sigma_3 \bar{\xi}\bar{\eta}}{R} = R\sigma_1 \qquad \text{and} \qquad \xi\eta + R\sigma_3(\bar{\xi} + \bar{\eta}) = R^2\sigma_2.$$

[1] F. V. Morley, The Three Bar Curve, *American Mathematical Monthly* (*1924*, 71–77).

The second of these equations is merely the conjugate of the first since

$$\bar{\sigma}_1 = \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = \frac{\sigma_2}{\sigma_3} \qquad \text{and} \qquad \bar{\sigma}_3 = \frac{1}{\sigma_3}\cdot$$

Hence two points ξ and η which are isogonal conjugates with respect to the triangle (z_1, z_2, z_3) are connected by the relation

$$\xi + \eta + \frac{\sigma_3\bar{\xi}\bar{\eta}}{R} = R\sigma_1. \tag{1}$$

The isogonal conjugate of the circumcenter $\xi = 0$ of the triangle is accordingly $\eta = R\sigma_1$ so that the point $R\sigma_1$ is the orthocenter. The fact that the three images (x_1, x_2, x_3) of a point on the circumcircle of a triangle in the sides of the triangle lie on a line through the orthocenter is equivalent to saying that the feet of the perpendiculars from a point on the circumcircle on the sides of a triangle lie on a straight line bisecting the join of the point on the circumcircle to the orthocenter.

When the isogonal conjugates ξ and η coincide, the point is one of the four in- or excenters of the triangle. These lie, therefore, on the curve

$$2\xi + \frac{\sigma_3\bar{\xi}^2}{R} = R\sigma_1,$$

which may be written in the form

$$\bar{\xi}^2 = \frac{2R}{\sigma_3}\left(\frac{R\sigma_1}{2} - \xi\right)\cdot \tag{2}$$

Equating the moduli of these expressions we have the equation [1]

$$\overline{OI}^2 = 2R\cdot\overline{NI},$$

where $N = \dfrac{R\sigma_1}{2}$ is the center of the nine point circle.

Upon taking the conjugate of (2) we have

$$\xi^2 = 2R\sigma_3\left(\frac{R\bar{\sigma}_1}{2} - \bar{\xi}\right)$$

and eliminating σ_3 between this and (2) we obtain

$$\xi^2\bar{\xi}^2 = 4R^2\left(\xi - \frac{R\sigma_1}{2}\right)\left(\bar{\xi} - \frac{R\bar{\sigma}_1}{2}\right),$$

which is Mr. Weaver's equation (3). It may be observed that we obtain, on equating the arguments of both sides of (2), the result

$$\arg\sigma_3 - 2\arg\xi = \arg\left(\frac{R\sigma_1}{2} - \xi\right),$$

so that $2\theta + \varphi$ is the same for each of the four in- and excenters; θ and φ being the angles made by OI and IN respectively with an arbitrary line.

[1] *Cf. Mathesis*, Tome 38, p. 18 (1924).

The equation of the straight line joining the three images (x_1, x_2, x_3) of a point Rt on the circumcircle is found by noting that the argument of $x_1 - R\sigma_1$ is one half the argument of

$$\frac{x_1 - R\sigma_1}{\bar{x}_1 - R\bar{\sigma}_1} = \frac{\sigma_3}{t}\cdot$$

The required equation is, accordingly,

$$\frac{x_1 - R\sigma_1}{\bar{x}_1 - R\bar{\sigma}_1} = \frac{\sigma_3}{t}\cdot \tag{3}$$

To find the equation of the Simpson line of the point Rt, denote any point on it by y so that $2y = x + Rt$ and we find

$$\frac{2y - Rt - R\sigma_1}{2\bar{y} - R\bar{t} - R\bar{\sigma}_1} = \frac{\sigma_3}{t}$$

or

$$2yt - 2\sigma_3\bar{y} = R\left(t^2 - \frac{\sigma_3}{t}\right) + R\sigma_1 t - R\sigma_2. \tag{4}$$

The Simpson line of a second point $R\tau$ on the circumcircle is

$$2y\tau - 2\sigma_3\bar{y} = R\left(\tau^2 - \frac{\sigma_3}{\tau}\right) + R\sigma_1\tau - R\sigma_2$$

and their point of intersection is given by

$$2y = R\left(t + \tau + \frac{\sigma_3}{t\tau} + \sigma_1\right)\cdot \tag{5}$$

Now the nine point circle is the locus of points half way from the orthocenter to the circumcenter; in fact if x is such a point

$$2x = R\sigma_1 + Rt,$$

from which we obtain on eliminating t

$$4\left(x - \frac{R\sigma_1}{2}\right)\left(\bar{x} - R\frac{\bar{\sigma}_1}{2}\right) = R^2,$$

which is a circle with center at $N = \dfrac{R\sigma_1}{2}$ and radius $\dfrac{R}{2}\cdot$ Since

$$\frac{R\sigma_3}{t\tau} = \frac{Rt_1t_2t_3}{t\tau}$$

is on the circumcircle, the point

$$\tfrac{1}{2}R\left(\sigma_1 + \frac{\sigma_3}{t\tau}\right)$$

is on the nine point circle. As σ_3 varies, the point given by (5) traces out a circle

obtained from the nine point circle by displacing it so that its center moves from $\frac{R\sigma_1}{2}$ to $\frac{R}{2}(\sigma_1 + t + \tau)$; the point $\frac{R(t+\tau)}{2}$ being the midpoint of the join of the two points Rt and $R\tau$ on the circumcircle whose Simpson lines are being discussed. The argument of the Simpson line of the point Rt being one half the argument of $\frac{\sigma_3}{t}$, we see that the two Simpson lines intersect at an angle $\frac{1}{2}(\arg \tau - \arg t)$. In particular when $\tau = -t$, so that the two points Rt and $R\tau$ are diametrically opposite to each other on the circumcircle, we have the well-known result that their two Simpson lines intersect at right angles on the nine point circle.

II. Apropos of Egyptian Mathematics.

By L. C. Karpinski, University of Michigan.

In the recent extensive review (*1924*, 247–251) of Peet's *The Rhind Mathematical Papyrus*, a misleading partial quotation (*loc. cit.*, p. 251) is from a recent article by the eminent Egyptologist, J. H. Breasted. The note is concerned with the alleged practical character of Egyptian mathematics and Professor Archibald states that "On the evidence of the Edwin Smith Medical Papyrus alone, J. H. Breasted expressed the belief that the evidence of Egyptian interest in pure science, as such, is perfectly conclusive." The whole paragraph from the *Quarterly Bulletin of the New York Historical Society* of April, 1922 (vol. VI, pp. 4–31), is as follows:

> The current conclusion regarding the mind of the ancient Egyptian, a conclusion in which I have myself heretofore shared, has been that he was interested in scientific principles, if at all, solely because of the unavoidable necessity of applying them in practical life;—that if he discussed the superficial content of a many-sided geometrical figure or the cubical content of a hemisphere it was because he was obliged to measure fields for taxation purposes and to compute the content of granaries. In the field of Egyptian mathematics Professor Karpinski of the University of Michigan has long insisted that the surviving mathematical papyri clearly demonstrate the Egyptians' scientific interest in pure mathematics for its own sake. I have now no doubt that Professor Karpinski is right, for the evidence of interest in pure science, as such, is perfectly conclusive in the Edwin Smith Medical Papyrus.

It is to be noted that Professor Archibald takes a portion of one sentence, after the manner in which Scripture is occasionally misused. The reference by Breasted is to my article, "Algebraical Developments among the Egyptians and Babylonians," in this Monthly (*1917*, 257–265) in which I expressed views recently confirmed in the Moscow Papyrus, first described by Touraeff in *Ancient Egypt*, 1917, and later by the writer in *Science* ("Egyptian Mathematical Papyrus in Moscow"), vol. 57, pp. 528–529, May 4, 1923.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

Geometry of the Complex Domain. By JULIAN COOLIDGE. Oxford University Press, 1924. 242 pages. Price $6.00.

While the geometry in the complex domain has occupied the attention of such leaders as Laguerre, Segre and Study, it has not received the attention of geometers in general, which it deserves. This is the first attempt to write the subject up from the beginning in English and hence can be considered as a real contribution to the literature of mathematics in the English language. In any language the book literature on the subject is small.

The study of complex geometry divides itself into two parts: The first is concerned with the representation of imaginary points by real elements, the second with the study of the geometry of complex points on an analytic curve or surface.

The representation of imaginary points by real elements has quite an extensive literature connected with such names as Poncelet, Laguerre, Gauss, Weierstrass, Lie, Marie, Segre, Klein, Study, and many others. The representations which receive most attention in this book are those of Laguerre, Marie, Segre, Klein and Study. Marie and Laguerre used a real point pair of a plane to represent conjugate imaginaries, while Segre represented the imaginary points of a plane by the points on a real variety of four dimensions lying in a space of eight dimensions. Others represented the imaginary points of the plane by means of the real lines in a space of three dimensions. Of these methods Segre's is the only one without exceptional elements. This phase of the subject is treated in Chapters I, III, IV, VII. Chapter I is an excellent historical introduction to the subject. I shall not however attempt to give an ordered review of the contents of the book simply because I wish to tell the readers of the MONTHLY some of the big problems discussed.

The second great division of the subject was started by Segre who began the study of threads of an algebraic curve. When one begins to reflect that a straight line, for example, contains a two-parameter set of complex points, his first inquiry would well be, has this any one-parameter sets of points having properties analogous to straight lines in the real plane. This would naturally be a set of points which, by projection, could be sent into the real points, and since cross-ratio is preserved, the set of points sought is that having a real cross-ratio for any set of four points on it. This set of points is called a chain and corresponds to the circle in the Argand plane. The parametric equations of the chain are linear in terms of the parameter. Since the definition of the chain is projective, its properties with respect to collineations and anti-collineations are naturally investigated. This leads again to a study of the geometry of the Hermitian and hyperalgebraic forms, which define an algebraic set of points which depend on a single real parameter.

These things can be extended to the complex plane and complex space. The

extension of the chain is the chain congruence which is the analogue of the pencil of lines. The plane has a four-parameter set of complex points, so we have one-, two- and three-parametered sets of points to consider. In space we have also four- and five-parametered sets. A curve in a plane contains a two-fold infinity of complex points and one question is when will a set of points depending on two real parameters be the points belonging to an analytic curve. Similar questions arise in three-space.

The next natural extension is to the differential geometry of threads, etc.

These subjects and many more are treated in the book under discussion. It is impossible in a short review to touch many of the interesting topics but one more deserves mention, *viz.*, that of the non-euclidean geometry defined by Fubini. The measurement is set up with respect to an Hermitian metric. This is very fascinating and needs to be studied to be appreciated.

The last chapter contains a postulational foundation for complex geometry. This seems to the reviewer to be a queer place to put the foundation of the subject. In fact there are several places where one naturally wishes to ask questions. I will, however, refrain here, for I have so much more to commend than to condemn. There are quite a number of misprints, but these will only serve to make the reader read more carefully and when he has completed his task he will feel amply repaid for his pains.

The author's style is very characteristic and the subject has lost none of its fascination at his hands. I hope to see many more such books by American mathematicians.

C. L. E. MOORE.

A Companion to Elementary School Mathematics. By F. C. BOON. London, Longmans, Green and Co., 1924. 302 pages. Price $4.50.

There is in the English language no series that more effectively seeks to set forth the best in the modern movements in the teaching of mathematics, at least from the standpoint of content, than the one of which this volume forms a part. Edited by Mr. P. Abbott and Dr. F. S. Macaulay, it contains a number of works that depart from the conventional, while, as might be expected of both the editors and the authors whom they have selected, it maintains a higher grade of mathematical scholarship than is found in most other books intended chiefly for teachers.

Mr. Boon's work is no exception to the general rule. It is not a textbook in algebra, or geometry, or trigonometry, or any other special branch of mathematics; it is rather what the name indicates—a companion for the teacher and the student. It is not a book that is intended as an aid to "cramming" for examinations, nor is it one that "meets the requirements" of any of those rigid courses of study that do so much to deaden all interest on the part of the learner and all enthusiasm on the part of the teacher. As the author says, it seeks to stimulate "the desire to explore byways with alluring vistas," and to open the field of mathematical imagination. "From some eminence, remote from the

high-road," the author adds, "the explorer obtains a better observation of his journey and its destination, and glimpses open to him of spacious prospects which are not for those who march in the dust between hedges." Such is doubtless the experience of every successful trainer of teachers, and of every successful teacher of the youth of our generation, and an aid of this kind is of great value to all who are concerned with elementary education.

Now this book puts in readable form just the kind of material that the teacher and the mathematically-inclined student need for the purpose of getting some idea of what lies beyond the conventional boundaries of the courses which they are running. Such topics as the significance of Euclid's postulates, the squaring of the circle, the various proofs of the Pythagorean theorem, symmetry, continuity, complex numbers, limits, and induction are here set forth in a non-technical way, and few students in the high school will find themselves unable to read with understanding most of the pages which Mr. Boon has prepared. There is also a short chapter on paradoxes and fallacies that will be found helpful in preparing the lighter work in the programs of mathematics clubs—organizations that will profit much by having access to such a work.

It is not to be expected that the book is without its flaws. This is true of every book. The historical-biographical part of this one is probably its least satisfactory portion, but the two or three chapters on the subject will not be taken very seriously and are not likely to do any material harm. The details of the criticism which will naturally be made upon them by any reader who has given even a little attention to the subject are not worth mentioning in a review.

The book may be highly recommended for every high school library and for all teachers of elementary mathematics.

DAVID EUGENE SMITH.

NOTE ON THE MATHEMATICS OF THE ARABS.

The most distinguished active contributor to our knowledge of the mathematical sciences of the Arabs is undoubtedly Dr. Karl Schoy of Essen, formerly lecturer at the University of Bonn.

In addition to a number of authoritative articles, Dr. Schoy is the author of two books on Arabic science, the one published and the other announced for early publication. The monumental work, *Die Gnomonik der Araber,* was published in 1923 (Berlin, Verein. Wissen. Verl.), as Volume I, *Die Geschichte der Zeitmessung,* by Ernst von Basserman Jordan. In this there is given a detailed account of the sun-dials of the Arabs, with the relationship indicated to trigonometry. The third chapter relates to shadows and the shadow tables, *i.e.,* tangent and cotangent tables. A table of tangents was constructed by Ibn Junus. This proceeds by ten-minute intervals, being calculated to a radius of sixty parts, the values being given to parts, minutes and seconds. This table is reproduced by Dr. Schoy following a Leyden Manuscript.

Recently Dr. Schoy's booklet, *Ueber den Gnomonschatten und die Schattentafeln der arabischen Trigonometrie Beitrag zur Arabischen Trigonometrie nach unedierten arabischen Handschriften,* was published by the Orient-Buchhandlung

in Hannover.[1] This gives the most complete information which we have concerning the tangent function among the Arabs and also interpolation processes, particularly as found in the mathematical work of al-Biruni, c. 1000 A.D., two of whose general works were published in English translation in England. The applications of the tangent tables as given by al-Nairizi and Abu'l Wefa, as well as in the so-called Hakemute tables of Ibn Junus, are also given.

A further intensive study of al-Biruni is promised for early publication by the same publishers. This is entitled: *Die trigonometrischen Lehren des ostarabischen Astronomen Muh. ibn Ahmad, Abu'l-Rihan al-Biruni, dargestellt nach al-Qanun al-Masudi.* This study is founded on the third book of al-Biruni's Qanun Masudi, the famous Arabian scholar, as this book contains his system of trigonometry which is as varied as it is original. From its contents may be quoted: the calculation of the side of the regular polygon of nine sides, the number π, the sin 1^0, the shadow and its tables, the statement of ten conditions on the trisection of the angle, the deduction of several relations concerning the spherical trigonometry, etc.

As al-Biruni does not treat the regular polygon of seven sides, this is presented following the work of Tabit ibn Qurra who preserved in Arabic for us the work of Archimedes. Included also is the fine work of Ibn al-Haitam (died 1059) who was probably the first to recognize the problem as a cubic. Omar Khayyam refers particularly to the latter in his Algebra.

Dr. Schoy gives the sine and tangent of Ulug-Beg (died 1499), rounding out a series of substantial contributions to Arabic mathematical science, placing particularly the trigonometry upon a surer foundation.

L. C. KARPINSKI.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

AMERICAN JOURNAL OF MATHEMATICS, volume 46, no. 4, October, 1924: "A general class of problems in approximation" by D. Jackson, 215–234; "On a rational plane quintic curve with four real cusps" by P. Field, 235–240; "Projective properties of a ruled surface in the neighborhood of a ruling" by A. F. Carpenter, 241–257; "Complete characterization of dynamical trajectories in n-space" by L. M. Kells, 258–272; "Rods of constant or variable circular cross section" by C. A. Garabedian, 273–287.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 26, no. 3, July, 1924: "The general theory of a class of linear partial q-difference equations" by C. R. Adams, 283–312; "The summability of the triple Fourier series at points of discontinuity of the function developed" by B. M. Eversull, 313–334; "An unusual type of expansion problem" by M. H. Stone, 335–355; "On the independence of principal minors of determinants" by E. B. Stouffer, 356–368; "A necessary and sufficient condition that two surfaces be applicable" by W. C. Graustein and B. O. Koopman, 369–372; "Extensions of relative tensors" by O. Veblen and T. Y. Thomas, 373–377; "Geometrics of paths for which the equations of the paths admit a quadratic first integral" by L. P. Eisenhart, 378–384; "A general mean-value theorem" by D. V. Widder, 385–394.

[1] Heinz Lafaire, Hannover Rathenauplatz, 5; 2.50 marks.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3107. Proposed by A. S. WIENER, Brooklyn, New York.

A certain city is divided into rectangular blocks by two systems of parallel streets, one system running north and south. A man standing at the intersection of two streets wishes to reach the intersection of two other streets m blocks north and n blocks east. In how many ways can he do so?

Also what is the probability that he will pass the corner which is i blocks north and j blocks east ($i \leqq m, j \leqq n$), assuming that he is just as likely to take one route as another?

3108. Proposed by M. KURTZ, New York City.

Prove, or disprove, that

$$a^n = n! + n(a-1)^n - \frac{n(n-1)}{1\cdot 2}(a-2)^n + \cdots + (-1)^n(a-n)^n.$$

holds for all values of n. Also prove or disprove the following corollary:

$$n(1^x) - \frac{n(n-1)}{1\cdot 2}(2^x) + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(3^x) + \cdots + (-1)^{n-1}(n^x) = y,$$

where $y = 1$ when $x = 0$, $y = 0$ when $x = 1, 2, 3, \cdots (n-1)$, and $y = -n!$ when $x = n$, both x and n being integral.

3109. Proposed by the late J. W. NICHOLSON.

Find two rational numbers which separate the roots of the equation $x^3 - ax^2 + bx - c = 0$.

3110. Proposed by J. L. RILEY, Stephensville, Texas.

If the curves $a_1x^m + b_1y^m + c_1 = 0$ and $a_2x^m + b_2y^m + c_2 = 0$ touch at a point, find the ratio of their radii of curvature at that point.

3111. Proposed by H. A. SIMMONS, University of Pittsburgh.

A right circular cylinder, radius r and length l, is closed at both ends. Its axis is horizontal and it has a hole one inch in diameter in its convex surface. If we fill the cylinder with water and set it rotating about its axis once every second, the rotation starting when the hole is over the axis, how long will it take the water to run out?

3112. Proposed by F. HENROTEAU, Dominion Observatory, Ottawa, Canada.

Given a triangle and two points anywhere in its plane, draw three conic sections passing through the two points and tangent respectively to each pair of sides of the triangle as well as to the other two conics.

SOLUTIONS.

3036 [1923, 337]. Proposed by F. M. GARNETT, Savannah, Georgia.

The inside dimensions of a chest are $l \times w \times d$; find the greatest length of a rectangular piece of timber with the cross-section $a \times b$ which fits in the closed chest. Numerical application: $l = 6$ feet, $w = 3$ feet, $d = 2$ feet, $a = b = 2$ inches.

SOLUTION BY WALTER B. CARVER, Cornell University.

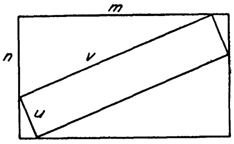
Consider the corresponding plane problem. If a rectangle of dimensions u and v is "inscribed" in another rectangle of dimensions m and n (each vertex of the first rectangle on one side of the second), we have the relation

$$(m^2 + n^2)(u^2 + v^2) - 4mnuv - (u^2 - v^2)^2 = 0,$$

or

$$v^4 - (m^2 + n^2 + 2u^2)v^2 + 4mnuv + (u^4 - m^2u^2 - n^2u^2) = 0.$$

If m, n $(m \geqq n)$, and u are given, we have an equation of the 4th degree in v. Making the natural restrictions $0 < u < n$, this equation has one negative root, and either one or three positive roots. All of the positive roots of this equation may be actual solutions of the problem. For instance, for $m = 21$, $n = 20$, and $u = 19$, we find three values for v, lying respectively between 10 and 11, 17 and 18, and 20 and 21; and these are all solutions of the problem.



But in attempting to find the longest rectangle of width u which can be placed in a rectangle of dimensions m and n, one would not be interested in values of v less than m, since a rectangle of length m could evidently be inserted. If, however, u is small, or, explicitly, if

$$0 < u < \frac{n}{3},$$

then equation (1) in v will always have *one and only one root greater than* m,[1] and this root will be less than $\sqrt{m^2 + n^2}$. For the equation whose roots are those of (1) diminished by m is

$$v^4 + 4mv^3 + (5m^2 - n^2 - 2u^2)v^2 + (2m^3 + 4mnu - 2mn^2 - 4mu^2)v + (u^4 + 4m^2nu - 3m^2u^2 - n^2u^2 - m^2n^2) = 0;$$

and, under the restrictions

$$0 < u < \frac{n}{3} \leqq \frac{m}{3},$$

it is readily seen that the constant term in this equation is negative, while all the rest of the coefficients are positive; and that, therefore, the equation has one and only one positive root. Similarly, diminishing the roots by $\sqrt{m^2 + n^2}$, it is found that the new equation has no positive or zero root.

Turning now to the problem in space, let the dimensions of the box be $l \geqq w \geqq d$, the two given dimensions of the timber $a \geqq b$, and the third dimension of the timber, to be determined, z. Let us assume, as axiomatic, that

When the longest possible timber of given cross-section is placed in the box, all of its vertices must lie in faces of the box.

There being eight vertices of the timber and only six faces of the box, at least two vertices must lie in some faces; and if more than two vertices lie in any face, exactly four vertices lie in that face. There are therefore the following five possible positions of the timber in the box:

1. Four vertices lie in one face of the box, and hence the remaining four vertices lie in the opposite face.
2. The four edges of the timber of the length z to be determined lie, one each, in four faces of the box, these four faces being two pairs of opposite faces.
3. The four edges of the timber of one of the given lengths, say a, lie, one each, in four such faces of the box.

[1] It may also be shown that, under these restrictions, the equation has no positive root less than or equal to m, except in the special case $m = n$, when $v = u$ is a double root.

4. Two opposite edges of the timber of the length z lie, one each, in two opposite faces of the box, and the remaining four vertices lie, one each, in the four remaining faces.

5. Like position 4, except that the edges which lie in faces of the box shall be of one of the given lengths, say a.

If the timber is in either the 1st or 2d position, its length [1] will evidently be one of the dimensions of the box.

If the timber is in the 3d position, the problem is equivalent to the plane case. If the edges of length a lie in the faces lw and ld of the box, then z will satisfy the equation

$$(w^2 + d^2)(b^2 + z^2) - 4wdbz - (b^2 - z^2)^2 = 0.$$

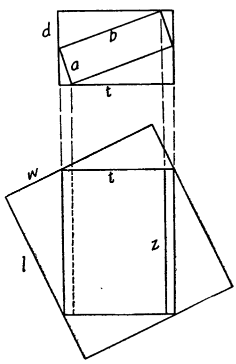
If the timber is in the 4th position, its length z may be found by two applications of equation (1). Suppose, for instance, that the edge of unknown length z lies in the face lw of the box. We may first find t as one of the two roots of the equation

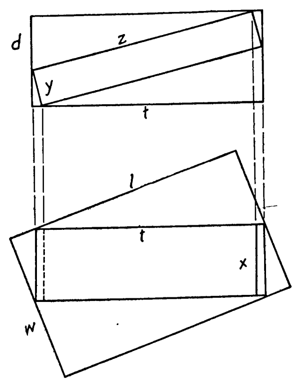
$$(a^2 + b^2)(d^2 + t^2) - 4abdt - (a^2 - b^2)^2 = 0,$$

and then z from the equation

$$(l^2 + w^2)(t^2 + z^2) - 4lwtz - (t^2 - z^2)^2 = 0.$$

The timber cannot be in this 4th position if $\sqrt{a^2 + b^2} < d$.





Finally, if the timber is in the 5th position, its length may again be found by two applications of equation (1). If, for instance, two edges of length a lie in the faces lw of the box, we may find t as one of the roots of the equation

$$(l^2 + w^2)(a^2 + t^2) - 4lwat - (a^2 - t^2)^2 = 0,$$

and then z from the equation

$$(d^2 + t^2)(b^2 + z^2) - 4dtbz - (b^2 - z^2)^2 = 0.$$

If now we make the restriction

$$0 < b \leqq a < \frac{d}{3} \leqq \frac{w}{3} \leqq \frac{l}{3},$$

cases 2 and 4 cannot arise at all; and the length of the timber in either of the 1st or 3d positions is less than the length in the 5th position. Hence we need only consider the 5th position.

Since, in this 5th position, two edges of either of the two given lengths a or b may lie in any one of the three pairs of opposite faces, there are, in general, six cases, yielding six different lengths for the timber. In the numerical example suggested, there are only three cases, since $a = b = 2$. The results for the three cases are, to the second decimal place,

$z = 81.45$, with edge lying in the face 72″ by 36″;
$z = 81.39$, " " " " " " 72″ by 24″;
$z = 81.36$, " " " " " " 36″ by 24″.

Hence the greatest length of timber in this example is, approximately, 81.45 inches.

Trial of several numerical cases seems to indicate that the greatest length will be obtained

[1] By *length* here is meant the dimension z to be determined, which is not necessarily greater than the two given dimensions.

when the shorter of the two given dimensions of the timber lies in the face of the box having the two greatest dimensions, but it seems to be a difficult matter to prove that this is true.

NOTE BY THE EDITORS: A discussion of the number of rectangles with a given side which can be inscribed in a given rectangle may be found in this journal (*1920*, 327–330). This discussion covers all the cases of rectangles considered above.

Also solved by F. L. WILMER.

3063 [1924, 147]. Proposed by H. A. BENDER, University of Illinois.

Show that

$$\sum_{r=1}^{r=b} (p^a-1)(p^a-p)\cdots(p^a-p^{r-1})\frac{(p^b-1)(p^{b-1}-1)\cdots(p^{b-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots(p-1)} = p^{ab}-1, \quad (a \geqq b)$$

is an identity in p.

SOLUTION BY HARRY LANGMAN, New York City.

Consider the following expansions:

$$(1+px)(1+p^2x)\cdots(1+p^bx) = \sum_{r=0}^{b} A_r x^r; \tag{1}$$

$$1 \div [(1+px)(1+p^2x)\cdots(1+p^bx)\cdots] = \sum_{r=0}^{\infty} B_r x^r; \tag{2}$$

$$1 \div [(1+p^{b+1}x)(1+p^{b+2}x)\cdots] = \sum_{r=0}^{\infty} C_r x^r. \tag{3}$$

To evaluate the coefficients A, we may proceed as follows:[1] Replacing x by px in (1),

$$(1+p^2x)(1+p^3x)\cdots(1+p^{b+1}x) = \sum_{r=0}^{b} p^r A_r x^r,$$

from which, with (1),

$$(1+px)\sum_{r=0}^{b} p^r A_r x^r = (1+p^{b+1}x)\sum_{r=0}^{b} A_r x^r,$$

giving

$$p^r A_r + p^r A_{r-1} = A_r + p^{b+1} A_{r-1}.$$

Hence we have,

$$\begin{aligned} A_0 = 1; \quad A_r &= \frac{p^{b-r+1}-1}{p^r-1}p^r A_{r-1} \\ &= \frac{(p^b-1)(p^{b-1}-1)\cdots(p^{b-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots(p-1)}p^{r(r+1)/2}; \qquad r = 1, 2, \cdots, b. \end{aligned} \tag{4}$$

In analogous manner, we get

$$\begin{aligned} B_0 = 1, \quad B_r &= \frac{p^r}{(p^r-1)(p^{r-1}-1)\cdots(p-1)}; \\ C_0 = 1, \quad C_r &= \frac{p^{r(b+1)}}{(p^r-1)(p^{r-1}-1)\cdots(p-1)}. \end{aligned} \tag{5}$$

The expansion (3) is evidently the product of the series (1) and (2). Hence, we may write

$$\begin{aligned} C_s &= \sum_{r=0}^{s} A_r B_{s-r}, \qquad s \leqq b; \\ C_s &= \sum_{r=0}^{b} A_r B_{s-r}, \qquad s \geqq b, \end{aligned} \tag{6}$$

giving

$$C_a = B_a + \sum_{r=0}^{b} A_r B_{a-r},$$

or

$$\frac{p^{a(b+1)}}{(p^a-1)(p^{a-1}-1)\cdots(p-1)} = \frac{p^a}{(p^a-1)(p^{a-1}-1)\cdots(p-1)}$$
$$+\sum_{r=1}^{b}\frac{(p^b-1)(p^{b-1}-1)\cdots(p^{b-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots(p-1)}\cdot\frac{(p^a-1)(p^{a-1}-1)\cdots(p^{a-r+1}-1)}{(p^a-1)(p^{a-1}-1)\cdots(p-1)}p^{[r(r-1)/2]+a}$$

[1] *Cf.* Chrystal, *Algebra*, II, p. 316.

Multiplying by $\dfrac{(p^a - 1)(p^{a-1} - 1)\cdots(p - 1)}{p^a}$, this yields the identity required.

NOTE BY THE EDITORS: For the convergence of (2) and (3) we may assume that p is less than unity in absolute value. After having established the desired result for such values of p, it then follows from the elementary theory of polynomials that it is true for all values of p.

3066 [1924, 147]. Proposed by B. F. FINKEL, Drury College.

What is the amount of work done in pulling a spool of thread, weight w, up an inclined plane whose length is l and inclination α, the spool to be pulled up the plane by taking hold of the outer end of the thread and allowing the thread to unwind? We assume that we may neglect the weight of the thread unwound, that friction is large enough to prevent slipping, and that the axis of the spool remains horizontal.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Clearly if the spool comes to rest at the top of the plane, the work done is just that of lifting the weight w through a height $l \sin \alpha$ or $wl \sin \alpha$. If the spool reaches the top with velocity v, the work done must be enough more to generate the corresponding kinetic energy $\frac{1}{2}\frac{w}{g}v^2 + \frac{1}{2}I\omega^2$, or if r is the radius of the spool and k its principal radius of gyration, in this case the work done is, since $\omega = \frac{v}{r}$ and $I = \frac{w}{g}k^2$,

$$\text{Work} = wl \sin \alpha + \frac{1}{2}\frac{w}{g}\left(1 + \frac{k^2}{r^2}\right)v^2.$$

Or one might reason as follows: Let F be the force exerted on the thread parallel to the plane. Taking moments about the line of contact of the spool with the plane

$$F(2r) - w \sin \alpha r = I'\frac{d\omega}{dt} \qquad \text{where} \qquad I' = I + \frac{w}{g}r^2 = \frac{w}{g}(k^2 + r^2)$$

and $\omega =$ the angular velocity $= \frac{v}{r}$; whence $\frac{d\omega}{dt} = \frac{1}{r}\frac{dv}{dl} = \frac{1}{r}\frac{vdv}{dx}$, x being the distance the spool has moved up the plane. This gives

$$F = \frac{w}{2}\sin \alpha + \frac{w}{2g}\left(1 + \frac{k^2}{r^2}\right)\frac{vdv}{dx}.$$

The force F must be applied through a distance $2l$ to bring the spool to the top of the plane so that W, the work done, is

$$W = \int_{x=0}^{x=l} F(2dx) = \int_0^l w \sin \alpha dx + \frac{w}{g}\left(1 + \frac{k^2}{r^2}\right)vdv = w \sin \alpha l + \frac{w}{2g}\left(1 + \frac{k^2}{r^2}\right)v^2,$$

the same as before, v being the final velocity upon reaching the top of the plane.

Also solved by EDWARD CONDON.

3068 [1924, 148]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that

$$\sum_{n=1}^{n=\infty} u_n/n! = (e - 1)^2/2$$

if $u_1 = 0$, $u_n = 2u_{n-1} + 1$, $(n = 2, 3, 4, \cdots)$.

SOLUTION BY EDWARD CONDON, University of California.

The proof of the equality in the problem is most easily obtained by developing the right side. Thus

$$\frac{(e-1)^2}{2} = \frac{e^2}{2} - e + \frac{1}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^n}{n!} - \sum_{n=0}^{\infty}\frac{1}{n!} + \frac{1}{2},$$
$$= \sum^{\infty}\frac{2^{n-1} - 1}{n!}.$$

It is easily verified that if we set $u_n = 2^{n-1} - 1$ the given equation for u_n is satisfied, and also that this gives $u_1 = 0$. Moreover this is the only solution of the equation.

If the value of the infinite series had not been given in the problem, thus furnishing the value of u_n, this value could be found as follows: Sum both sides of the equation $2^k u_{n-k} = 2^{k+1}u_{n-k-1} + 2^k$ from $k = 0$ to $k = n - 2$. After certain cancellations there results

$$u_n = \sum_{k=0}^{n-2} 2^k = 2^{n-1} - 1.$$

Then by reversing the work above, the value of the series may be found.

Also solved by H. HALPERIN, A. M. HARDING, HARRY LANGMAN, M. MORRIS, E. J. OGLESBY, H. L. OLSON, A. PELLETIER, W. E. ROUTH, G. E. RAYNOR, H. L. SLOBIN, A. S. WIENER, and W. M. WHYBURN.

3069 [1924, 148]. Proposed by J. ROSENBAUM, Milford, Conn.

Given the mid-points of the sides of an inscribed quadrilateral and the radius of the circumscribed circle, to construct the quadrilateral.

SOLUTION BY A. PELLETIER, Montreal, Can.

Let A, B, C, D be the mid-points in order. Then $AB = DC = \frac{1}{2}$ diagonal MN of the quadrilateral, and each segment is parallel to it. Likewise $BC = AD = \frac{1}{2}$ diagonal RS and each segment is parallel to it.

Hence, we know the two diagonals of the inscribed quadrilateral and their angle.

The construction is as follows: Describe a circle with the given radius. Draw in it any chord $M'N'$ equal to MN or $2AB$; then another chord $R'S'$ equal to RS or $2BC$, making with MN the angle of the parallelogram $ABCD$. We thus determine the quadrilateral $M'R'N'S'$, with the mid-points A', B', C', D'. Now, $MRNS$ and $M'R'N'S'$ are congruent and may be placed so as to coincide.

NOTE BY THE EDITORS: If the diagonals of the quadrilateral are perpendicular, or if one of them is a diameter, there is only one solution; otherwise there are two.

Also solved by PHILIP FITCH, H. E. H. GREENLEAF, HARRY LANGMAN, W. E. ROUTH, A. S. WIENER, and the PROPOSER.

3061 [1924, 101]. Proposed by NORMAN ANNING, University of Michigan.

The cubic $48y = 25x - x^3$ fits the sine curve $y = \sin(\pi x/6)$ for $x = -3, -1, 0, 1, 3$. For what x $(-3 < x < 3)$ is the fit the poorest?

SOLUTION BY J. K. WHITTEMORE, New Haven, Conn.

We write $f(x) = \dfrac{x^3}{48} + \sin(\pi x/6) - \dfrac{25}{48}x$ and consider only positive values of x since f is an odd function. We have

$$f' = \frac{x^2}{16} + \frac{\pi}{6}\cos(\pi x/6) - \frac{25}{48}, \qquad f'' = \frac{x}{8} - \frac{\pi^2}{36}\sin(\pi x/6).$$

Since f vanishes for $x = 0, 1, 3$ and f'' vanishes only once when x is between 0 and 3, f' vanishes just once when x is between 0 and 1 and just once when x is between 1 and 3. These values of x give maximum and minimum values to f or maximum values to the numerical difference of $\sin(\pi x/6)$ and $(25x - x^3)/48$.

The equation $f' = 0$, solved by trial, interpolation and two applications of Newton's method in each case, the work being carried out with 5 place logarithms, gives $x = 0.563$ and 2.375.

We find

$$f(0.563) = +0.0010, \qquad f(2.375) = -0.0110.$$

Hence, the "poorest fit" is for $x = 2.375$.

Also solved by a contributor who failed to sign his name to his solution and by MAURICE BAUDIN.

3062 [1924, 147]. Proposed by HARRY LANGMAN, New York City.

Let A be the vertex of a hyperbola. Draw APQ, any line through A cutting the curve again in P and either asymptote in Q. Draw PR parallel to the other asymptote, cutting the first in R. Show that the length RQ is constant and give a simple mechanical construction of the curve based on this property.

SOLUTION BY S. E. FIELD, University of Michigan.

Given the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with one vertex $A = (a, 0)$ and the line APQ having as its equation $y = m(x - a)$. The intersection of APQ with the asymptote $bx + ay = 0$ is the point

$$Q = \left(\frac{a^2m}{b + am}, -\frac{abm}{b + am}\right)$$

and with the hyperbola is the point

$$P = \left(\frac{a(a^2m^2 + b^2)}{a^2m^2 - b^2}, \frac{2amb^2}{a^2m^2 - b^2}\right).$$

The equation of PR parallel to the other asymptote, $bx - ay = 0$, is

$$bx - ay + \frac{ab(b - am)}{b + am} = 0,$$

which gives, on $bx + ay = 0$, the point

$$R = \left(\frac{a(am - b)}{2(b + am)}, -\frac{b(am - b)}{2(b + am)}\right)$$

and $RQ = \frac{1}{2}\sqrt{a^2 + b^2}$, a constant.

To construct the curve by means of this property we have only to draw the axes and asymptotes in position and draw a pencil of lines through a vertex A, cutting one asymptote in the points Q_1, Q_2, Q_3, $\cdots$. From these points lay off distances Q_1R_1, Q_2R_2, $\cdots$ equal to $\frac{1}{2}\sqrt{a^2 + b^2}$, the half diagonal of the rectangle whose sides are the semi axes, and draw lines through R_1, R_2, $\cdots$ parallel to the second asymptote. The intersections, P_1, P_2, P_3, $\cdots$, of these lines with the corresponding lines of the pencil through A will be points of the curve.

NOTE BY THE EDITORS: It is simpler to use the equation of the hyperbola in oblique coordinates $xy = k^2$ (1). A line through the vertex A is given by an equation $y - k = m(x - k)$ (2); it cuts the x-axis in Q, where $OQ = k - k/m$. Solving (1) and (2) for x, we find $OR = -k/m$. Therefore $RQ = k$.

This property is true for *any* point A on the hyperbola. For let QAP cut the other asymptote in S. Then $PS = QA$. Through P draw PT parallel to the asymptote ORQ, where O is the center, and cutting OS in T. Draw AU parallel to OS cutting the asymptote in U. Then $UQ = TP = OR$. Therefore $RQ = OU$ (a constant).

Also solved by A. G. CLARK, F. HENROTEAU, A. PELLETIER, J. B. REYNOLDS, and J. H. WEAVER.

3064 [1924, 147]. Proposed by BURRELL MORGAN, Panther, W. Va.

One of the parallel sides of a trapezoid containing $17\frac{1}{2}$ acres is 80 rods and the two non-parallel sides are 22 and 48 rods respectively; what is the length of the remaining side?

NOTE BY OTTO DUNKEL, Washington University.

A trapezoid whose two parallel sides are 80 and 150 and whose altitude is 22 contains 2530 sq. units. But this trapezoid is seen, geometrically, to be larger than the desired trapezoid which is to contain 2,800 sq. units (taking one acre as equal to 160 sq. rods). Hence no real solution is possible.

3065 [1924, 147]. Proposed by A. S. WIENER, Cornell University.

Prove the following identity:

$$\begin{vmatrix} a^{m+1}+c^m(a+b), & b(bc^{m-1}+a^m), & b(a^{m-1}c+a^m+c^m) \\ c(ab^{m-1}+a^m+b^m), & b^{m+1}+a^m(b+c), & c(a^{m-1}c+b^m) \\ a(ab^{m-1}+c^m), & a(bc^{m-1}+b^m+c^m), & c^{m+1}+b^m(c+a) \end{vmatrix}$$

$$\times \begin{vmatrix} a^{n+1}+c^n(a+b), & b(bc^{n-1}+a^n), & b(a^{n-1}c+a^n+c^n) \\ c(ab^{n-1}+a^n+b^n), & b^{n+1}+a^n(b+c), & c(a^{n-1}c+b^n) \\ a(ab^{n-1}+c^n), & a(bc^{n-1}+b^n+c^n), & c^{n+1}+b^n(c+a) \end{vmatrix}$$

$$= 8 \begin{vmatrix} a^{m+n+2}+c^{m+n}(a^2+b^2), & b^2(b^2c^{m+n-2}+a^{m+n}), & b^2(a^{m+n-2}c^2+a^{m+n}+c^{m+n}) \\ c^2(a^2b^{m+n-2}+a^{m+n}+b^{m+n}), & b^{m+n+2}+a^{m+n}(b^2+c^2), & c^2(a^{m+n-2}c^2+b^{m+n}) \\ a^2(a^2b^{m+n-2}+c^{m+n}), & a^2(b^2c^{m+n-2}+b^{m+n}+c^{m+n}), & c^{m+n+2}+b^{m+n}(c^2+a^2) \end{vmatrix}.$$

Solution by C. G. Latimer, Tulane University.

Let $\Delta = D(\alpha, \beta, k;\ p, s)$ be the determinant obtained by replacing in the first determinant on the left a, b, c by α, β, γ, where $\alpha\beta\gamma = k$; and the indices $m, m \pm 1, 1$, by $p, p \pm s, s$. Then if $k = abc$, the problem may be stated: Prove

$$D(a, b, k;\ m, 1)\cdot D(a, b, k;\ n, 1) \equiv 8D(a, b, k;\ m+n, 2). \tag{1}$$

If $k = 0$, $\Delta = 0$. Assume $k \neq 0$. In Δ, multiply the elements of the first, second, third columns by $\beta^s, \gamma^s, \alpha^s$, respectively, and divide the elements of the corresponding rows by β^s, γ^s, α^s, respectively. Next, subtract the elements of the second and third columns from the corresponding elements of the first. Then add one half the elements of the first column to the corresponding elements of the second and third. Removal of the common factors from the elements of the columns gives

$$\Delta \equiv -2k^s \begin{vmatrix} \alpha^p & \gamma^p & \alpha^p+\gamma^p \\ \alpha^p & \alpha^p+\beta^p & \beta^p \\ \beta^p+\gamma^p & \gamma^p & \beta^p \end{vmatrix} \equiv 8k^{p+s}. \tag{2}$$

From (2) and from the first sentence of the above paragraph we obtain the following:

$$\prod_{i=1}^{n} D(\alpha_i, \beta_i, k;\ q_i, t_i) \equiv 8^{n-1}D(\alpha', \beta', k;\ \sum_{i=1}^{n} q_i - r, \sum_{i=1}^{n} t_i + r). \tag{3}$$

(1) is a special case of (3).

Also solved by Lida B. May, Hazel E. Schoonmaker, and H. L. Slobin.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. Burgess, c/o Western Electric Co., 195 Broadway, New York City.

Professor H. Bateman, of the California Institute of Technology, has been elected a member of the American Philosophical Society.

At the University of Vermont, Instructor F. W. Householder has been promoted to an assistant professorship, and Mr. H. A. Giddings, of New Hampshire State University, has been appointed instructor of mathematics.

Associate Professor A. Kiernan, of the United States Naval Academy, has accepted a position with Ginn and Co., New York City.

At the United States Naval Academy, Dr. L. M. Kells and Mr. W. A. Conrad have been promoted to assistant professorships.

Dr. P. E. HEMKE, of the United States Naval Academy, has accepted a position as mathematical analyst at Langley Field, Va.

At the Agricultural and Mechanical College of Texas, Dr. W. P. UDINSKI has been promoted to an assistant professorship, Assistant Professors P. K. SMITH and F. W. SPARKS have resigned, and Messers F. AYRES, A. A. BLUMBERG, and W. P. STEVENS have been appointed instructors.

At the University of the Philippines, Mr. TELESFORO TIENZO, recently returned from the States, has been promoted from instructor in mathematics to assistant professor.

At the University of North Carolina, Professor Archibald Henderson, head of the department of mathematics, who has been on leave of absence studying in England and Germany, has resumed his duties; also the following promotions and appointments are announced: as professors, J. W. Lasley, Jr., and A. W. Hobbs; as associate professor, A. S. Winsor; as assistant professor of applied mathematics, J. B. Linker; as instructor, C. H. Benson. Assistant Professor E. L. Mackie has been granted leave of absence for the academic year 1924–25, and is studying at the University of Chicago. Mr. M. A. Hill, who has been doing graduate work at the University of Chicago, has resumed his duties as instructor.

To the list of doctorates with mathematics as major subject conferred in 1923 as announced in the MONTHLY (*1924*, 361), the following name should be added: Fredrick Wood, Wisconsin, "Group velocity and the propagation of disturbances in dispersive media."

J. P. Burton, A.B., for the past two years instructor in mathematics in Randolph-Macon College, is now engaged in graduate studies at Columbia.

S. T. Arnold and W. S. McClintic have been appointed instructors in mathematics at Randolph-Macon College, Ashland, Va.

T. McN. Simpson, Jr., professor of mathematics in Randolph-Macon College, is at present secretary of the Association of Virginia Colleges.

The annual meeting of the Mathematics Section of the Virginia Educational Conference was held at John Marshall High School, Richmond, Va., on Thanksgiving morning with a good attendance.

The general topic for discussion was "Present tendencies in the teaching of mathematics." The papers were as follows:

In teachers colleges and normal schools, Professor W. N. Hamlet, State Teachers College, Fredericksburg.

In the high school, J. G. Scott, head of the department of mathematics, Petersburg High School.

In the grammar grades, Miss Lila London, head of the department of mathematics, State Teachers College, Farmville.

In the elementary grades, Miss Katherine M. Anthony, supervisor of teacher training, State Teachers College, Harrisonburg.

The general discussion of the papers and the topic was led by Professor T. McN. Simpson, Jr., of Randolph-Macon College.

Other items of the program included a historical summary of the work of the section for the past five years by the secretary, Miss Gillie A. Larew, professor of mathematics, Randolph-Macon Woman's College, and a brief and informal presentation of some arithmetical methods of approximation to roots by Professor J. E. Rowe of the College of William and Mary.

Officers of the section for the coming year are: President, Dr. H. A. Converse, State Teachers College, Harrisonburg; secretary, Dr. Gillie A. Larew, Randolph-Macon Woman's College. Both are reëlections. The next meeting of the section will be held in Norfolk, Va., during the session of the Virginia Educational Conference next Thanksgiving week.

The article on Benjamin Peirce in this issue will be reprinted with introductory matter and several additional portrait reproductions. These reprints in paper covers will be for sale to libraries and individuals through the Secretary of the Association at the nominal price of $0.75 each ($0.50 to members of the Association).

The same reprints bound in board covers will be sold at $1.00 each by the Open Court Publishing Company, 122 South Michigan Avenue, Chicago, Illinois.

THE CARUS MATHEMATICAL MONOGRAPHS.

The first one of the Carus Mathematical Monographs will have come from the press before this number of the MONTHLY reaches its readers. This marks the beginning of the second stage in the development of this important undertaking, the first being the announcement of the notable gift by MRS. CARUS making possible the publication of the monographs. The scope of this gift has now been extended to cover increased costs of printing and definite arrangements for distribution.

In this first monograph, Professor G. A. BLISS has attempted to make the essential features of the *Calculus of Variations* accessible and attractive to persons who have an interest in mathematics but who may not be specialists in this field. The first chapter describes the historical setting out of which the theory grew, and explains some of the simpler problems. The next three chapters show the development in detail of the known results for three special problems which illustrate in excellent fashion the essential characteristics of the general theory as contained in the fifth chapter with which the book concludes. The book contains an ample bibliography which will be especially useful for those readers who may be interested to follow up the literature of the subject.

Every member of the Association who subscribes for these monographs will not only contribute directly to the progress and prestige of the Association but will

also receive full value for his investment. All receipts from members will go directly toward building up a publication fund for the Association.

Members of the Association (both individual and institutional) who have ordered monograph number one should receive copies very soon. Members who have not yet ordered may do so, remitting the cost price ($1.25) to the Secretary-Treasurer, W. D. CAIRNS, Oberlin, Ohio. Each member is entitled to receive one copy at this special price.

Subscribers to the MONTHLY, individuals or libraries, who are not members of the Association, should order copies directly from the Open Court Publishing Company, Chicago, Ill., at the regular sale price of $2.00 per copy.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 7–8, 1925;

Tenth Annual Meeting, Kansas City, Mo., December, 1925.

The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3, 1924
INDIANA, Purdue Univ., April
IOWA, State College, Ames, May 2–3, 1924
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA–MISSISSIPPI, Baton Rouge, March 1, 1924
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Baltimore, December 22, 1924
MICHIGAN, Ann Arbor, April 3, 1924
MINNESOTA, Hamline Univ., St. Paul, May 24, 1924
MISSOURI, Kansas City, November 15, 1924
NEBRASKA, Creighton Univ., Omaha, May 2, 1924
OHIO, Ohio Wesleyan Univ., Delaware, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 21
TEXAS, San Antonio, November 28, 1924

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THE REGULAR MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The fifteenth regular meeting of the Section was held May 17, 1924, in the Thomson School Building, Washington, D. C. The members of the Section were guests at luncheon of the District of Columbia Members.

There were forty-three (43) present including the following members of the Association: O. S. Adams, J. J. Arnaud, R. N. Ashmun, Sarah Beall, G. A. Bingley, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, A. Dillingham, H. English, H. W. Ficken, W. M. Hamilton, W. E. Heal, P. E. Hemke, L. S. Hulburt, G. H. Keulegan, W. D. Lambert, A. E. Landry, N. J. McKnight, Frank Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, R. E. Root, W. F. Shenton, E. W. Woolard.

The following officers were elected for the ensuing year: Chairman, F. D. Murnaghan; Secretary-Treasurer, Harry English; Member of the Executive Committee, J. A. Bullard.

The following program was given, abstracts numbered to correspond being supplied thereafter. The discussions of the various papers were lively and thought-provoking.

(1) "The axes of an n-line" by Professor Frank Morley of Johns Hopkins University.

(2) "A modern presentation of determinants" by Professor F. D. Murnaghan of Johns Hopkins University.

(3) "A remark on Viviani's enigma" by Professor J. A. Bullard of the U. S. Naval Academy.

(4) "Vectors in the foundations of geometry" by Dr. G. Y. Rainich of Johns Hopkins University (by invitation).

(5) "A problem in probabilities" by Professor C. C. Bramble of the U. S. Naval Academy.

(6) "Gothic window tracery curves" by Professor E. C. Phillips of Woodstock College.

(7) "Curve fitting" by Professor J. R. MUSSELMAN of Johns Hopkins University.

(8) "On Kichewsky's method of fitting frequency curves" by E. W. WOOLARD of the U. S. Weather Bureau.

2. Professor Murnaghan gave an outline of a presentation of the theory of determinants using the generalized Kronecker symbol $\delta^{r_1\cdots r_m}_{s_1\cdots s_m}$. This symbol is defined by the following properties. The letters $r_1\cdots r_m$, $s_1\cdots s_m$, each being capable of taking any one of the n values $1, 2, \cdots, n$ $(m \leqq n)$, $\delta^{r_1\cdots r_m}_{s_1\cdots s_m}$ is alternating both in the superscripts and subscripts; *i.e.*, an interchange of any two superscripts or of any two subscripts changes the sign but not the numerical value of the symbol. If the values assigned to $(r_1\cdots r_m)$ are all distinct, as are also the values assigned to $(s_1\cdots s_m)$, the symbol has the value zero if the group $(r_1\cdots r_m)$ is different from the group $(s_1\cdots s_m)$ and if these groups are the same, it has the value ± 1 according as the permutations $(r_1\cdots r_m)$, $(s_1\cdots s_m)$ of the same m numbers are of the same class or not. Then the determinant $|a_{rs}|$ may be defined as $= \delta^{\alpha_1\cdots\alpha_n}_{s_1\cdots s_n} a_{\alpha_1 s_1}\cdots a_{\alpha_n s_n}$ where the $\alpha_1\cdots\alpha_n$ are summation or umbral symbols. The theorems on the differentiation of determinants, Laplace's expansion, product of two determinants, etc., follow immediately from this definition. Conversely, the symbol

$$\delta^{r_1\cdots r_m}_{s_1\cdots s_m} = \begin{vmatrix} \delta^{r_1}_{s_1} & \cdots & \delta^{r_1}_{s_n} \\ \cdot & \cdot & \cdot \\ \delta^{r_m}_{s_1} & \cdots & \delta^{r_m}_{s_m} \end{vmatrix},$$

where δ^r_s is the usual Kronecker symbol. (*To be published in the* MONTHLY.)

3. Vincenzo Viviani, a Florentine mathematician, proposed a problem to his contemporaries in the form of an enigma: To cut out four equal windows from a hemispherical dome so that the area of the surface left should be exactly that of a square (*Acta Eruditorum*, 5 April, 1692). Leibnitz, James Bernoulli, John Wallis, David Gregory reached the solution and Viviani gave a construction without proof. This solution now appears in the elementary calculus text in some such form as this: To show that the cylinders $x^2 + y^2 = \pm Rx$, cut out from the hemisphere $z = +\sqrt{R^2 - x^2 - y^2}$ such an area that the remainder is $4R^2$, *i.e.*, the square on a diameter. Leibnitz, Bernoulli and the rest placed the cylinders with axes in the base of the hemisphere $(x^2 + z^2 = \pm Rx)$, thus cutting windows of somewhat semicircular appearance.

Taking the origin at the center of the sphere and using ordinary cylindrical coördinates, we see that the integral $R\int_0^{2\pi}\sqrt{R^2 - [f(\theta)]^2}d\theta$ gives the area of the surface of the hemisphere outside the cylinder $\rho = f(\theta)$ which stands upon the base of the hemisphere. Considering the equation

$$4R^2 = R\int_0^{2\pi}\sqrt{R^2 - [f(\theta)]^2}d\theta,$$

we see that $f(\theta) = R\cos 2n\theta$ is a solution, and that the cylinder $\rho = R\cos 2\theta$

cuts four equal windows from the dome and that this solution is more satisfactory than the classic one. It is interesting to note that the cylinders $\rho = R \cos 2n\theta$ (n any integer) cut a constant area (and also constant volume) from the hemisphere $\rho^2 + z^2 = R^2$. Similarly do the cylinders $\rho = R \cos (2n + 1)\theta$. Thus it is possible to cut any even number of windows and leave a remainder $4R^2$. In case $n = 0$ we get the classic solution of the problem.

4. Euclidean geometry can be introduced by a system of axioms involving vectors. The fundamental objects are points $A, B, C, \cdots$. Vectors are introduced as differences between points, $B - A$, $C - A$, $\cdots$. Vectors having the same initial point, *i.e.*, the same subtrahend, form a bundle; operations of addition and multiplication for the vectors of the same bundle are characterized by the laws of these operations. A vector $B - A$ is called equal to the vector $D - C$ if $B - A = D - A - (C - A)$. The last two axioms are these: From $A = B$ follows $B = A$ and from $A = B$, $B = C$ follows $A = C$. If we drop the last of these axioms, we obtain a much more general geometry of which the Riemann Geometry is a special case.

5. Professor Bramble presented a discussion of the probabilities of two players making certain scores with alternate throws of a single die. The particular interest of the problem lay in the variety of distinct methods of solution.

6. A brief historical introduction contained an account of the rise and development of the Gothic window and of the various forms of ornamentation used by successive generations of architects and culminating in the beautiful and intricate tracery of the great rose windows of the cathedrals built in the twelfth, thirteenth and fourteenth centuries. The rest of the paper dealt with the mathematical expression representing the general characteristics of that particular type of Gothic ornamentation which is known as "Flowing Tracery." The following parametric equations were found to represent quite faithfully this style of tracery:

$$\theta = \frac{c}{p}(t - \tfrac{1}{2} \sin 2t),$$

$$\rho = a(1 - k \cos t),$$

where θ and ρ are polar coördinates, c and p are relatively prime integers, a is an arbitrary constant unrestricted in value, and k is equal to or less than unity. It was shown that the character and shape of the curve depend on the three constants c, p and k, while the constant a affects only the size of the curve. A number of window designs plotted from the above equations were exhibited, the designs being cut out from paper and backed with a material resembling stained glass.

7. Dr. Musselman discussed the mathematical theory underlying the expression obtained by Charlier for fitting curves to statistical data. In particular, he presented the contributions to this theory by Thiele, Gram, and Charlier.

8. The two systems of skew curves in most general use for empirically fitting observed frequency distributions are the Pearson curves and the Gram-Charlier

expansion; a number of other systems have been proposed from time to time, and in this paper a recent method due to S. Krichewsky of the Egyptian Ministry of Public Works was described. An application to the frequency distribution of rainfall amounts at Washington, D. C., showed that the method gave as good a fit as that of Pearson, and with much less labor. Brief mention was made of the practical uses of frequency curves in meteorology, civil engineering, actuarial work, etc., and some precautions necessary to their intelligent use pointed out. The paper has been published in the *Monthly Weather Review*, 52, 91–94, February 1924.

The sixteenth meeting of the Section was held in Baltimore, December 6, 1924. The program will appear later.

HARRY ENGLISH, *Secretary-Treasurer.*

EIGHTH ANNUAL MEETING OF THE MISSOURI SECTION.

The eighth annual meeting of the Missouri Section of the Mathematical Association of America was held at the Junior College of Kansas City on Saturday morning, November 15, 1924, in affiliation with the annual meeting of the Missouri State Teachers Association.

The attendance was twenty-three including the following thirteen members of the Association: A. C. Andrews, Theodosia T. Callaway, L. H. Cutting, B. F. Finkel, R. R. Fleet, A. H. Huntington, W. A. Luby, A. D. Pierson, P. R. Rider, J. H. Scarborough, E. Stephens, R. A. Wells, Meta Wood.

The session was presided over by Professor R. R. Fleet, chairman of the Section. The following officers were elected for 1925: Chairman, R. A. WELLS, Park College; Vice-chairman, A. C. ANDREWS, Manual Training High School, Kansas City; Secretary-Treasurer, P. R. RIDER, Washington University. A number of the members of the Section attended the luncheon for mathematics teachers at the Coates House on Friday, November 14, and the meeting of the department of mathematics of the State Teachers Association, which was held at the Coates House on Friday afternoon.

The following four papers were read:

1. "Symbolic calculus" by Mr. EUGENE STEPHENS, Washington University.
2. "Service mathematics" by Professor THEODOSIA T. CALLAWAY, Stephens Junior College.
3. "How and what should freshmen be taught?" by Professor R. R. FLEET, William Jewell College.
4. "Simple illustrations of certain types of statistical series" by Professor P. R. RIDER, Washington University.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Mr. Stephens gave a historical introduction to the subject of symbolic

operators and their application to the operations of the calculus and to differential equations, together with a bibliography of papers prior to 1900. (This is an introductory paper only; others will follow, developing the subject completely.)

2. Mrs. Callaway's paper reviewed briefly the efforts made at Stephens College to locate and remove the mathematical difficulties of students in elementary clothing and foods classes. A careful study was made to determine definitely what mathematical concepts and processes are encountered in these courses and the degree of difficulty to which each process is carried. The study shows that no mathematics beyond that acquired in the first seven or eight grades is necessary. Experimentation has shown that tests and drill sheets based on the results of this study easily locate and remove difficulties in mathematical manipulation; but this inability to handle the mathematical processes is not nearly so prevalent as the inability to analyze problems and propose solutions. The paper ended with a plea for teachers of all subjects, and especially teachers of mathematics, to put more emphasis on methods of attacking problems, both general and mathematical.

3. Professor Fleet spoke of the present difficulty of preparing college courses in mathematics suitable to all classes of students enrolling in that subject. Local statistics were cited to show that high school students had not caught the spirit of the *Report of the National Committee* (1919) concerning the essentials necessary for the pursuit of a college course in mathematics. It was suggested that a list of those subjects in elementary mathematics needing especial emphasis be made known to the secondary schools.

4. Professor Rider's paper gave some simple examples of Bernoulli, Poisson, and Lexis series, and illustrated the formulas for mean and standard deviation in the three types.

P. R. RIDER, *Secretary-Treasurer.*

TENTH REGULAR MEETING OF THE KANSAS SECTION.

The tenth regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, February 2, 1924. All of the regular officers were absent on account of illness or removal from the state. The meeting was called to order by Professor C. H. Ashton, University of Kansas, and Professor W. H. Garrett, Baker University, was elected temporary Chairman and Professor G. W. Smith, University of Kansas, temporary Secretary.

Among those attending were the following twenty members of the M. A. A.: C. H. Ashton, Florence Black, Wealthy Babcock, R. H. Carpenter, Lucy T. Dougherty, E. F. Farner, W. H. Garrett, W. A. Harshbarger, W. C. Janes, S. Lefschetz, C. F. Lewis, O. B. Loewen, Thirza A. Mossman, H. S. Myers, B. L. Remick, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, Professor J. J. WHEELER, University of Kansas, Vice-chairman, LUCY T. DOUGHERTY,

Kansas City Junior College, Secretary-Treasurer, Professor U. G. MITCHELL, University of Kansas.

The following papers were presented:

(1) "Some space curves and their intrinsic equations" by Professor R. H. CARPENTER, Iola Junior College.

(2) "Holditch's theorem and Kempe's theorem for areas" by Professor W. T. STRATTON, Kansas State Agricultural College.

(3) "An hour with Einstein—Relativity: what it is and what it is not" by Professor S. LEFSCHETZ, University of Kansas.

A general discussion of Professor Lefschetz's paper was led by Professor E. B. Stouffer, University of Kansas.

Abstracts of papers, numbered as in the above list, follow:

1. This paper discussed certain cylindrical helices defined by their intrinsic equations and showed the nature of the surfaces upon which they lie.

2. Proofs of these two theorems were followed by a discussion of Woolhouse's and Elliott's extensions of the first and the many interesting variations which arise in connection with the second.

3. This paper consisted in a discussion of the growth of the Einstein theory, mainly in its more mathematical aspects, culminating in the assumption of the permanence of a ds^2 and some of the consequences of this assumption.

The eleventh regular meeting was held in Topeka in connection with the meeting of the State Council of Administration, February 7, 1925. Report of this meeting will be given at a later date.

U. G. MITCHELL, *Secretary-Treasurer.*

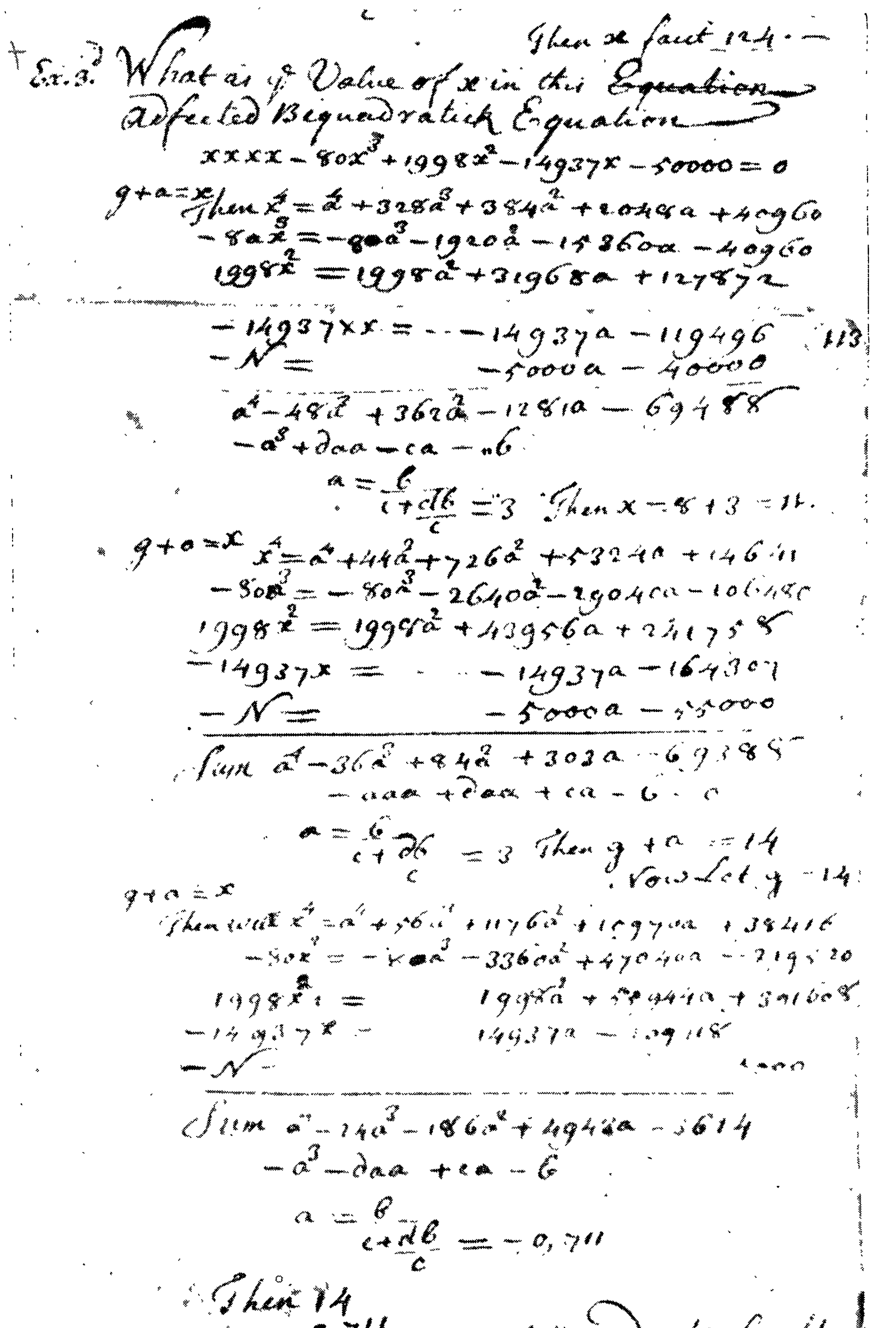
ALGEBRA AT HARVARD COLLEGE IN 1730.

By LAO GENEVRA SIMONS, Hunter College of the City of New York.

One of the earliest professors of mathematics in an American college was Isaac Greenwood who occupied the chair founded at Harvard College by Thomas Hollis. Greenwood was a graduate of Harvard of the class of 1721; he received his A.M. degree three years later and went to England for further study. He was installed as professor in February, 1728. In 1729, he published anonymously a work on arithmetic.[1] His connection with Harvard came to an end in July, 1738, on account of serious irregularities in his own life.

Direct evidence hitherto published of the work that this mathematics professor was doing at Harvard has been so meager as to be quite negligible. Such

[1] Conclusive evidence of the authorship of this arithmetic is to be found in the advertisement which follows: "[Boston] *Weekly News-Letter*, May 29, 1729. Just Published. Arithmetick Vulgar & Decimal; with the Application thereof to a Variety of Cases in Trade & Commerce. By Isaac Greenwood, A.M., Hollisian Professor of Mathematicks, and Philosophy. To be Sold by Thomas Hancock at the Bible & Three Crowns near the Town Dock, Boston." The copy of this work in the New York Public Library contains "Eliakim Willis his Book 1733." Willis graduated from Harvard in 1735 and no doubt used the book in his college course.



Then x facit 12.4 —

+ Ex. 3.d What is ye Value of x in this ~~Equation~~ Adfected Biquadratick Equation

$xxxx - 80x^3 + 1998x^2 - 14937x - 50000 = 0$

$8+a=x$ Then $x^4 = a^4 + 32a^3 + 384a^2 + 2048a + 4096$

$-80x^3 = -80a^3 - 1920a^2 - 15360a - 40960$

$1998x^2 = 1998a^2 + 31968a + 127872$

$-14937x = -14937a - 119496$ 113

$-N = -5000a - 40000$

$a^4 - 48a^3 + 362a^2 - 1281a - 69488$

$-a^3 + daa - ca - n6$

$a = \frac{6}{c + \frac{db}{c}} = 3$ Then $x = 8 + 3 = 11$.

$8+a=x$ $x^4 = a^4 + 44a^3 + 726a^2 + 5324a + 14641$

$-80x^3 = -80a^3 - 2640a^2 - 29040a - 106480$

$1998x^2 = 1998a^2 + 43956a + 241758$

$-14937x = -14937a - 164307$

$-N = -5000a - 50000$

Sum $a^4 - 36a^3 + 84a^2 + 303a - 69388$

$-aaa + daa + ca - 6 \cdot c$

$a = \frac{6}{c + \frac{db}{c}} = 3$ Then $g + a = 14$

Now Let $g = 14$

$g+a=x$

Then will $x^4 = a^4 + 56a^3 + 1176a^2 + 10976a + 38416$

$-80x^3 = -80a^3 - 3360a^2 + 47040a - 219520$

$1998x^2 = 1998a^2 + 55944a + 391608$

$-14937x = 14937a - 209118$

$-N =$ 50000

Sum $a^4 - 24a^3 - 186a^2 + 4943a - 5614$

$-a^3 - daa + ca - 6$

$a = \frac{6}{c + \frac{db}{c}} = -0{,}711$

Then 14
−0,711
13,289 = ye Biquadratic sought nearly

Solution of a biquadratic equation from the manuscript of a Harvard student, 1730.

evidence as there is probably passed under the eyes of a great many persons until one looked upon it who knew its worth. The discovery of a complete manuscript notebook on algebra dated 1739 greatly enriched our knowledge of the beginnings of algebra in America.[1] The manuscript referred to was found in the museum housed in the old jail in York Village, Maine. It is by Samuel Langdon who graduated from Harvard in 1740 and some years later became president of the College.

The value of this manuscript was considerably enhanced by the further discovery, in the *Manuscripts Americana* at Harvard University, of another manuscript which so closely resembles the Langdon one as to leave no doubt that the two notebooks were taken from the same original. The Harvard manuscript was written by James Diman, who graduated from Harvard in 1730, and who served the College as librarian from 1735–1737.

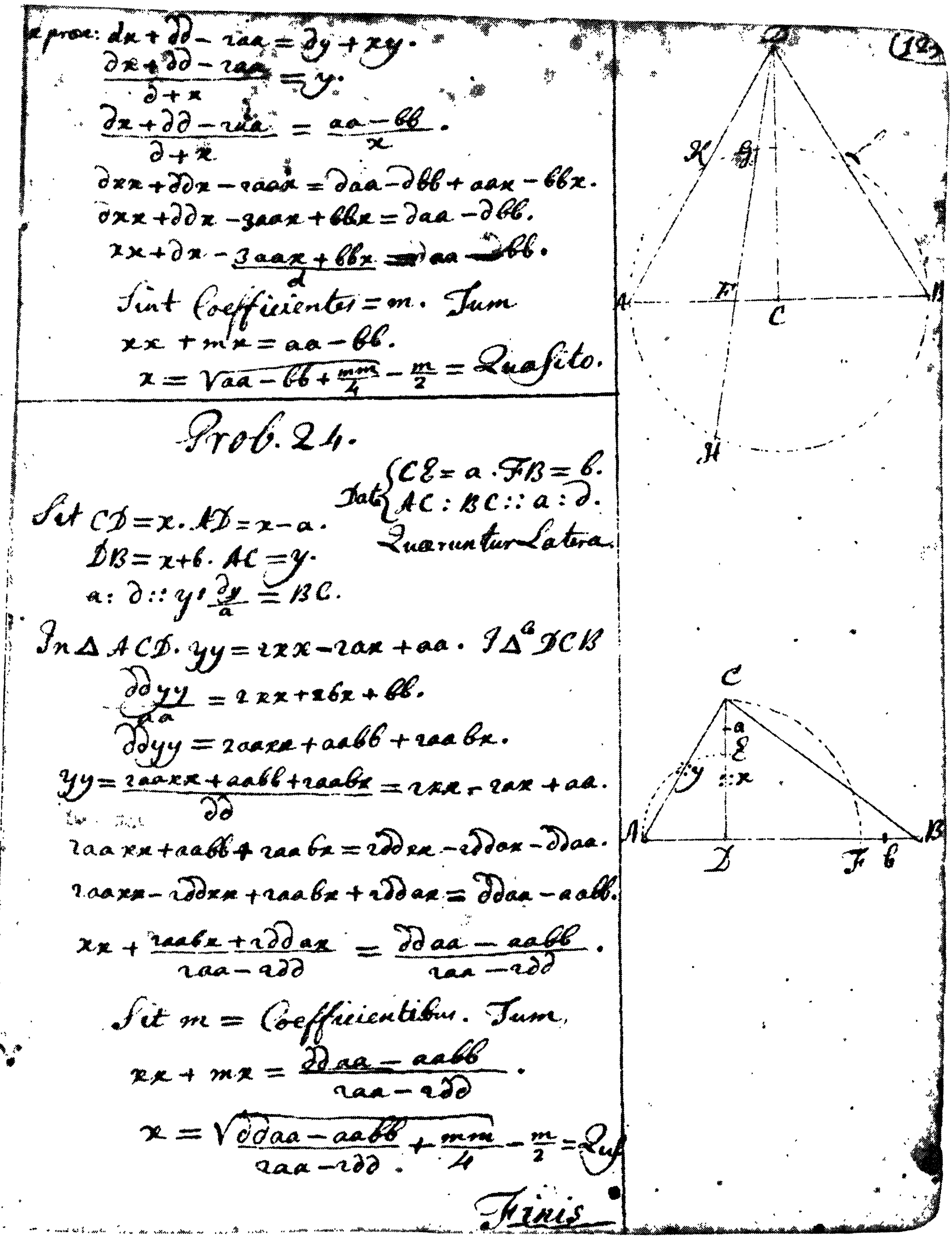
The Langdon manuscript consists of seventy-five numbered pages, with two unnumbered pages forming a front cover and eighteen unnumbered pages at the end. The latter contain no notes, except one leaf. The pages are 14.7 cm. by 18.7 cm., and the book consists of forty-eight leaves, three-fourths of one leaf having been cut out by the author. On the front cover appears the inscription: "Samuel Langdon's Book, July 25, 1739," and on the reverse of this leaf are the words: "Algebra by Isaac Greenwood, M.A. Began July 25, 1739." A colophon reads: "Finished writing Algebra August 17, 1739. Algebrae Finis."

The Diman manuscript consists of four unnumbered pages followed by one hundred and twenty-five numbered pages, 16 cm. by 19.3 cm. In the upper right-hand corner of the first page is the inscription: "James Diman's Book 1730/31." In large writing on this same page appears the title: "Algebra or Universal Mathematics reviewed 1738 with Notes and Additions." The third page has the following note: "Books perused in y^e review of my Algebra made in 1738. 1. Harris Lexicon Technicum. 2. Chambers Cyclopaedia. 3. Wolfius Elementa Matheseos Univers." The work ends on page 125 simply with "Finis."

The earlier date, 1730–31, on the Diman manuscript occurred during Greenwood's first years in his professorship, and hence there is every reason to believe that Greenwood was using this algebra material at least as early as 1730. The other date, August 1738, on this same manuscript was just after Greenwood's withdrawal from the College. A comparison of the two shows just a small section that is not found in the Langdon one and so Diman added little to the original in his review. Langdon's book was made about a year later in July 1739. Both manuscripts seem to be careful copies of work done earlier at Harvard.

The existence of two such manuscripts is a matter of importance in the history of American mathematics. One manuscript might have been the work given to a private pupil, but two similar manuscripts at different times during the same professorship afford unmistakable proof that such work was being taught at Harvard during the period of this particular man.

[1] David Eugene Smith, "A Glimpse at Early Colonial Algebra." *School and Society*, January 5, 1918: This article fully describes the manuscript and any reference to it herein contained will necessarily cover the same ground, although it is the result of a study made by the writer.



prox: $dx + dd - 2aa = dy + xy$.

$\frac{dx+dd-2aa}{d+x} = y.$

$\frac{dx+dd-2aa}{d+x} = \frac{aa-bb}{x}.$

$dxx+ddx-2aax = daa-dbb+aax-bbx.$

$dxx+ddx-3aax+bbx = daa-dbb.$

$xx+dx-\frac{3aax+bbx}{d} = aa-bb.$

Sint Coefficientes $= m$. Tum

$xx+mx = aa-bb.$

$x = \sqrt{aa-bb+\frac{mm}{4}}-\frac{m}{2} =$ Quaesito.

Prob. 24.

Dat. $CE = a$. $FB = b$. $AC : BC :: a : d$.

Quaeruntur Latera.

Sit $CD = x$. $AD = x-a$.

$DB = x+b$. $AC = y$.

$a : d :: y : \frac{dy}{a} = BC$.

In $\triangle ACD$. $yy = 2xx-2ax+aa$. In $\triangle^{a}DCB$

$\frac{ddyy}{aa} = 2xx+2bx+bb.$

$ddyy = 2aaxx+aabb+2aabx.$

$yy = \frac{2aaxx+aabb+2aabx}{dd} = 2xx-2ax+aa.$

$2aaxx+aabb+2aabx = 2ddxx-2ddax-ddaa.$

$2aaxx-2ddxx+2aabx+2ddax = ddaa-aabb.$

$xx+\frac{2aabx+2ddax}{2aa-2dd} = \frac{ddaa-aabb}{2aa-2dd}.$

Sit $m =$ Coefficientibus. Tum

$xx+mx = \frac{ddaa-aabb}{2aa-2dd}.$

$x = \sqrt{\frac{ddaa-aabb}{2aa-2dd}+\frac{mm}{4}}-\frac{m}{2} =$ Qu

Finis

Algebra applied to the solution of geometric problems from the Diman (1730) manuscript.

Some discussion of the subject matter of these notebooks will show the extent of the algebra presented by the first professor at Harvard. The introduction denotes an interest on the part of the author of the original in the history of the subject. It begins as follows:

This science is called Algebra from two or three words en ye Arabian Language, wch may be interpreted either the Art of Restitution, & Comparison; or ye Art of Resolution & Equation. It is also known by various other Names. The first yt wrote upon this Subject, in Europe, termed it ye Rule of Restitution & Opposition; Since, it has been called by some, the Analytick Art; by others, Specious Computation; Regula Rei et Census; ye great Art; Modern Geometry; Universal Mathematicks &c.[1]

The topics treated in the notebooks are almost identical except for variations in spelling. Given in the words of the Diman manuscript, they are:

Notation, Algebraical Caracters, Addition of Integers, Subtraction, [Langdon uses 'Substraction'[2]], Multiplication of Algebraic Integers, Division, Algebraical Fractions, Addition and Subtraction of Fractions, Multiplication of Fractions, Division of Fractions, Involution of Whole Quantities, Involution of Fractional Quantities, Evolution of Whole Quantities, Fractional Evolution, Binomial Quantities, Involution [Of Binomial Quantities], Promiscuous Examples, Multinomial Quantities, Involution [Of Multinomial Compound Quantities], Evolution, Surd Quantities, Notation [Of Surds], Reduction of Surds, Multiplication of Surds, Division of Surds, Addition and Subtraction of Surds, Compound Surds, Multiplication of Binomial Surds, Division in Compound Surds, ['Compound Surds' is the one topic in which there is a marked difference in the two manuscripts, the Langdon one containing much less material than the Diman], Equation, Reduction of Equations, Reduction by Addition, Reduction by Subtraction, Reduction by Multiplication, Reduction by Division, Reduction by Involution, Reduction by Evolution, Reduction by Analogies to Equations & e Contra, The Method of Resolving Algebraical Questions, General Rules concerning ye Reduction of Equations, Simple Equations, The Solution of Adfected Quadratick Equations, Mr. Oughtreds method of solving adfected Quadraticks, The Solution of Adfected Equations by taking away ye Second Term,[3] The Solution of Adfected Quadratick Equations by ye method of Compleating ye Square, Questions, The Resolution of Cubic Equations, Cubic Equations by Substitution, Cubic Equations by Tryalls and Depression, The Solution of Irregular Cubics, The Method of Converging Series, Mr. Raphson's Theorems for Simple Powers, Mr. Raphson's Theorems for Adfected Equations, Dr. Halley's Theorems for Solving Equations of all sorts, Concerning the Method of resolving Geometrical Problems algebraically.

A few interesting passages will be given to indicate the spirit of these textbooks.

Involution of Binomial Quantities. . . . Consequently, if ye Numeral figures of Coefficients could be found ye whole might be performed without multiplication and this is done by ye following problem. [Langdon says "This is done by ye following rule, given by Sr. Isaac Newton, see p. 139." [4]]. To find ye Coefficients in Binomial Powers. Rule. Multiply ye Coefficient into ye Index of ye Power and Divide that Product by ye Number of terms, counting from ye left hand, and ye Quotient will be ye Coefficient or Numeral Figure of ye next successive Quantity.

Irrational Quantities are noted thus $\sqrt{:}\,2$ wch is 2 wth ye Sign of Irrationality $\sqrt{:}$ before it. . . . There is also another way of marking surd Quantities where Roots are expressed without ye

[1] Quotations are from the earlier manuscript, that by Diman; citations from the Langdon one would be practically the same. The latter appeared in: David Eugene Smith, "A Glimpse at Early Colonial Algebra," *loc. cit.*

[2] For a discussion of the origin of this word, see David Eugene Smith, *History of Mathematics*, II, 95, Boston, 1924.

[3] In this method $y - \frac{1}{2}d$ is substituted in the equation $x^2 + dx = m$; whence $y^2 - yd + \frac{1}{4}d^2 = x^2$ and $y^2 = m + \frac{1}{4}d^2$ or $y = \sqrt{m + \frac{1}{4}d^2}$.

[4] The work referred to is entitled: *The Elements of that Mathematical Art commonly called Algebra* by John Kersey, London, 1673.

Radical sign by their Index, this is founded upon yᵉ manner of expressing Powers, thus x^2, x^3, x^4, signifie yᵉ Square, Cube & Biquadratick of x: so $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{4}}$ will accordingly signifie yᵉ Square, Cube and Biquadratick Root of x. . . . and wⁿ at any time there is yᵉ Sign of Irrationality prefixt to mixt Quantities with yᵉ sign of Inseperation over yᵐ thus $\sqrt{:}\ ^3\overline{7 + \sqrt{:2}}$ it is called a Universal Root.

Reduction of Analogies to Equations & e Contra.
Reduce yᵉ Analogy $x : 4 : 2x$, $4 \times 4 = 16$, $x \times 2x = 2xx$,

$$\frac{2xx}{2} = \frac{16}{2} = 8 \text{ per } 16 : 6 \text{ Euclid, } xx = 8.$$

Sets of problems follow the rules for the solution of simple and quadratic equations. It is of especial interest to note that in connection with the solution of question 7 of the set under quadratic equations, the Langdon manuscript (p. 49) gives an imaginary number in the result. This is the only approach to an imaginary in either of the manuscripts, and it indicates scholarship on the part of the author of the notes as well as ability on the part of his young pupil.

The advanced nature of the work done at Harvard in 1730 in the subject of algebra is indicated in the treatment of cubic and higher degree equations. Three methods are given for the solution of cubic equations. The first is that "By Substitution, Deduction & Division." The second method is by "Tryalls and Depression," and the author tells that "The Resolution of Cubic Equations by Tryalls may be performed two ways." The first way suggests the beginning of the application of the Remainder Theorem, since divisors of the absolute term, which must always be transposed to the first member of the equation, are sought out. Then the cubic expression is to be divided by $x +$ or $x -$ this divisor. The quotient will be a quadratic expression which may be treated as such or the above process may be repeated. The other way again takes factors of the absolute term and substitutes these factors for the unknown, one by one, until one is found which makes the first member of the equation amount to the absolute term.

With the third method, "The Method of Converging Series," the more difficult handling of cubic equations appears. As defined in the book:

> The Method of Converging Series is an Approximation, or orderly approach nearer & nearer yᵉ Truth yᵉ more one works on ad Infinitum, by wᶜʰ means any Equation wᵗsoever either Quadratick, Cubic, Biquadratick, Sursolid, etc. may be answer'd to any degree of exactness.

After some further general remarks on this method, there follow "Mʳ. Raphson's[1] Theorems for Simple Powers." "For yᵉ Biquadratick $a^4 = n - (n - g^4)/4ggg = x$." It is applied "To extract yᵉ Biquadrate of 90."

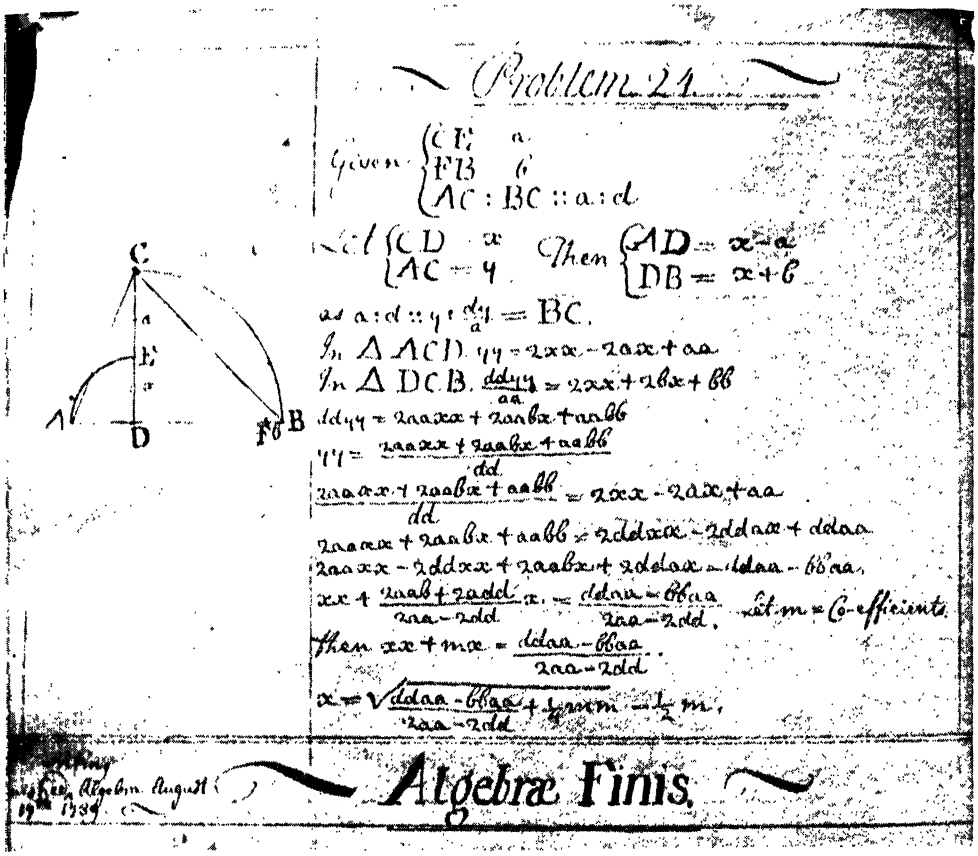
In addition to the theorems for simple powers, there are "Mʳ. Raphson's Theorems for adfected Equations." For the solution of the "Biquadrate $a^4 \pm pa^3 = N$, Theorem $(N - g^4 \mp pg^3)/(4g^3 \pm 3pg^2) = x$" and the theorems for the equations of higher degrees are correspondingly difficult.

[1] Joseph Raphson, who published in 1690 a work entitled *Analysis aequationum universalis*, in which he modified Newton's method of finding the approximate roots of a numerical equation. For discussion, see F. Cajori, *Oughtred*, p. 140.

The last section of this part of the work is devoted to "D^r^. Halley's[1] Theorems for Solving Equations of all sorts." As the text indicates:

> D^r^. Halley has found out two universal Theorems which answer all cases Imaginable of any parts whatsoever, w^th^ greater ease and expedition y^n^ y^e^ others.

The theorems are applied to the solution of several equations.



Algebra applied to the solution of a geometric problem from the Langdon (1739) manuscript.

The notebooks close with a section "Concerning the Method of resolving Geometrical Problems Algebraically," without which no algebra textbook of the eighteenth century was complete.

It is one of the significant topics of these two manuscripts.

In the opening paragraph, the author writes:

> But in Geometrical Problems is tho't sufficient to note only such Particulars as are necessary to lead y^e^ Geometritian to some known Theorem, whereby y^e^ solution may be made. And to facilitate this I must advise y^e^ Student, always in Geometrical affairs, to consider y^e^ unknown

[1] Edmund Halley, the great astronomer. He was also deeply interested in algebra and geometry.

Quantity as really known. Then Compairing y• several Quantities in y• Problem, note how they are related either directly or by Consequence of any of Euclid, &c. Demonstrations.

Twenty-four problems of a geometric nature follow. Diman uses Latin throughout this set, while Langdon continues to use the English language. The problems are the same and are numbered alike in the two manuscripts. No statement of the problem in words precedes the solution, except in one instance. The diagram in each case is clearly marked, and the data given by reference to the diagram. There are scattered throughout the exercises many facts from Euclid, both definitely referred to and implied. We cannot doubt that the pupils who studied this algebra, writing out these manuscripts as textbooks, had already taken a good course in geometry.

The two manuscripts under discussion bear a close resemblance to each other, but they differ enough to indicate that the individualities of the students themselves played a part in the final productions. They are both unquestionably based on a set of notes prepared by Professor Greenwood to use in a course at Harvard. Had he lived out his professional life, it is altogether probable that we should have had from his pen the first printed work on algebra written by a native American. Work of the character shown in these manuscripts is evidence that mathematics of a collegiate nature measured by present day standards was taught to students at Harvard College in 1730.

ON THE CORRELATION BETWEEN TWO VARIATES x AND $y = kx^s$.

By KARL PEARSON, University of London, England.

In a paper recently published in the MONTHLY (*1924*, 227–231), Professor P. R. Rider discusses the above problem in the special case of the normal distribution of the variate x. As a practical statistician I should like to consider the problem from a rather more general standpoint with simpler analysis.

Let $\mu_s' = S(x^s)/N$ be the sth moment coefficient about any fixed origin of the character x in a population of N, the symbol S denoting summation for all individual values. Then: $\mu_1' = S(x)/N =$ *mean value of* $x = \bar{x}$, say. Let μ_s be the moment sth coefficient about the mean. Then it follows from a simple application of the binomial theorem that

$$\mu_s' = \mu_s + s\bar{x}\mu_{s-1} + \frac{s(s-1)}{1\cdot 2}\bar{x}^2\mu_{s-2} + \text{etc.} \qquad (i)$$

If r be the correlation coefficient of x and y,

$$r = \frac{S(kx^{s+1}) - \bar{x}kS(x^s)}{N\sigma_x\sigma_y},$$

where $\sigma_x =$ standard deviation of $x = \sqrt{\mu_2}$, while

$$\sigma_y^2 = \frac{S(k^2x^{2s}) - k^2[S(x^s)]^2/N}{N},$$

and accordingly

$$\sigma_y^2 = k^2(\mu_{2s}' - \mu_s'^2) \qquad (ii)$$

and

$$r = \frac{\mu_{s+1}' - \bar{x}\mu_s'}{\sqrt{\mu_2(\mu_{2s}' - \mu_s'^2)}}. \qquad (iii)$$

This is the simple expression for r. By aid of (i) we can always convert the moments about the fixed origin into moments about the mean.

Thus:

$$r = \frac{\mu_{s+1} + s\bar{x}\mu_s + \frac{(s+1)s}{1\cdot 2}\bar{x}^2\mu_{s-1} + \cdots - \bar{x}\left(s\bar{x}\mu_{s-1} + \frac{s(s-1)}{1\cdot 2}\bar{x}^2\mu_{s-2} + \cdots\right)}{\left\{\mu_2\left(\mu_{2s} + 2s\bar{x}\mu_{2s-1} + \frac{2s(2s-1)}{1\cdot 2}\bar{x}^2\mu_{2s-2} + \cdots - \left(\mu_s + s\bar{x}\mu_{s-1} + \frac{s(s-1)}{1\cdot 2}\bar{x}^2\mu_{s-2} + \cdots\right)^2\right)\right\}^{1/2}}. \qquad (iv)$$

Illustrations:

(a) $s = 1$:

$$r = \frac{\mu_2 + \bar{x}^2 - \bar{x}^2}{\{\mu_2(\mu_2 + \bar{x}^2 - \bar{x}^2)\}^{1/2}} = 1,$$

since $\mu_1 = 0$ and $\mu_0 = 1$.

There is therefore perfect correlation between x and kx, as might be anticipated.

(b) $s = 2$:

Let us write as usual $\sqrt{\beta_1} = \mu^3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$. Then applying (iv) we find at once:

$$r = \frac{\frac{\bar{x}}{\sigma_x} + \frac{1}{2}\sqrt{\beta_1}}{\sqrt{\left(\frac{\bar{x}}{\sigma_x} + \frac{1}{2}\sqrt{\beta_1}\right)^2 + \frac{1}{4}(\beta_2 - \beta_1 - 1)}}.$$

As β_2 is always $\geqq \beta_1 + 1$, this is always less than unity. Expanding:

$$r = 1 - \tfrac{1}{8}(\beta_2 - \beta_1 - 1)\frac{1}{\left(\frac{\bar{x}}{\sigma_x} + \frac{1}{2}\sqrt{\beta_1}\right)^2} + \text{etc.}$$

Now $\bar{x}/\sigma_x = 100/V_x$, where V_x is the so-called coefficient of variation. Now in the usual anthropometric characters in man V_x is of the order 3 to 5, and though exceptions can of course be found, this is much the order of variation in plant and animal life. Accordingly, the important term is of the order 1/1000 for the sort of value of β_1 and β_2 usually met with, or $r = .999$.

(c) $s = 3$:

Let us put $\beta_3' = \mu_s/\sigma_x^5$, $\beta_4 = \mu_6/\mu_2^3$, then we have

$$r = \frac{\left(\frac{\bar{x}}{\sigma_x}\right)^2 + \sqrt{\beta_1}\frac{\bar{x}}{\sigma_x} + \frac{1}{3}\beta_2}{D},$$

where

$$D = \left\{\left(\left(\frac{\bar{x}}{\sigma_x}\right)^2 + \sqrt{\beta_1}\frac{\bar{x}}{\sigma_x} + \tfrac{1}{3}\beta_2\right)^2 + \frac{\bar{x}^2}{\sigma x^2}(\beta_2 - \beta_1 - 1) \right.$$
$$\left. + \tfrac{2}{3}\frac{\bar{x}}{\sigma_x}[\beta_3' - \sqrt{\beta_1}(1 + \beta_2)] + \tfrac{1}{9}(\beta_4 - \beta_2 - \beta_1)\right\}^{1/2}.$$

The value of r as thus determined is nearly equivalent to the following:

$$r = 1 - \tfrac{1}{2}\frac{\dfrac{\bar{x}^2}{\sigma_x^2}(\beta_2 - \beta_1 - 1) + \tfrac{2}{3}\dfrac{\bar{x}}{\sigma_x}[\beta_3' - \sqrt{\beta_1}(1 + \beta_2)] + \tfrac{1}{9}(\beta_4 - \beta_2 - \beta_1)}{\left\{\left(\dfrac{\bar{x}}{\sigma_x}\right)^2 + \sqrt{\beta_1}\dfrac{\bar{x}}{\sigma_x} + \tfrac{1}{3}\beta_2\right\}^2}.$$

To appreciate the order of the deviation from unity in most practical statistical work, let us apply the result to normal distribution for x. We have $\beta_2 = 3$, $\beta_4 = 15$, $\beta_1 = \beta_3' = 0$ and

$$r = 1 - \frac{\dfrac{\bar{x}^2}{\sigma_x^2} + \tfrac{2}{3}}{\left(\dfrac{\bar{x}^2}{\sigma_x^2} + 1\right)^2} = 1 - \frac{1}{\dfrac{\bar{x}^2}{\sigma_x^2} + 1}, \text{ nearly,} \qquad = 1 - \frac{1}{\dfrac{(100)^2}{V_x^2} + 1}, \text{ nearly.}$$

If V_x be of the order 4,

$$r = 1 - \frac{1}{625} = .9984,$$

which is perfect correlation for all practical purposes.

Long ago[1] I showed that nothing was gained practically in intensity of correlation by correlating intelligence with weight per stature squared or cubed in the case of adults instead of with weight per simple stature, and those scientists who invent indices of (weight)/(stature)^n as practical measures are wasting their energies. It is of course true that the correlation *coefficient* of x and kx^s is not unity. But it is a very familiar result that r is not unity for perfect correlation unless the regression is *linear*. In all cases of skew regression where the correlation is perfect, as in the case of x and kx^s, what is unity is the correlation ratio y, and no statistician of today lays any stress on the correlation coefficient until he has satisfied himself by computing $\eta_{x\cdot y}$ and $\eta_{y\cdot x}$ and seeing that they do not differ sensibly from r, that the regression is for the range of his problem linear. Then, and then only, r is of practical significance. When, however, the coefficient of variation is of the order 3 to 6, then the portion of the regression curve $y = kx^s$ which actually comes into consideration is sensibly straight and r will not differ from unity by more than 1 or 2 in 1000. Even for considerably higher values of V, r will not differ from unity by more than 1 per cent., which is adequate approximation to unity for most statistical treatment.

[1] *Proc. Royal Soc.*, vol. 71 (1902), p. 111.

There is a good deal of idle talk about a second character failing to follow a normal distribution if it be a function of a first character which does. There is nothing in this criticism, if the coefficient of variation does not take unusual values. If it does take unusual values, then the statistician has been warned of this because his correlation ratios differ sensibly from his correlation coefficient. He then knows that his regression is non-linear and therefore that no normal Gaussian surface can describe the frequency distribution, and accordingly the marginal totals, one or both of them, cannot be normal.

But the deviation from normality is usually very small, the frequency curve to a first approximation being:

$$z = z_0 e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}\left[1 + \left(\frac{1}{s} - 1\right)y/\bar{y}\right],$$

where the origin is at the mean of $\bar{y}$. This may be written

$$z = z_0' e^{-\frac{1}{2}\left(\frac{y}{\sigma y} + \frac{s-1}{s}v_y\right)^2},$$

where z_0' is another constant. We have therefore, still to the same approximation, the curve of frequency, a normal curve, but with its origin not at its mean, but shifted through a distance $-\dfrac{(s-1)}{s}v_y\sigma_y$ or

$$-\frac{(s-1)}{s}v_y^2\bar{y} = -s(s-1)v_x^2\bar{y},$$

since $v_y^2 = s^2 v_x^2$ closely.

If s be not greater than 3, then for a value of a coefficient of variation 4—*i.e.*, $V_x = 100v_x = 4$—this shifting is only through 1 per cent. of $\bar{y}$. If s be fractional, it is far less. Of course there are occasionally higher values of V_x, but the practical statistician will always protect himself against error if he compares his correlation ratios with his correlation coefficient, and bears in mind the magnitude of his coefficients of variation. In actual data I should not be at all surprised if the weights of apples gave quite as good (or bad) a fit to a normal frequency distribution curve as their diameters. There are so many other factors of variation that such a mathematical relation as: weight varies as the cube of the diameter, plays but a small part in the total effect.

NOTE ON THE GEOMETRIC ASPECTS OF EINSTEIN'S THEORY.

By C. N. REYNOLDS, West Virginia University.

A most valuable feature of Professor Pierpont's recent exposition[1] of the geometric aspects of the theory of relativity was the inclusion of a sequence of examples illustrative of the various sections of his paper. These will be of value to many who have occasion to teach elementary courses in the theory of relativity. In my belief that the best approach to this subject is through the two-dimensional case, I suggest the consideration of the following additional exercises, throughout which we shall make use of Pierpont's notation and take $n = 2$.

*A*1. Evaluate (12, 12) explicitly.

*A*2. Prove $aG_{\alpha\mu} = -a_{\alpha\mu}(12, 12), \quad \alpha, \mu = 1, 2.$

*A*3. Prove $G = -\frac{2}{a}(12, 12).$

*A*4. Prove $G_{ij} - \frac{1}{2}a_{ij}G = 0, \quad i, j = 1, 2.$

If H is invariant and $H_{\sigma\tau} = Ha_{\sigma\tau}$, then

$$A5.\ H_{\sigma\tau/\nu} = a_{\sigma\tau}\frac{\partial H}{\partial x_\nu};\ \text{div}\, H_{\sigma\tau} = \sum_\tau a_{\tau\tau}\frac{\partial H}{\partial x_\tau};\ \text{and div}\, H_\tau^\rho = \frac{\partial H}{\partial x_\tau}.$$

*A*6. If $a \neq 0$, then the vanishing of either of the above divergences implies that H is constant.

Replacing equation (86) of Pierpont's paper by

$$T^{\rho\tau} = \rho\frac{dx_\rho}{ds}\frac{dx_\tau}{ds} - pa^{\rho\tau}, \quad \text{or} \quad T_{\rho\sigma} = \rho a_{\alpha\sigma}a_{\rho\tau}\frac{dx_\alpha}{ds}\frac{dx_\tau}{ds} - pa_{\rho\sigma},$$

where p is the hydrostatic pressure,[2] we have, from physical considerations,

$$\text{div}\, T^{\rho\tau} = \text{div}\, T_{\rho\sigma} = 0.$$

*B*1. Prove $T = \rho - 2p$; and $T_{\rho\sigma}\frac{dx_\rho}{ds}\frac{dx_\sigma}{ds} = \rho - p.$

*B*2. Show that if $T_{\rho\sigma}$ be set equal to a linear combination of metric covariants of the second order of the form $aGa_{\rho\sigma} + bG_{\rho\sigma} + ca_{\rho\sigma}$ and if $\lambda \equiv \left(a + \frac{b}{2}\right)G + c$, where a, b, and c are constants, then

(*a*) $T_{\rho\sigma} = \lambda a_{\rho\sigma}$.

(*b*) λ is constant and equal to minus p.

(*c*) our metric is that of a spherical surface of radius $\sqrt{\frac{2a + b}{c - \lambda}}$.

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[1] *Annals of Mathematics*, 1923, Series II, vol. 23, p. 228.

[2] *Cf.* Birkhoff: *Relativity and Modern Physics*, chap. VII.

Turning now to the case of a static field, we set

$$x_1 = r, \qquad x_2 = t, \qquad \alpha_{12} = 0, \qquad \frac{\partial a_{11}}{\partial t} = \frac{\partial a_{22}}{\partial t} = 0.$$

Since $ds^2 = 0$ for a light impulse, we shall set $a_{22} = - C^2(r)a_{11}$, where $C(r)$ is the observed velocity of light.

$C1$. The equations of a geodesic are equivalent to

$$\begin{cases} a_{22}\dfrac{dt}{ds} = A, \\ a_{22}\left[- C^{-2}\left(\dfrac{dr}{ds}\right)^2 + \left(\dfrac{dt}{ds}\right)^2\right] = 1, \end{cases}$$

where A is a constant of integration.

$C2$. The four equations $G_{ij} = 0$ $(i, j = 1, 2)$ are equivalent to the single equation $(12, 12) = 0$ when $a \neq 0$. Integrating this equation, we have

$$a_{22} = k^2 e^{\pm 2c\int \frac{dr}{C(r)}}.$$

$C3$. The equations of a geodesic imply that $\dfrac{\frac{dr}{dt}}{C(r)} = \tanh c(t - t_0)$, whence $a_{22} = A^2 \operatorname{sech}^2 c(t - t_0)$.

$C4$. Show that our metric is that of a right circular cylinder of radius $\dfrac{2A}{c}$. Express the cylindrical coördinates in terms of the observed velocity of light, $C(r)$.[1]

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems (especially new problems) which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS.

52. Is there any simple treatment of the regular pentagon constructions the proof of which involves neither medial section nor trigonometrical formulæ?

I. REPLY BY NORMAN ANNING, University of Michigan.

1. Introduction. In order to construct a regular pentagon without explicit use of medial section or trigonometry, it will be necessary to invoke the aid of

[1] At Professor Pierpont's suggestion I take this opportunity to point out a serious misprint which he found in line 11, page 250, of his paper in volume 23 of the *Annals of Mathematics* to which reference has been made. This line now reads: "by the vanishing of the energy-momentum tensor T," whereas it should read: "by the vanishing of the *divergence* of the energy-momentum tensor T," as is clearly shown by the context.

theorems in which the trigonometry is latent. This note employs Ptolemy's theorem and a moment-of-inertia theorem neither of which requires for its proof any proposition outside of the first six books of Euclid.

2. Analysis. Suppose that a regular decagon $AD'BE'CA'DB'EC'$ has been constructed in a circle whose center is O and whose radius is r. Let $A'B = x$ and $A'C = y$.

The centroid of unit masses at the vertices of the regular pentagon $ABCDE$ is O. For the moment of inertia about an axis through A' perpendicular to the plane of the pentagon, we have

$$\Sigma(AA')^2 = \Sigma AO^2 + 5(OA')^2;$$
$$(2r)^2 + x^2 + y^2 + y^2 + x^2 = 5r^2 + 5r^2. \tag{1}$$
$$x^2 + y^2 = 3r^2.$$

Applying Ptolemy's theorem to the inscribed quadrilateral $A'CE'B$,

$$y^2 + xy = CB^2 = AB^2 = 4r^2 - x^2.$$
$$xy + x^2 + y^2 = 4r^2.$$

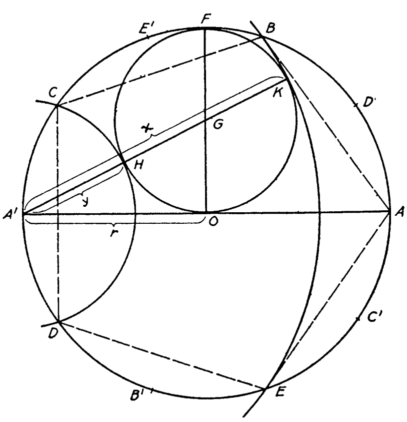
Using (1),

$$xy = r^2. \tag{2}$$

From (1) and (2) by elementary algebra,

$$x - y = r. \tag{3}$$

Two segments x and y which satisfy equations (2) and (3) can be constructed by using any circle in which a chord of length r can be placed. We shall find it convenient to use a circle whose diameter is r.



3. Synthesis. With center O describe a circle of radius r; draw any diameter AOA' and draw OF perpendicular to AA' to meet the circle in F. Draw the circle whose diameter is OF and whose center is G. Draw $A'G$ to meet this latter circle in H and K.

Since $A'K$ is a secant of a circle to which $A'O$ is tangent and since $HK = OF = r$, it follows that $A'K$ and $A'H$ are, respectively, the x and y of the above analysis.

With center A' and radius x describe a circle cutting the original circle in B and E, and, with radius y, one cutting it in C and D. $ABCDE$ is the required regular pentagon.

II. Reply by C. H. Chepmell, Hove, England.

THEOREM. (Fig. 1.) *If ABD be a circle (centre C, radius a), and E and F any points in AC and BC respectively, and DF the ordinate at F; then, joining ED,*

and laying off, along AB, EG $(= ED)$, we have[1]

$$a^2 - CG^2 = 2 \cdot EC \cdot FG.$$

For

$$EG^2 = ED^2 = EF^2 + a^2 - CF^2 = EC^2 + 2EC \cdot CF + a^2,$$

also

$$EG^2 = EC^2 + 2EC(CF + FG) + CG^2,$$

and equating the two values of EG^2,

$$EC^2 + 2EC \cdot CF + a^2 = EC^2 + 2EC \cdot CF + 2EC \cdot FG + CG^2$$

or

$$a^2 - CG^2 = 2EC \cdot FG.$$

Q.E.D.

(It will be noted that the magnitude or existence of CF does not affect this result.)

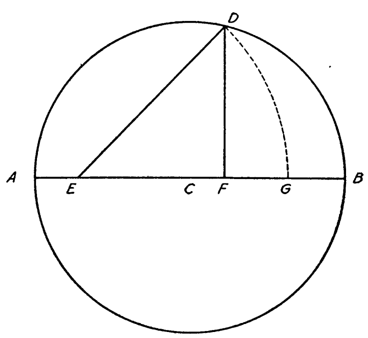


FIG. 1.

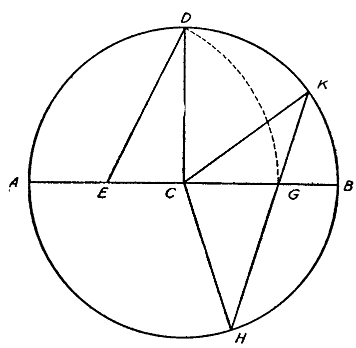


FIG. 2.

Now (Fig. 2), to construct the Pentagon, etc.: if we take E so that $EC = \frac{a}{2}$, and F so as to coincide with C, we have by our theorem above that $a^2 - CG^2 = a \cdot CG$. Draw GH $(= a)$ and produce to K, and join CH, CK. Then $a \cdot GK = HG \cdot GK = AG \cdot GB = a^2 - CG^2 = a \cdot CG$. Hence $GK = CG$; and if we call $\angle\, KCG$ θ, then $\angle\, CKG = \theta = \angle\, CHG$; and $\angle\, GCH = \angle\, CGH = 2\theta$; and $5\theta = \pi$.

$$\therefore \quad \angle\, KCB = 36° \quad \text{and} \quad \angle\, BCH = 72°.$$

This proof has involved no Trigonometry and does not invoke the special relation—"medial section"—but a theorem of much wider and more general application, and capable (see above) of the very simplest proof.

[1] This theorem may be used to solve some problems very neatly; for instance, *Given two sides of a triangle, to construct it so that the angle opposite the greater shall be five times the angle opposite the less.*

DISCUSSIONS.

I. Note on "Limit Proofs in Solid Geometry."

By Paul Capron, United States Naval Academy.

Readers may be interested to learn that Prof. Longley's "Limit Proofs" [*1924*, 196] were for many years taught at the U. S. Naval Academy. How long ago they were introduced into the course, I cannot say, but I think probably about 1865—possibly earlier, probably not later. They were discontinued seven or eight years ago in accordance with a plan to make the course of study at the Naval Academy as much as possible like the conventional courses in the smaller colleges of this country. Some of them were published, but only in texts designed for the use of Midshipmen.

All the proofs were given as Prof. Longley gives them, except that each was worked for a circumscribed set as well as for an inscribed set, and the two sets shown to have a common limit.

II. An Elementary Solution of a Problem of Diophantus.

By A. A. Bennett, University of Texas.

An interesting, elementary, and very ingenious treatment of a problem in Diophantine analysis was given recently by H. E. J. Curzon, in this Monthly.[1] Curzon gives incidentally the solution due to Diophantus, but is engaged chiefly in describing a new solution depending upon three integers, l, m, n, which enter the solutions homogeneously. Their sum is furthermore restricted to be zero, so that we have there essentially only one independent *rational* parameter, although the use of all three parameters makes possible an elegant and symmetrical form for the answer. Three of the solutions that I here offer for the same problem contain two independent rational parameters, and so may be naturally regarded as more extensive than that of Curzon. A new one-parameter solution is also given. To a slight extent necessary as well as sufficient conditions are examined, but the possibility of other two-parameter solutions is not excluded and other independent one-parameter solutions may well exist.

The *form* of the problem may be generalized somewhat from that previously given without altering essentially the character of the solutions. It will be stated as follows. *Given three rational numbers, k_1, k_2, k_3, it is required to find three distinct rational numbers, a, b, c, such that $ab + 2d$, $ab - 2d$, $bc + 2d$, $bc - 2d$, $ca + 2d$, $ca - 2d$ shall all be squares of rational numbers, d being defined as $k_1a + k_2b + k_3c$.*

We shall obtain solutions of the form,

$$\begin{cases} a = cA(s, t), \\ b = cB(s, t), \\ d = c^2D(s, t), \qquad D \text{ not identically zero,} \end{cases}$$

[1] Page 200, June, 1924. This problem is quoted in Carmichael's *Diophantine Analysis* as number 5 of the terminal miscellaneous exercises.

where s and t are arbitrary rational parameters not homogeneously related and where A, B, D are rational functions of s, t, with numerical coefficients. To satisfy the conditions of the problem, we shall also have $d = c(k_1A + k_2B + k_3)$, whence

$$c = (k_1A + k_2B + k_3)/D.$$

Since the coefficients, k, enter nowhere else in the problem, we may dismiss further discussion of them with the remark that in the form in which Diophantus stated the problem these coefficients were all equal to $\frac{1}{2}$, $2d$ being therefore the sum $a + b + c$.

As Curzon notes,[1] it is *sufficient* that rational numbers x, y, z be found such that

$$\begin{cases} bc = x^2 + \dfrac{d^2}{x^2}, \\ ca = y^2 + \dfrac{d^2}{y^2}, \\ ab = z^2 + \dfrac{d^2}{z^2}. \end{cases} \tag{1}$$

We shall now show that this condition is also necessary. By hypothesis, $\sqrt{bc + 2d}$, $\sqrt{bc - 2d}$ are rational, the positive sign being used before the radical to denote a positive number. Then the sum $\sqrt{bc + 2d} + \sqrt{bc - 2d}$ is rational and will be denoted by $2x$. Similarly $2y$ and $2z$ are defined. Now

$$\frac{2d}{x} = \frac{4d}{\sqrt{bc + 2d} + \sqrt{bc - 2d}} = \frac{4d(\sqrt{bc + 2d} - \sqrt{bc - 2d})}{4d} \tag{2}$$
$$= \sqrt{bc + 2d} - \sqrt{bc - 2d}.$$

Thus

$$x + \frac{d}{x} = \sqrt{bc + 2d}, \qquad x - \frac{d}{x} = \sqrt{bc - 2d}.$$

Squaring and adding, we have, as desired, $bc = x^2 + \dfrac{d^2}{x^2}$, and the other relations of (1) follow in like manner.

From the fact that $a = (abc)/(bc)$, we may write

$$a = \frac{P}{\left(x^2 + \dfrac{d^2}{x^2}\right)}, \qquad b = \frac{P}{\left(y^2 + \dfrac{d^2}{y^2}\right)}, \qquad c = \frac{P}{\left(z^2 + \dfrac{d^2}{z^2}\right)}, \tag{3}$$

where P^2 is the product $\left(x^2 + \dfrac{d^2}{x^2}\right) \cdot \left(y^2 + \dfrac{d^2}{y^2}\right) \cdot \left(z^2 + \dfrac{d^2}{z^2}\right)$. Three possibilities may be imagined. All three, only one, or else none at all of the three

[1] Curzon takes d in the special form of the product, $-xyz$, so that the relations are throughout more specific. All of his work can be carried through with the more general symbol, d, and with perhaps a slight gain in brevity and symmetry but without essential increase in generality.

expressions $\left(x^2+\frac{d^2}{x^2}\right)$, $\left(y^2+\frac{d^2}{y^2}\right)$, $\left(z^2+\frac{d^2}{z^2}\right)$ may be the square of a rational number. Diophantus uses the hypothesis under which each of these is a square. Curzon mentions the example for which $x = 40$, $y = 24$, $z = 15$, $d = 1680$, so that $d/x = 42$, $d/y = 70$, $d/z = 112$. It is impossible that exactly two of these expressions should be squares. We shall not make use of the separate negations in these hypotheses but shall consider only the question of the product of the three being the square of a rational number, thus including all three possibilities.

Curzon's method of finding a solution may be interpreted as leading to the following conclusion. There exists a symmetric function $F(x, y, z)$ such that the product of the three expressions, $x^2+\frac{d^2}{x^2}+F$, $y^2+\frac{d^2}{y^2}+F$, $z^2+\frac{d^2}{z^2}+F$, is the square of a rational function, P, of x, y, z. A choice for this function, F, is found to be given by

$$rF(x, y, z) = d^2p + d(pq - r) + qr,$$

where p, q, r are the symmetric functions $x+y+z$, $xy+yz+zx$, xyz, respectively. This choice gives for P,

$$P = (x+y)(y+z)(z+x)\left(1+\frac{d}{xy}\right)\left(1+\frac{d}{yz}\right)\left(1+\frac{d}{zx}\right).$$

The solution follows as in (3) by so choosing x, y, z that F shall vanish, which suggests the methods of use in multiplicative domains. These remarks upon Curzon's method are given to show for the first time the common background for these two types of solution. The methods here used make use of this common start.

We have in any case

$$\left(x^2+\frac{d^2}{x^2}\right)\left(y^2+\frac{d^2}{y^2}\right) = c^2\left(z^2+\frac{d^2}{z^2}\right). \tag{4}$$

This equation we may write in the form

$$\left(xy+\frac{d^2}{xy}\right)^2 + d^2\left(\frac{x}{y}-\frac{y}{x}\right)^2 = c^2z^2 + d^2\frac{c^2}{z^2}. \tag{5}$$

From this stage on we shall consciously restrict[1] our investigations to such solutions, if any, as satisfy the relations, satisfied by (5) but not necessary to it, namely,

$$\begin{cases} xy+\dfrac{d^2}{xy} = cz, \\ \dfrac{x}{y}-\dfrac{y}{x} = \dfrac{c}{z}. \end{cases} \tag{6}$$

[1] This restriction excludes Curzon's elementary and very ingenious solution.

It may be noted that had the terms been paired in the other order, the only effect would have been to replace z by d/z, and so to yield the same solution for (1). It would have been equally convenient to replace (5) by

$$\left(xy - \frac{d^2}{xy}\right)^2 + d^2\left(\frac{x}{y} + \frac{y}{x}\right)^2 = c^2z^2 + d^2\frac{c^2}{z^2}, \tag{5'}$$

or in (6) to consider other combinations of plus and minus signs for the two right hand members. This is in fact sometimes necessary if we wish to obtain directly answers that are necessarily positive. In practice, however, it is just as well to confine attention to a definite algebraic choice of signs, and then at the end to change all the signs in case that negative numbers are obtained.

From (6) we obtain, by multiplication and removal of factors,

$$\left(1 + \frac{d^2}{x^2y^2}\right)\left(\frac{x^2}{y^2} - 1\right) = \frac{c^2}{y^2}. \tag{7}$$

The left hand member of (7) is then the square of a rational number. Two obvious cases leading to different solutions suggest themselves, from the factored form of the left hand member. Still other solutions might be obtained when the left hand member is expanded and the terms rearranged so as to suggest other devices. One of these will be carried through in (24). For the present we may discuss the alternatives,

(i): $1 + \dfrac{d^2}{x^2y^2}$ and $\dfrac{x^2}{y^2} - 1$ are individually squares of rational numbers,

(ii): $1 + \dfrac{d^2}{x^2y^2} = e\left[\dfrac{x}{y} - 1\right]$, where e is a rational number, and $e\left[\dfrac{x}{y} + 1\right]$ is the square of a rational number.

For (i) solutions of (7) are at once obtained in the form

$$\left\{\begin{array}{ll} \dfrac{x}{y} = \dfrac{s^2+1}{s^2-1}, & \dfrac{x^2}{y^2} - 1 = \left(\dfrac{2s}{s^2-1}\right)^2, \\ \dfrac{d}{xy} = \dfrac{2t}{t^2-1}, & 1 + \dfrac{d^2}{x^2y^2} = \left(\dfrac{t^2+1}{t^2-1}\right)^2. \end{array}\right. \tag{8}$$

Thence choosing a particular sign for the square root, we have

$$\frac{t^2+1}{t^2-1} \cdot \frac{2s}{s^2-1} = \frac{c}{y}. \tag{9}$$

From (8) and (9) by elimination, we have

$$\left\{\begin{array}{l} x = c\cdot\dfrac{s^2+1}{2s}\cdot\dfrac{t^2-1}{t^2+1}, \\ y = c\cdot\dfrac{s^2-1}{2s}\cdot\dfrac{t^2-1}{t^2+1}, \\ d = c^2\cdot\dfrac{s^2-1}{2s}\cdot\dfrac{s^2+1}{2s}\cdot\dfrac{2t}{t^2+1}\cdot\dfrac{t^2-1}{t^2+1}. \end{array}\right. \tag{10}$$

From these and by use of either one of the equations (6), we obtain the same expression for the value of z, namely,

$$z = c\cdot \frac{s^2-1}{2s}\cdot\frac{s^2+1}{2s}\cdot$$

By substitution in (1), we have finally as the solution under assumption (i), the following:

$$\left\{\begin{aligned} a &= c\left[\left(\frac{s^2+1}{2s}\right)^2-\left(\frac{t^2-1}{t^2+1}\right)^2\right] \\ &\qquad = c\,\frac{[(s-1)^2t^2+(s+1)^2][(s+1)^2t^2+(s-1)^2]}{(2s)^2(t^2+1)^2}, \\ b &= c\left[\left(\frac{s^2-1}{2s}\right)^2+\left(\frac{t^2-1}{t^2+1}\right)^2\right] \\ &\qquad = c\,\frac{[(st-1)^2+(s-t)^2][(st+1)^2+(s+t)^2]}{(2s)^2(t^2+1)^2}, \\ d &= c^2\left[\frac{s^2-1}{2s}\cdot\frac{s^2+1}{2s}\cdot\frac{2t}{t^2+1}\cdot\frac{t^2-1}{t^2+1}\right], \qquad (d\neq 0). \end{aligned}\right. \tag{11}$$

For case (ii) we have analogously

$$\left\{\begin{aligned} e\left(\frac{x}{y}-1\right) &= 1+\frac{d^2}{x^2y^2}, \\ e\left(\frac{x}{y}+1\right) &= s^2, \end{aligned}\right. \tag{12}$$

where s is to be an arbitrary rational parameter. Hence

$$s^2 = \frac{d^2}{x^2y^2}+1+2e.$$

We shall introduce a second arbitrary rational parameter, t, by the condition

$$t = d/(xy). \tag{13}$$

Thus $2e = s^2-t^2-1$, and from (12), $(s^2-t^2-1)\left(\frac{x}{y}+1\right) = 2s^2$ or

$$\frac{x}{y} = \frac{2s^2}{s^2-t^2-1}-1 = \frac{s^2+t^2+1}{s^2-t^2-1}\cdot \tag{14}$$

Relation (7) now becomes

$$\frac{(2s)^2(1+t^2)^2}{(s^2-t^2-1)^2} = \frac{c^2}{y^2},$$

whence, choosing a sign for y, we have

$$y = c\frac{s^2 - t^2 - 1}{2s(t^2 + 1)}, \tag{15}$$

From (14) we have therefore

$$x = c\frac{s^2 + t^2 + 1}{2s(t^2 + 1)}. \tag{16}$$

whence by (13) we have

$$d = c^2\frac{1}{(2s)^2}\cdot\frac{t}{(t^2 + 1)^2}\cdot(s^2 - t^2 - 1)(s^2 + t^2 + 1). \tag{17}$$

From either equation in (6) we obtain the same expression for z, namely,

$$z = c\frac{1}{(2s)^2(t^2 + 1)^2}(s^2 - t^2 - 1)(s^2 + t^2 + 1). \tag{18}$$

By substitution in (1), we obtain for the system of solutions in case (ii), the following:

$$\begin{cases} a = c\dfrac{[(s + 1)^2 + t^2][(s - 1)^2 + t^2]}{(2s)^2(t^2 + 1)}, \\ b = c\dfrac{[(s + t)^2 + 1][(s - t)^2 + 1]}{(2s)^2(t^2 + 1)}, \\ d = c^2\dfrac{t[(s^2 - t^2 - 1)(s^2 + t^2 + 1)]}{(2s)^2(t^2 + 1)^2}, \qquad (d \neq 0). \end{cases} \tag{19}$$

It may be noted that the solutions for (i) and (ii) are shown in closely analogous but noticeably distinct forms. However these two have a common subcase that we may denote by (iii). It will not be easily inferred from the final forms of the solutions but is readily discovered by retracing the steps in the determination of the solution. Thus we may replace, in (i), s by $[s(t^2 - 1)]/(t^2 + 1)$, while leaving t unaltered, or replace, in (ii), t by $(2t)/(t^2 - 1)$, while leaving s unaltered, and obtain the same solution in the two cases. The formulas are further simplified if s is then replaced by $s/(t^2 - 1)$. It is to be noted that this common solution continues to exhibit two independent rational parameters, although not containing all of the solutions of either of the two cases already given. This fact is verified by noting the increase in the degree to which these parameters are raised in the formulas

$$\begin{cases} a = c\dfrac{[(s + t^2 - 1)^2 + 4t^2][(s - t^2 + 1)^2 + 4t^2]}{(2s)^2(t^2 + 1)^2}, \\ b = c\dfrac{[(s + 2t)^2 + (t^2 - 1)^2][(s - 2t)^2 + (t^2 - 1)^2]}{(2s)^2(t^2 + 1)^2}, \\ d = c^2\dfrac{2t}{t^2 - 1}\cdot\dfrac{[s^2 - (t^2 + 1)^2][s^2 + (t^2 + 1)^2]}{(2s)^2(t^2 + 1)^2}, \qquad (d \neq 0). \end{cases} \tag{20}$$

Let us examine for a moment an explicit numerical solution for cases (i) and (ii). Since of necessity d does not vanish, we cannot give to s or t the values 0, 1, or -1 in (11). Putting, then, $s = t = 2$, we have from (11)

$$\begin{cases} a = c\dfrac{13\cdot 37}{16\cdot 25} = c\dfrac{481}{400}, \\ b = c\dfrac{9\cdot 41}{16\cdot 25} = c\dfrac{369}{400}, \\ d = c^2\dfrac{9}{20}, \end{cases}$$

whence

$$\begin{aligned} ac + 2d &= c^2\left(\frac{29}{20}\right)^2, & ac - 2d &= c^2\left(\frac{11}{30}\right)^2, \\ bc + 2d &= c^2\left(\frac{27}{20}\right)^2, & bc - 2d &= c^2\left(\frac{3}{20}\right)^2, \\ ab + 2d &= c^2\left(\frac{567}{400}\right)^2, & ab - 2d &= c^2\left(\frac{183}{400}\right)^2. \end{aligned}$$

If c is to be so determined that $2d = a + b + c$, in accordance with the condition of Diophantus, we shall have

$$c = \frac{125}{36}.$$

Similarly from (19),

$$a = c\frac{13\cdot 5}{16\cdot 5} = c\frac{13}{16}, \qquad b = c\frac{17\cdot 1}{16\cdot 5} = c\frac{17}{80}, \qquad d = c^2\frac{2 - (-1)\cdot 9}{16\cdot 25} = -c^2\frac{9}{200},$$

whence

$$\begin{aligned} ab + 2d &= c^2\left(\frac{23}{80}\right)^2, & ab - 2d &= c^2\left(\frac{41}{80}\right)^2, \\ bc + 2d &= c^2\left(\frac{7}{20}\right)^2, & bc - 2d &= c^2\left(\frac{11}{20}\right)^2, \\ ca + 2d &= c^2\left(\frac{17}{40}\right)^2, & ca - 2d &= c^2\left(\frac{19}{20}\right)^2, \end{aligned}$$

and if c is to be determined as above,

$$c = -45.$$

The sign of c may be here changed with a resulting change in the numerical representation of a, and b, and without altering the fundamental relations.

Such a change merely corresponds to a different determination in the signs of one or more of the radicals used.

Let us now start afresh from equation (7), writing it in the form

$$\frac{d^2}{y^2} + \left(x^2 - \frac{d^2}{x^2}\right) = c^2 + y^2. \tag{21}$$

For this we seek a solution of the form [1]

$$\frac{d}{y} = c, \qquad x^2 - \frac{d^2}{x^2} = y^2. \tag{22}$$

The second of these reduces to

$$(xy)^2 + d^2 = x^4. \tag{23}$$

This has the family of solutions,

$$xy = 2mn, \qquad d = n^2 - m^2, \qquad x^2 = n^2 + m^2,$$

where $m = n\dfrac{2s}{s^2-1}$, giving

$$x = n\frac{s^2+1}{s^2-1}, \qquad y = n\frac{4s}{s^2+1}, \qquad d = n^2\frac{s^4-6s^2+1}{(s^2-1)^2}.$$

Then from the first equation of (22)

$$n = c\frac{4s(s^2-1)^2}{(s^2+1)(s^4-6s^2+1)},$$

whence

$$\begin{cases} x = c\dfrac{4s(s^2-1)}{s^4-6s^2+1}, \\ y = c\dfrac{16s^2(s^2-1)^2}{(s^2+1)^2(s^4-6s^2+1)}, \\ d = c^2\dfrac{16s^2(s^2-1)^2}{(s^2+1)^2(s^4-6s^2+1)}. \end{cases}$$

Inserting in either equation of (6), we shall have

$$z = c\frac{(s^4-6s^2+1)^2}{4s(s^2-1)(s^2+1)^2}.$$

This gives finally by (1) a new one-parameter solution, which may be directly verified by expansion, namely,

$$\begin{cases} a = c\dfrac{[(s^2-1)^4+(2s)^4]^2+12(s^2-1)^4(2s)^4}{[(s^2-1)^4-(2s)^4]^2}, \\ b = c\dfrac{8(s^2-1)^2(2s)^2[(s^2-1)^4+(2s)^4]}{[(s^2-1)^4-(2s)^4]^2}, \\ d = c^2\dfrac{4(s^2-1)^2(2s)^2}{[(s^2-1)^4-(2s)^4]}. \end{cases} \tag{24}$$

[1] The other pairing fails to give a solution.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

S. Lie, *Gesammelte Abhandlungen* herausgegeben von Friedrich Engel und Poul Heegaard; Leipzig, B. G. Teubner; Kristiania, H. Aschehoug & Co., Volumes 3 and 5, 1922 and 1924, 16 + 789 and 12 + 776 pages. Subscription prices 25 and 35 Norwegian crowns.

Niels Henrik Abel (1802–1829) and Marius Sophus Lie (1842–1899) are two of the world's great mathematicians whom Norway has produced. Editions of Abel's collected papers appeared in 1839 and in 1881; indeed for the latter edition, Lie was one of the editors. And now, about twenty-five years after Lie's death, we have two volumes of the fine edition of his collected papers now being published by the Norwegian Mathematical Society with the assistance of grants from the Norwegian Research Fund of 1919, from the Society of Sciences of Christiania, and from the Academy of Sciences at Leipzig [compare *1921*, 318]. It is proposed that the collected papers shall be comprised within seven volumes of which the first two shall be devoted to geometrical works; the third and fourth to memoirs on differential equations; the fifth and sixth to memoirs on groups of transformations; and the seventh to posthumous works. Hence the volumes which have now appeared are the first on differential equations and the first on groups of transformations; for the editing of both, Engel is chiefly responsible.

These volumes are not, by any means, merely reprints of the papers as written; they have been edited with meticulous and satisfying detail. At any point of the new edition of a paper the page of the original source may be at once found; Lie's "Selbstanzeige" of each paper as it appeared in the *Jahrbuch über die Fortschritte der Mathematik* is printed along with the paper; 44 pages are taken to list the corrections made in the originals of the 42 papers (published 1872–83) in volume 3, and of the 24 (published 1874–89) in volume 5; 330 pages are occupied with critical remarks on the papers, and 32 with an account of the beginning of Lie's theory of transformation groups as contained in his correspondence with Adolph Mayer; 38 pages are filled with various indexes, errata lists, etc. The contents of some of these and of later papers will be comparatively familiar to mathematicians through Lie's treatises: *Theorie der Transformationsgruppen* edited by Engel in three large volumes (1888–93), *Vorlesungen über Differentialgleichungen mit bekannten infinitesimal Transformationen* edited by Scheffers (1891), *Vorlesungen über kontinuierliche Gruppen mit geometrischen und anderen Anwendungen* edited by Scheffers (1893), and *Geometrie der Berührungstransformationen* by S. Lie and G. Scheffers of which only the first volume appeared (1896). All of these (except the last two volumes of the work first mentioned) were reviewed in the *Bulletin* of the New York, or American, Mathematical Society.[1]

[1] C. H. Chapman in *Bulletin of the New York Mathematical Society*, vol. 2, pp. 61–71, 1892;

Two English treatises have widened the circle of those familiar with some of Lie's ideas. One of these is J. E. Campbell's *Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups* (Oxford, 1903); the other is America's contribution, namely, A. Cohen's *An Introduction to the Lie Theory of One-Parameter Groups* [1] (Boston, 1911). A brief general account of Lie's work, in English, was given in Klein's *Lectures on Mathematics* (New York, 1894, pp. 9–24) delivered in 1893 at the Evanston Colloquium. (Compare G. A. Miller in this MONTHLY, *1896*, 295–6.) Then E. O. Lovett's English translation of Darboux's article on Lie was published in *Bulletin of the American Mathematical Society* (vol. 5, pp. 367–370, 1899) and in this MONTHLY, *1899*, 99–101, where a portrait, and sketch by G. B. Halsted, are also published (*1899*, 97–98), as well as interesting reminiscences of one of Lie's pupils, G. A. Miller (*1899*, 191–193).

A biographical sketch and an elaborate bibliography (1869–98) by Engel appeared in *Bibliotheca Mathematica* (series 3, vol. 1, 1900, pp. 166–204); and for Max Noether's very valuable memoir one turns to *Mathematische Annalen* (vol. 53, pp. 1–41, 1900). Noether concludes: "Mit ihm schied der originalste und shöpferischste Vertreter der geometrischen Wissenschaft der letzten drei Decennien dieses Jahrhunderts dahin."

Engel's careful editorial work [2] was evidently a labor of love, and mathematicians must be grateful for considerable new light which he has already shed on some of Lie's ideas. Volume 6, the second on transformation groups, is now in the press.

It is to be hoped that the Norwegian Mathematical Society will be encouraged in its undertaking by receiving subscriptions for Lie's collected papers from many libraries and individuals in America. With the present low rate of Norwegian exchange the subscription prices to the volumes make them exceptionally cheap. The regular prices are about one third higher. Subscribers should write to H. Aschehoug & Co., Sehesteds plads 3, Christiania, Norway.

R. C. ARCHIBALD.

Methodik des mathematischen Unterrichts, 3. Teil. Didaktik der angewandten Mathematik. By DR. W. LIETZMANN. Leipzig, Quelle and Meyer, 1924. xii + 234 pages. Price 10 gold marks.

This volume forms part of the seventh volume of the monumental *Handbuch des naturwissenschaftlichen und mathematischen Unterrichts*, edited by Dr. J. Norrenberg, well known as one of the foremost men in educational reform in Prussia. The author, Dr. Lietzmann, is connected with the Reform-Realgymnasium in Göttingen, was the most prominent of the younger German workers in the International Commission on the Teaching of Mathematics, and is editor of one of the leading educational-mathematical periodicals of Europe. The

E. O. Lovett in *Bulletin of the American Mathematical Society*, vol. 4, pp. 155–167, 1898; J. M. Brooks in *B. A. M. S.*, vol. 1, pp. 241–248 (1895); E. O. Lovett in *B. A. M. S.*, vol. 3, pp. 321–350, 1897.

[1] Reviewed by E. J. Wilczynski in *Bulletin of the American Math. Soc.*, vol. 18, pp. 514–5, 1912.

[2] In this Professor Heegaard was doubtless an able assistant.

book is dedicated to Professor Klein and pays just tribute to him for his influence in the reform of mathematical teaching, not only in his native land, but in the world at large; and in the development of the subject this influence appears throughout the book.

The work is a continuation of the one already reviewed in the MONTHLY (*1914*, 252), but this portion is unique in that it approaches the subject of applied mathematics from the pedagogical standpoint, giving the educational problems the serious consideration which has thus far been so rare. Up to the present time the subject has generally been approached from the standpoint of a collector of problems or of a worker in some special field of industry. The book therefore forms a worthy addition to the educational literature of the day—the more worthy because it is written by one who knows the material as well as the pedagogical side of mathematics.

The work is divided into seven chapters, the topics considered being drawing, the question of instruction in industries, geodesy and astronomy, civic economics, mechanics, physics, and philosophy. It contains numerous illustrations, in half-tone and line engraving, showing the models and apparatus used in the best German schools when presenting such topics as central perspective, solid geometry, hydrostatics, geodesy, and astronomy. Dr. Lietzmann has further made his readers his debtors by his indexes—(1) of apparatus, (2) of names and (3) of subjects—and by his convenient although too brief table of biographical notes concerning various contributors to the field of mathematical teaching. The work is made the more valuable by a large number of bibliographical notes, referring largely to the recent literature of the subject. These references are almost entirely to German sources, but the covering of the international field could hardly be expected; indeed, a reader goes to a German book for the German point of view, and it is not an unmixed evil that this point of view is emphasized.

As to the text itself, it will be an inspiration to our American readers. Our high schools confessedly lack the scholarship of those found in Europe, our educational problem being different from that of the older countries—more democratic and less given to an aristocracy of learning. It is a hopeful sign, however, that we are beginning a scientific differentiation of our students, and this will eventually lead to a much higher grade of scholarship in classes composed of the intellectually capable. With this will come the need for such direction as Dr. Lietzmann's work furnishes, and the book will have a wholesome influence upon the present and the coming generation of teachers in this country.

DAVID EUGENE SMITH.

L'analysis situs et la géométrie algébrique. (Collection de monographies sur la théorie des functions publiée sous la direction de M. Emile Borel.) By S. LEFSCHETZ. Paris, Gauthier-Villars and Co., 1924. 154 pages. Price 20 francs.

In line with his numerous articles in various journals, and in particular with his recent prize memoir, Professor Lefschetz here presents a brief but scholarly treatise, fully worthy of the excellent company found in the Borel series.

the genera of the two curves. These results had been previously found by different methods.

The fifth chapter, algebraic varieties of more than two dimensions, contains a resumé of many of the ideas developed in the prize memoir. This is so recent and so generally accessible that it will not be necessary to consider the chapter further, except that two pretty proofs will be noticed, first the generalization of the Zeuthen-Segre invariant, first established by Alexander, and second, the proof that a complete base of the general surface of order greater than 3 is a plane section (Noether).

The last chapter, analysis situs and abelian functions, begins with the existence theorem, then discusses periodic functions and those with multipliers, like the theta functions, and closes with the determination of the number ρ (Picard-Severi) for abelian varieties.

Then follow twenty-five pages as an appendix, devoted to a discussion of double integrals of the second kind and of simple integrals of the third kind on algebraic surfaces. The idea was not needed in the earlier discussion, but the methods there employed are here extended to the other types for the sake of completeness. The most important contribution is the investigation of the improper double integral of the second kind. A second note gives a description of the Volterra models, with illustrations.

The work is richly provided with references; the treatment is everywhere concise, and considerable maturity is presupposed on the part of the reader, who is supposed to be familiar with the Picard transcendental theory. At times it is also necessary to consult the memoirs of Severi, and to refer to several papers on analysis situs. What the reader does gain is an impression of a curious unity in the new presentation and that no one earlier method is capable of developing this beautiful theory alone.

The book is almost free of typographical errors; the few not found in the short list of errata will cause no confusion to the reader.

VIRGIL SNYDER.

NOTES ON RECENT PUBLICATIONS.

The Biometric Laboratory, University College, London, England, is issuing a twenty-place table of logarithms called *Logarithmetica Britannica*. The computation is being done by Mr. A. J. Thompson and the printing by the Cambridge University Press. The work will appear in nine parts at ten shillings a part and will form, when finished, a quarto volume about two-and-a-half inches thick. Part IX, 90,000 to 100,000, is out now and the remaining parts should appear within the next four years. Professor Karl Pearson says in the preface:

> The success of the undertaking—which it is needless to say is not an enterprise of profit—can only be assured if a large section of the mathematical world shows a readiness to commemorate with us in this fashion the tercentenary of Briggs' *Arithmetica Logarithmica*.

A historical note on Briggs by Professor Pearson will appear in an early number of the MONTHLY.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

EDUCATION, volume 45, no. 2, October, 1924: "Mathematics or hieroglyphics" by A. W. Forbes, 105–106.

JOURNAL OF THE FRANKLIN INSTITUTE, volume 198, no. 5, November, 1924: "Heaviside's operators in engineering and physics" by E. J. Berg, 647–702.

MONIST, volume 34, no. 3, July, 1924: "Number: An introduction to the theory of analytic functions" by G. Mittag-Leffler, 321–357.

SCIENTIFIC MONTHLY, volume 29, no. 5, November, 1924: "Five fundamental concepts of pure mathematics" by G. A. Miller, 496–501.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 7, October, 1924: "Mathematics in the Senior High School" by F. Allen, 716–726.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Md. [*1922*, 178.]

There were two lectures by outside speakers during 1923–24. Dr. H. H. Lloyd of the Goucher chemistry department lectured on the use of mathematics in chemistry. Dr. Harlow Shapley, Director of the Harvard College Observatory, lectured on the "Evolution of the stars." Dr. Shapley's lecture was open to the whole college. A third faculty talk was given by Dr. Clara L. Bacon of our mathematics department on "The mathematical fourth dimension."

The club meets on an average of every two weeks, and the program regularly consists of papers by students. Two of these this year were on "Magic squares" and the "Theory and use of the planimeter."

(Report by Professor F. P. Lewis.)

THE MATHEMATICS CLUB OF CONNECTICUT COLLEGE, New London, Conn. [*1922*, 24.]

All meetings are open for discussion of topics closely or more remotely connected with mathematics. The paper of the evening furnishes a principal topic. All members are urged to bring in all items of mathematical interest, whether they appear as news items, changes in curricula or entrance requirements, book reviews, jokes, puzzles, or campus talk, and this feature is frequently the backbone of the evening program. Refreshments are served at all meetings.

October 10, 1921. Organization meeting. Election of officers was completed. Officers for the year were: President, Dorothy S. Wheeler '22; Secretary, Marcia Langley '23; Treasurer, Ellen Willcox '24; Chairman of Program Committee, Ruth Wells '24.

November 22, 1921. "The discovery of the decimal system by Stevin" by Marcia Langley '23; "The geometric solution of trigonometric equations" by Dr. David D. Leib.

January 16, 1922. Open meeting. "Some interesting bits of history" by Dr. Leib.

February 20, 1922. Washington's birthday party. Mathematical and other recreations in charge of Ruth Wells '23.

March 21, 1922. "Mathematics in Asia" by Elizabeth Hall '23.

April 19, 1922. "Beginnings of mathematics in Arabia" by Olive Holcombe '23; "Summary of and extracts from 'Religio Mathematici' by D. E. Smith" by Augusta O'Sullivan '23.

May 23, 1922. Annual picnic in Bolleswood. Election of President and Secretary for 1922–1923.

October 4, 1922. Organization meeting. Election of officers was completed. Officers for the year were: President, Marcia Langley '23; Secretary, Olive Holcombe '23; Treasurer, Helena R. Wulf '23; Chairman of Program Committee, Marie Jester '24.

Dr. Leib told of the work and purpose of the Club. Miss Ruth E. Irwin told of the Smith Club.

November 13, 1922. Keyser's *Mathematical Philosophy*. A review by Florence A. Hopkins. All members present then took the Thorndike algebra test.

December 11, 1922. Christmas party at the home of Dr. and Mrs. Leib.

January 8, 1923. "Magic squares" by Ellen Willcox '24; "Theory of numbers" by Florence A. Hopkins '23.

February 13, 1923. Valentine party at the home of Dr. and Mrs. Leib. "Card tricks" by Dorothy McFarland '25; "Some well-known problems" by Helena Wulf '23.

March 12, 1923. "The development of arithmetic" by Sadie Kenig; "Mapmaking—merits and limitations of various types" by Dr. Leib.

April 16, 1923. Annual open meeting. "The fourth dimension—mathematical and philosophical points of view" by Dr. Leib.

May 29, 1923. Annual picnic and election of President for 1923–24.

October 22, 1923. Organization meeting. Election of officers was completed. Officers for the year were: President, Louise Hall '24; Secretary, Marie Jester '24; Treasurer, Verna A. Kelsey '25; Chairman of the Program Committee, Mary Courtney '24.

November 12, 1923. "Mathematical terms in daily life" by Grace D. Byron '24; "Mathematical poems" by Doris E. Miner '24.

December 10, 1923. Christmas party.

January 17, 1923. "The making of magic squares and other arrays" by Jessie Josolowitz '25.

February 11, 1924. "American mathematics and mathematicians" by Elsie Marquardt '24.

March 10, 1924. "Some fallacies in the study of probabilities" by Verna Kelsey '25; "Some troublesome ideas" by Dr. Leib.

April 14, 1924. Open meeting. "Egyptian mathematics" by Professor R. C. Archibald, Brown University.

May 22, 1924. Business meeting. Election of President and Secretary for 1924–25.

May, 1924. Annual picnic. Due to the weather it was held at the home of Dr. Leib.

October 18, 1924. Organization meeting. Officers for the year are: President, Aileen Fowler '25; Secretary, Eleanor Kelly '25; Treasurer, Jessie Josolowitz '25; Chairman of Program Committee, Clarissa Lord '26.

Dr. Leib spoke of the history and purpose of the club and re-read the paper given at the first meeting of the C. C. Mathematics Club on "Dimensionality." Clarissa Lord '26 proposed a number of puzzles, problems and recreations to test the skill and acumen of the club members after the summer vacation.

(Report by Professor Leib.)

Grinnell College Mathematics Club, Grinnell, Iowa.
[*1922*, 416.]

November 20, 1923: "Arithmetic," Mr. R. B. McClennon. (In two divisions—historical and problems.) "Practical applications of arithmetic," Mr. Gilbert Pullen.

December 18, 1923: "Geometry," Mr. Dana Corrough. "Zeno's paradoxes," Mr. Edwin Kingery.

January 22, 1924: "Mathematical recreations," Miss Alice Clifton. (Problems of arithmetic, algebra, geometry and chess.) "Paper folding," Miss Marian Piersol.

March 4, 1924: "A problem in projective geometry," Mr. Loren Gray. "Reliability of teacher's marks," Miss Ethel Warnick.

March 18, 1924: "The spring constellations," Miss Margaret Field. "The hyperboloid of one sheet" Mr. Sibert Dave.

April 15, 1924: Problems of arithmetic, algebra, and geometry submitted by members for solution.

The average attendance at the meetings for the year was eleven.

Mathematics Club of the University of Oklahoma, Norman, Okla. [*1923*, 335.]

The Mathematics Club of the University of Oklahoma was organized on October 26, 1916, by Dr. W. W. Reaves, head of the mathematics department, and a group of students interested in mathematics. The purpose of the organization was to promote interest and stimulate activity in the study of mathematics by offering an opportunity for the presentation and discussion of mathematical subjects of interest and has functioned as such for eight years.

The total membership for the year 1923–24 was thirty-two. The officers were as follows: President, Varnakale Jones '25, Vice-president, Raymond Dragoo '24, Secretary-Treasurer, Mabel Erwin '24.

During the year the following papers were read:
"Applications of mathematics to astronomy" by Dr. J. O. Hassler.
"Magic squares" by Miss Jewel Conkling.
"Magic cubes" by Miss Sara Beth Barbour.
"Flatland and the fourth dimension" by Miss Margaret Rayburn.
"History and use of the abacus" by Miss Ester Billows.
"Theory and applications of the slide rule," by Varnakale Jones.
"How mathematical theory verifies astronomical data" by Dr. J. O. Hassler.
"Different proofs of the Pythagorian proposition" by Miss Mabel Erwin.
"Mathematical fallacies" by Mr. Carl Wilson.
"The great problem of Apollonius" by Dr. N. A. Court.

(Report by Professor Court.)

Mathematical Club of the University of Arkansas, Fayetteville, Ark.

Topics for 1923–24.

"Magic squares," Elmer Nichols.
"What is mathematics?" Assistant Prof. A. D. Campbell.
"The sun and the moon," Dr. A. M. Harding.
"The solution of simultaneous quadratics," Sam Byrd.
"The nine-point circle," Dean G. W. Droke.
"Cycloid and hypocycloid," Marvin Leeper.
"Hyperbolic functions," W. H. Taylor.
"Mathematical wrinkles," Miss Myrtle Farmer.
"The solution of third and fourth degree equations," Bolling Dunn.
"Exponential functions," Miss Emily Heston.
"Application of the binomial theorem to physics," Marshall Hickman.

(Report by Asst. Professor Campbell.)

The Mathematics Club of the University of Maine, Orono, Maine. [*1923*, 335.]

The following officers were elected for the year 1923–24. President: Vera Savage '24, Vice-president: Ethelyn Percival '24, Secretary-treasurer: Donald Trouant '24, Faculty adviser: Professor Bryan.

The meetings were held as follows:
September 26: "Talk on text books" by Professor N. R. Bryan.
October 10: "Books added to the library" by Ethelyn Percival '24.
November 21: "Who's who in mathematics" by Edwin Hadlock '24 and H. L. Bowen '24.
December 12: "Eclipses of the sun" by Dean J. N. Hart.
January 9: "The meaning of a group in mathematics" by Mr. W. S. Lucas.
February 13: "Periodicals devoted to mathematical subjects" by Mary Copeland '24, Madeline McPhelies '25, and Doris Shorey '26.
February 27: "The mathematical theory of the game of Nim" by Mr. E. C. Brown.

April 23: "The correlation of high school and college work" by Prof. W. S. Evans.
May 14: "The prediction of eclipses" by Mr. W. S. Lucas.

The following were selected for officers for the coming year: President, Donald Trouant '25; Vice-president, Madeline McPhelies '25; Secretary-treasurer, Elizabeth Laughlin '26.

(Report by Elizabeth H. Laughlin.)

The Mathematics Club of The College of the Ozarks, Clarksville, Arkansas.

The Mathematics Club of The College of the Ozarks was organized in October 1921 for the purpose of arousing interest and stimulating activity in the field of mathematics. The present officers of the Club are: president, John Neal '25; vice-pres., Opal Johnson '26; secretaries, Roy Burgess '24, and Annie Moon '26; treasurer, Viola Stegall '27; chaplain, James Boudra '27. Meetings are held monthly in the College Science Hall. Most of the members of the Club are majoring in mathematics and consequently the discussions and reports are mainly mathematical.

Some of our recent discussions, reports and papers were:
"Mathematical philosophy," Roy Burgess '24.
"Introduction to Einstein's theory," John Neal '25.
"History of geometry," Era King '27.
"Pedagogy of mathematics," Annie Moon '26.
"Great contributions to mathematics," Opal Johnson '26.
"Mathematics and religion," James Boudra '27.
"Count Korzybski's concept of man," Ralph Pitts '26.
"Geometry and faith," Viola Stegall '27.

(Report by Professor Chester R. Hillard.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3113. Proposed by A. S. WIENER, Cornell University.

A man is paying off a mortgage of N dollars, interest i per cent. annually, by monthly installments of a dollars (where $a > iN/1200$, the first month's interest). How many months will it take him to pay off the mortgage?

3114. Proposed by B. R. HEADSTROM, Boston, Mass.

What is the per cent. error in the following supposed method of trisecting an angle? Given the angle ABC with its vertex at B, lay off on its sides the equal lengths BD and BE and draw the line DE. With this latter length as diameter draw a semi-circle on the side opposite to B; also draw arcs of circles with D and E as centers and radii equal to that of the semi-circle cutting the latter in M and N, respectively. Then the lines BM and BN trisect the given angle.

3115. Proposed by H. W. REDDICK, Cooper Union Institute of Technology.

Given a plane triangle and a system of coördinate axes in a 3-space, find the number of ways of placing the triangle so that its vertices lie on the axes.

3116. Proposed by A. A. BENNETT, University of Texas.

Show that if a number is k times the number obtained by reversing the order of its digits, and neither has a zero for leading digit, then $k = 1, 4$, or 9. Show by example that 4 and 9 are actually possible.

3117. Proposed by W. J. SIDIS, New York City.

It is well known that, except for a possible factor 3, all prime factors of $m^2 \pm mn + n^2$ must be of the form $6k + 1$, if m and n are prime to each other. Prove that, if n is an odd prime, all prime factors of

$$x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1},$$

and

$$x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1},$$

excluding a possible factor n, must be of the form $2kn + 1$, provided x and y are prime to each other.

3118. Proposed by HARRY LANGMAN, New York City.

If $n > 2$ and ϵ is a primitive root of $\epsilon^n = 1$, show that

$$\begin{vmatrix} \epsilon & \epsilon^2 & \epsilon^3 & \cdots & \epsilon^{n-1} \\ \epsilon^2 & \epsilon^4 & \epsilon^6 & \cdots & \epsilon^{2(n-1)} \\ \epsilon^3 & \epsilon^6 & \epsilon^9 & \cdots & \epsilon^{3(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \epsilon^{n-1} & \epsilon^{2(n-1)} & \epsilon^{3(n-1)} & \cdots & \epsilon^{(n-1)(n-1)} \end{vmatrix} = (-1)^{(n-1)(n-2)/4} n^{(n-2)/2}.$$

3119. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Through a given point to draw a line meeting the sides BC, CA, AB of a given triangle ABC in the points P, Q, R such that $PQ = kQR$, where k is a given constant.

SOLUTIONS.

2957 [1922, 81]. Proposed by J. L. WALSH, Harvard University.

The envelope of the circles of curvature of a curve is, in part at least, the curve itself. What further curves, if any, are parts of this envelope?

SOLUTION BY THE PROPOSER.

Let us use rectangular coördinates (α, β) for points on the curve which we take as $\beta = f(\alpha)$, (1). The equation giving the circles of curvature is

$$\{x - \alpha + (f' + f'^3)/f''\}^2 + \{y - \beta - (1 - f'^2)/f''\}^2 = (1 + f'^2)^3/f''^2. \tag{2}$$

We differentiate (2) partially with respect to α taking account of (1), thereby obtaining the equation $f'(x - \alpha) - (y - \beta) = 0$, (3) after we have suppressed the factor $3f'f''^2 - f''' - f'^2 f'''$. This factor corresponds to the differential equation satisfied by a circle. If (1) represents a circle, (2) represents the same circle, so no extraneous curves can be introduced.

It remains to eliminate α and β from (1), (2), and (3). Substitution of the value of $(y - \beta)$ from (3) in (2) gives an equation which reduces to $(x - \alpha)^2(1 + f'^2) = 0$. If we are dealing with a real curve, the second factor is never zero, so

$$x = \alpha, \qquad y = \beta, \qquad y = f(x).$$

Hence *the entire envelope of the circles of curvature of a curve is precisely the curve itself.*

2972 [1922, 224]. Proposed by J. H. M. WEDDERBURN, Princeton University.

In *Mathematical Education*, p. 361, B. Branford gives the following approximate construction for the trisection of an angle ABC. Take any point D on AB produced and make $BC = BD$; through C draw a parallel to BA and on it lay off a length $CP = CD$; then ABP is approximately $\frac{1}{3}ABC$. Show that this construction is rendered very much more accurate by laying off on CP a distance $CQ = 2BD$ and taking in place of P the point R which divides PQ in the ratio $5 : 4$. Also show that the error does not exceed $10''$ for angles less than $45°$.

SOLUTION BY OTTO DUNKEL, Washington University.

Let the given angle ABC be denoted by θ and take $BD = BC = 1$. Then $DC = CP = 2\cos\theta/2$. In the first construction the approximation is too large and we shall set angle $ABP = x = \theta/3 + h$, where h is the error. The projection of BCP on BA is $\cos\theta + 2\cos\theta/2$, and hence

$$\tan x = \frac{\sin\theta}{\cos\theta + 2\cos\frac{\theta}{2}}\cdot \tag{1}$$

Also from $h = x - \theta/3$, we have

$$\tan h = \frac{\cos\frac{\theta}{3}\tan x - \sin\frac{\theta}{3}}{\cos\frac{\theta}{3} + \tan x\sin\frac{\theta}{3}}\cdot \tag{2}$$

Inserting in (2) the value of $\tan x$ above and after making certain simple trigonometric reductions we find

$$\begin{aligned}\tan h &= \frac{\sin\frac{2\theta}{3} - 2\sin\frac{\theta}{3}\cos\frac{\theta}{2}}{\cos\frac{2\theta}{3} + 2\cos\frac{\theta}{3}\cos\frac{\theta}{2}},\\ &= \frac{4\sin\frac{\theta}{12}\sin\frac{\theta}{3}\sin\frac{5\theta}{12}}{4\cos\frac{\theta}{12}\cos\frac{\theta}{3}\cos\frac{5\theta}{12} - 1}\cdot\end{aligned} \tag{3}$$

From the first expression above it is clear that the denominator does not vanish for θ less than or equal to 135° and from either we see that it decreases as θ increases up to this value. Also from the second it is seen that the numerator increases with θ from $\theta = 0$ to $\theta = 216°$. Hence the error h is positive and it increases with θ. It will be found that the denominator is not zero until $\theta = 156° \, 35.72'$, hence h increases from 0 to 90° as θ increases from 0 to $156° \, 35.72'$.

It should be noticed that it is only necessary to consider angles θ less than or equal to 45°. For if θ is greater than 90°, we trisect the supplementary angle θ' and subtract $\theta'/3$ from 60°. If θ is greater than 45°, we trisect the complementary angle θ'' and subtract $\theta''/3$ from 30°.

For $\theta = 45°$ we find from (1) or (3) $h = 28' \, 13.5''$, and the above discussion shows that as θ decreases the error decreases from this amount. If θ is very small and all the angles are in radians, then approximately

$$h = \frac{5\theta^3}{324}\cdot \tag{4}$$

Since $h < \tan h$, it is easy to obtain upper limits for h when the upper limit of θ is given. Thus if $\theta \leqq \pi/4$,

$$h < \frac{\theta^3}{57}\cdot \tag{5}$$

This results from replacing the sine by the angle and computing the rest of formula (3) for $\theta = \pi/4$.

The Second Construction. Here we have $CR = (4CP + 5CQ)/9 = (8\cos\theta/2 + 10)/9$. Setting angle $ABR = x$, we shall here set $h = \theta/3 - x$, since it will appear later that x is too small. Just as before we find

$$\tan x = \frac{9\sin\theta}{9\cos\theta + 8\cos\frac{\theta}{2} + 10}, \quad \tan h = \frac{\sin\frac{\theta}{3} - \cos\frac{\theta}{3}\tan x}{\cos\frac{\theta}{3} + \sin\frac{\theta}{3}\tan x}\cdot \tag{6}$$

Eliminating $\tan x$ from these two equations and making certain trigonometric transformations, we find

$$\tan h = \frac{16\sin\frac{\theta}{3}\sin^4\frac{\theta}{12}\left(7 + 8\cos\frac{\theta}{6}\right)}{2\cos\frac{\theta}{3}\left(9\cos\frac{\theta}{3} + 4\cos\frac{\theta}{2} + 5\right) - 9}\cdot \tag{7}$$

Here the denominator is positive for $\theta \leqq 180°$ and hence h is positive. In order to examine the right-hand side we separate it into two factors one of which may be written

$$\frac{8 + 7\sec\dfrac{\theta}{6}}{2\cos\dfrac{\theta}{3}\sec\dfrac{\theta}{6}\left(9\cos\dfrac{\theta}{3} + 4\cos\dfrac{\theta}{2} + 5\right) - 9\sec\dfrac{\theta}{6}}. \tag{8}$$

The numerator of (8) increases with θ; the part of the denominator in parentheses decreases as θ increases; $-9\sec\theta/6$ decreases as θ increases; and finally $\cos\theta/3\sec\theta/6 = 2\cos\theta/6 - \sec\theta/6$ also decreases as θ increases. Hence (8) increases as θ increases up to 180°, and, since the other factor of (7), *i.e.*, $16\sin\theta/3\sin^4\theta/12$ increases as θ increases, we see that h increases from 0 as θ increases from 0 to 180°.

The first value of θ for which the denominator is zero is $\theta = 182° 52' 6''$ and for this value $h = 90°$.

When $\theta = 45°$, (6) or (7) gives $h = 9.6''$. If then the angle θ is less than or equal to 45°, the error is less than or equal to this amount. If θ and h are in radians, then for small angles we have approximately

$$h = \frac{10\theta^5}{9\cdot 6^5}. \tag{9}$$

Just as before by replacing the sine by the angle and computing the remaining parts of (7) for $\theta = 45°$, we find

$$h < \frac{\theta^5}{6403}, \qquad \theta \leqq \frac{\pi}{4}.$$

It is obvious from the above that the second approximation is very much better than the first and that it gives a very high degree of accuracy.

3070 [1924, 206]. Proposed by J. L. RILEY, Tarleton Station, Texas.

A triangle circumscribes the circle $x^2 + y^2 = r^2$ and two of its vertices lie on the circle $(x - a)_2 + y^2 = R^2$; find the condition necessary that the third vertex may also lie on this circle.

SOLUTION BY THEODORE BENNETT, University of Illinois.

Call the first circle C_1 and the second C_2; then C_1 may be written

$$x = r\frac{1 - t^2}{1 + t^2}, \qquad y = r\frac{2t}{1 + t^2}. \tag{1}$$

If we draw tangents to C_1 at the points whose parameters are t and T, it is easily verified that they intersect at

$$x = r\frac{1 - tT}{1 + tT}, \qquad y = r\frac{t + T}{1 + tT}. \tag{2}$$

By substituting these values of x and y in the equation of C_2 and rearranging, we obtain the condition that this point lie on C_2 in the form

$$T^2(\alpha t^2 + r^2) + 2tT(a^2 - R^2) + (r^2t^2 - \beta) = 0, \tag{3}$$

where $\alpha = (r + a)^2 - R^2$, $\beta = (r - a)^2 - R^2$.

Now consider t fixed; this fixes a tangent to C_1 which cuts C_2 in A_1 and A_2. Through A_1 there is a second line l_1 tangent to C_1, and similarly through A_2 a tangent l_2; the points where l_1 and l_2 touch C_1 are given by (1) for $t = t_1$ and $t = t_2$, where t_1 and t_2 are obtained by solving (3) for T. The intersection of l_1 and l_2 can be written down at once from (2) by substituting the values of t_1t_2 and $t_1 + t_2$, which can be read off from (3); the result is

$$x = r\frac{t^2(\alpha - r^2) - (\beta - r^2)}{t^2(\alpha + r^2) + (\beta + r^2)}, \qquad y = \frac{-2rt(a^2 - R^2)}{t^2(\alpha + r^2) + (\beta + r^2)}. \tag{4}$$

The problem requires the condition that this point lie on C_2.

Substituting these values of x and y in the equation of C_2, we obtain a quartic equation in t, which can be factored in the form

$$(t^2 + 1)(\alpha t^2 + \beta)(a^4 + R^4 - 2a^2R^2 - 4r^2R^2) = 0. \tag{5}$$

The first two factors equated to zero give the four intersections of the two circles (the first giving the circular points at infinity), as may be verified by substituting $t = \sqrt{-\beta/\alpha}$ in (1). At these points the conditions of the problem are obviously satisfied (by improper triangles). To have a triangle none of whose vertices are points of intersection of C_1 and C_2, the third factor of (5) must vanish. Since this puts a condition on C_1 and C_2 but not on t, we see that if there is one such triangle, then there is an infinite number of them, any point of C_2 being a vertex of some one triangle. Setting the third factor of (5) equal to zero and rearranging, we have

$$a^2 + r^2 = (r \pm R)^2.$$

This shows that if the y-axis cuts C_1 at P, then the distance from P to the center of C_2 must be either the sum or difference of the radii of C_1 and C_2. This gives a necessary and sufficient geometrical condition for the existence of such triangles in any two given circles.

NOTE BY THE EDITORS: The distance a between the centers of the circumscribed and inscribed circles of a triangle $a^2 = R^2 - 2rR$ is derived on page 79 of Chauvenet's *Trigonometry*, 9th ed. It is also shown there that the distance between the centers of the circumscribed and an escribed circle is $a^2 = R^2 + 2r'R$. See also Salmon's *Conic Sections*, page 343, 6th ed. These are therefore necessary conditions. If either one of these conditions is satisfied, it may be easily shown by elementary geometry that there are two isosceles triangles which are inscribed in R and which circumscribe r. Let ABC be one of these triangles and from any point A' of the circle R draw two tangents to the circle r cutting R in B', C', the points A', B', C' being taken in the same order of rotation as A, B, C. It will be shown that $B'C'$ must also be tangent to r. Let AB, AC cut $B'C'$ in M', N', respectively; $A'B'$, $A'C'$ cut BC in M, N, respectively. From A as a center project B, C, B', C' on $B'C'$; from A' as a center project the same points on BC. Then the ranges (M', N', B', C') and (B, C, M, N) are projective, and therefore (N, M, C, B) and (M', N', B', C') are also projective. From this it follows that the three points (NN', MM'), $(MB', N'C)$, $(NC', M'B)$ lie on a straight line. If we call the last two points P and P', then the hexagon $M'P'NMPN'$ has the first five sides tangent to r, and by Brianchon's Theorem the remaining side $N'M'$ (*i.e.*, $B'C'$) must also be tangent. By a slight rearrangement of this proof the two equalities above may be deduced as necessary conditions.

Also solved by MAURICE BAUDIN and J. H. WEAVER.

3074 [1924, 206]. Proposed by J. H. MURPHY, Pittsburgh, Pa.

Two circles of radii R_1, R_2 intersect. A third circle passes through their points of intersection. Find the radius of this third circle, when the sum of the areas of the two crescents cut from it by the other two circles is a maximum.

SOLUTION BY F. HENROTEAU, Dominion Observatory, Ottawa, Canada.

Let C_1, C_2, C_3 be the three circles, A, B, C, D the four points where C_1 and C_2 cut their common diameter and E and F their points of intersection. The third circle C_3 may be defined by E, F and a third point K anywhere on AD.

Suppose that K moves on AD from $-\infty$ to $+\infty$. We see easily that when K moves from $-\infty$ to A the sum of the areas of the two crescents is increasing.

When K is between A and D, the sum of the areas of the two crescents is equal to the area of C_3 plus the area of the crescent common to C_1 and C_2 which is a constant. From A to the intersection of AD with EF the area of C_3 decreases, from this last point to D it increases again—so it is the same for the sum of the areas of the two crescents. When K moves from D to $+\infty$, the sum of the areas is decreasing. There are thus two maxima and a minimum when $R = R_1$, $R = R_2$ and $R = EF/2$ respectively, for the sum of the areas of the two crescents.

Also solved by C. K. ROBBINS.

3079 [1924, 254]. Proposed by J. J. QUINN, Pittsburgh, Pa.

The line OP is equal to and coincident with the straight line segment AB. As O describes AB uniformly, OP rotates uniformly through 180°. Determine (a) the locus of P; (b) the length of the curve for a complete revolution; (c) the area of a loop.

SOLUTION BY P. J. DA CUNHA, Lisbon University.

Take as the x-axis the line AB and for the y-axis the perpendicular to it at A. Let $AB = a$ and set angle $BOP = t$. From the definition of the locus in the problem we have $OA/a = t/\pi$, and hence it results that

$$x = \frac{at}{\pi} + a \cos t, \qquad y = a \sin t,$$

are the parametric equations of that locus. By a change of axes it will be seen that the curve is the epitrochoid inverted. The location of the maximum breadth of the loop on AB is found by setting

$$\frac{dx}{dt} = a\left(\frac{1}{\pi} - \sin t\right) = 0,$$

which gives $t = 18° \ 33' \ 39''$ or $161° \ 26' \ 21''$. The loop is symmetric with respect to the perpendicular bisecting AB.

(b) For the length of the part of the loop above AB we have

$$s = \frac{2a\sqrt{1+\pi^2}}{\pi}\int_0^{\pi/2} \sqrt{1 - k \sin t}\, dt, \qquad k = \frac{2\pi}{1+\pi^2}.$$

Developing $\sqrt{1 - k \sin t}$ by the binomial theorem and integrating, we obtain

$$s = \frac{a\sqrt{1+\pi^2}}{\pi}\left[\pi - k - \left(\frac{1}{2}\right)^2\frac{\pi k^2}{4} - \frac{2k^3}{4\cdot 6} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{5\pi k^4}{6\cdot 8} - \cdots \right.$$
$$- \left(\frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\right)^2\frac{(2m+1)(2m+3)\cdots(4m-3)}{(2m+2)(2m+4)\cdots 4m}\pi k^{2m}$$
$$\left. - 2\frac{(2m+3)(2m+5)\cdots(4m-1)}{(2m+2)(2m+4)\cdots(4m+2)}k^{2m+1} - \cdots\right].$$

A calculation of the first ten terms gives the approximate value

$$s = 2.474a.$$

(c) The area of the part of the loop above AB is given by

$$A = 2a^2\int_0^{\pi/2}\left(\frac{t}{\pi} - \frac{1}{2} + \cos t\right)\cos t dt,$$
$$= .934a^2 \text{ (approximately)}.$$

To find the area of the entire loop the double point below AB must be found. This is given by $t = -42° \ 34.03'$. The area is then found to be $1.3467a^2$.

3080 [1924, 255]. Proposed by C. N. SCHMALL, New York City.

Given a cyclic quadrilateral $ABCD$ such that its diagonals AC, BD intersect at right angles in P. If O be the center of the circle, and perpendiculars be dropped on the sides of the quadrilateral from O and P, show that the eight feet of these perpendiculars are concyclic.

NOTE: The circle on which these points lie is somewhat analogous to the *nine-point* circle of a triangle. Its center is the middle point of OP.

SOLUTION BY C. H. CHEPMELL, Hove, Sussex, England.

Bisect OP in M. Let OS and PT be the perpendiculars on AB; let R be the radius of the circle $ABCD$. Then $MS = MT$. Draw OD and SP. Then, because APD is a right angle and S is the mid-point of AD, $SP = SD$; and $OS^2 + SP^2 = OS^2 + SD^2 = R^2$. Hence $MS = MT = \sqrt{(R^2/2) - OM^2} = L$ (a constant). Thus all the eight points such as S and T lie on the circle with center at M and radius L.

Also solved by THEODORE BENNETT, E. P. BUGDENOFF and ALEX WIESNER.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

The one hundredth anniversary of the announcement by Sadi Carnot of the second law of thermodynamics was celebrated in New York City under the auspices of the Engineering Foundation, on December 4, 1924. Addresses were delivered by Professor M. I. PUPIN, of Columbia University, and Dr. W. L. EMMET, of the General Electric Company, on Carnot's principle.

On the occasion of the centenary of the Franklin Institute, the University of Pennsylvania conferred honorary degrees on Sir W. H. BRAGG, Mr. W. C. L. ENGLIN, Dr. CHARLES FABRY, Sir CHARLES PARSONS, Dr. E. W. RICE, Jr., and Dr. PIETER ZEEMAN.

The General Electric Company has appropriated to Union College a fund of $25,000 in memory of Dr. C. P. STEINMETZ. The income is to be used for scholarships.

The Rensselaer Polytechnic Institute of Troy, the oldest school of science and engineering in any English-speaking country, celebrated on October 3–4, 1924, the centenary of its foundation. On this occasion the Institute conferred the honorary degree of Doctor of Science on Professor A. A. MICHELSON, President of the National Academy of Sciences.

Professor H. A. WILSON, of Rice Institute, has been appointed professor of natural philosophy at Glasgow University.

Dr. G. N. BAUER has been appointed associate professor of mathematics at the University of New Hampshire.

Professor H. B. MITCHELL, of Columbia University, has resigned to enter business.

Instructor VICTOR DOUSHKESS, of Lafayette College, has been promoted to an assistant professorship of mathematics.

Associate Professor O. J. RAMLER, of Catholic University, has been promoted to a full professorship.

Mrs. R. B. MONTGOMERY, of Lynchburg College, has been promoted to an assistant professorship of mathematics.

Mr. L. W. JARMON has been appointed head of the department of mathematics at Chicora College, Columbia, S. C.

At the University of South Carolina, Mr. W. L. WILLIAMS has been appointed adjunct professor and Mr. R. L. JONES instructor.

Professor E. A. BAILEY, of LaGrange College (Georgia), has been appointed dean.

The Founders' Day address by Professor R. M. WINGER as retiring president of the Chapter of Phi Beta Kappa at the University of Washington was printed in full in the *Phi Beta Kappa Key* for October, 1924.

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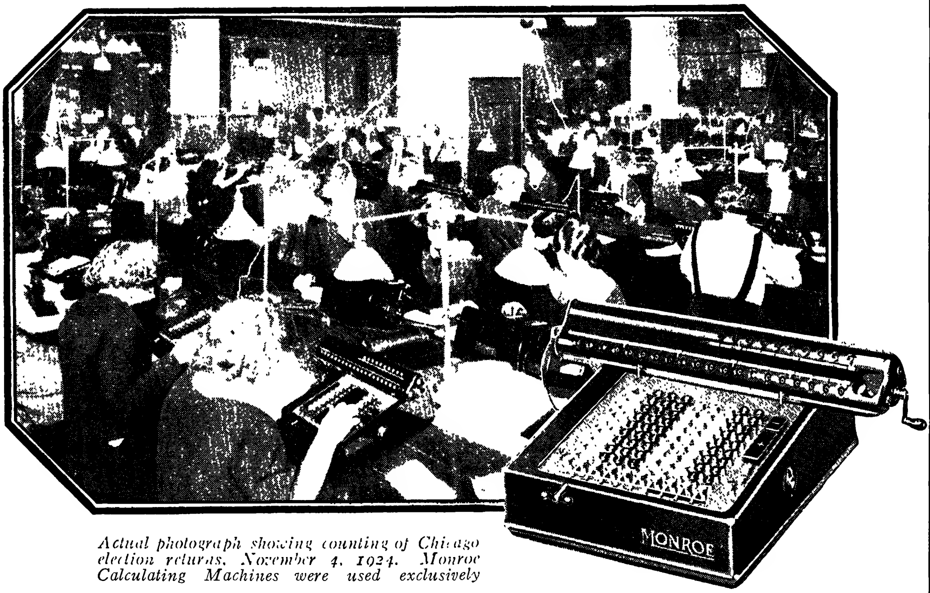

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BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 7–8, 1925;

Tenth Annual Meeting, Kansas City, Mo., December, 1925.

The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Elgin, May 2–3, 1924
INDIANA, Purdue Univ., April
IOWA, State College, Ames, May 2–3, 1924
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA–MISSISSIPPI, Baton Rouge, March 1, 1924
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Baltimore, December 22, 1924
MICHIGAN, Ann Arbor, April 3, 1924
MINNESOTA, Hamline Univ., St. Paul, May 24, 1924
MISSOURI, Kansas City, November 15, 1924
NEBRASKA, Creighton Univ., Omaha, May 2 1924
OHIO, Ohio State Univ., Columbus, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 21
TEXAS, Dallas, November 27–28, 1925

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THROUGH an oversight the Association has not reserved a sufficient number of copies of the MONTHLY for October, 1920, and for August, September, and October, 1921, also May, September, and October, 1924. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

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THE MAY MEETING OF THE IOWA SECTION.

The thirteenth regular meeting of the Iowa section of the Mathematical Association of America was held in Central Building, Iowa State College, Ames, Iowa, on May 2 and 3, 1924. There were forty-three present, including the following twenty-four members of the Association: Julia T. Colpitts, Marian E. Daniells, R. M. Deming, C. W. Emmons, Iva Ernsberger, Fay Farnum, C. Gouwens, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, I. F. Neff, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, W. J. Rusk, J. R. Sage, E. R. Smith, G. W. Snedecor, A. Helen Tappan, J. S. Turner, F. M. Weida, C. W. Wester, R. Woods.

On Friday evening the members and their friends enjoyed dinner together, arrangements being in charge of the mathematical fraternity Pi Mu Epsilon; Professor E. R. Smith acted as toastmaster.

Chairman McGaw presided at both the Friday afternoon and Saturday morning sessions, being relieved by vice-chairman McKelvey for a time.

Officers elected for this year are: Chairman, Professor E. R. Smith, Iowa State College; Vice-chairman, Professor I. F. Neff, Drake University; Secretary-treasurer, Professor J. F. Reilly, University of Iowa.

The next meeting of the Section will be held at Iowa State Teachers' College, Cedar Falls, Iowa, on May 1 and 2, 1925.

The following sixteen papers were presented:

(1) "Reduction of the equation of the general conic section" by Professor W. J. Rusk.

(2) "The preparation of high school teachers of mathematics in Iowa" by Professors E. W. Chittenden and F. M. Weida.

(3) "On the meaning, evaluation and application of the refund integral" by Professor F. M. Weida.

(4) "The zeros of the function $f_k(z) \equiv \sum_{n=0}^{\infty} n^k \frac{z^n}{n!}$, where k is a positive integer" by Professor Julia T. Colpitts.

(5) "Note on the average number of brothers and of sisters of the boys in families of n children" by Professor H. L. Rietz.

(6) "Relation of surface-level of a rotating fluid to the shape of the container" by Professor E. S. Allen.

(7) "Determination of the foci and directrices of a conic section" by Professor Allen.

(8) "Construction of tangents of an ellipse" by Professor Allen.

(9) "A problem in mathematical teaching" by Professor F. M. McGaw.

(10) "The isobaric differential equation in an elementary course" by Professor J. F. Reilly.

(11) "A new interpolation formula" by Professor Reilly.

(12) "A theorem in spherical trigonometry" by Professor J. S. Turner.

(13) "A new property of the circle" by Professor TURNER.

(14) "A theorem in geometry" by Professor ROSCOE WOODS.

(15) "The solution of q-difference equations by means of infinite determinants" by Professor E. R. SMITH.

(16) "Certain transformations of frequency distributions" by Professor G. W. SNEDECOR.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. By rotating the axes first and translating to a new origin afterwards Professor Rusk was able to give a reduction in which the parabola is treated directly as a special case of the general reduction.

2. The paper of Professors Chittenden and Weida gave the findings of a questionnaire that had been sent to high school teachers of mathematics in Iowa for the purpose of determining the degree of preparation they had made for their work. It was found that only a comparatively small number had met the "tentative standard" suggested by the National Committee on Mathematical Requirements for a senior high school.

3. In his paper Professor Weida expressed the present value of the refund benefit as a definite integral, the refund being the sum of the annuity payments received by the annuitant less his share of the purchase price. Assuming Makeham's law of mortality, he evaluated this integral in terms of the hypergeometric series. He also showed how to evaluate the integral by using a best fitting parabola in the sense of the theory of least squares.

4. In her paper, Professor Colpitts proved that

$$\sum_{n=0}^{\infty} n^k \frac{z^n}{n!} \equiv \sum_{r=1}^{k} a_r z^r e^z,$$

where

$$a_r = \frac{r^k + \sum_{m=0}^{r-1} (-1)^m {}_rC_m (r-m)^k}{r!}$$

and then discussed the nature of the zeros of this function.

5. In this paper, Professor Rietz dealt with a question brought to his attention by Professor J. McKeen Cattell. It seems to most people obvious that if there are on the average equal numbers of boys and girls in families, a boy must on the average have one more sister than brother. Professor Cattell points out the fact that this commonly accepted view is incorrect and that a boy has as many brothers as sisters. He demonstrates the fact for families of two children. In the present note, the fact stated by Cattell is proved for families of any size and the value of the average (AM) is obtained.

6. In his first paper, Professor Allen showed that if a vessel, bounded by a surface of revolution whose cross-section is $x = f(y)$, is filled to the level $y = 0$, and rotated with velocity ω, the ordinate of the center of the surface is a function $\varphi(\omega)$ depending on f. He derived the differential equation and developed several cases where f is derived from a known φ, and vice versa.

7. In his second paper, Professor Allen obtained the coördinates of the foci, the equations of the directrices, and the eccentricity of a conic section directly from the general equation of the second degree without change of coördinate axes.

8. In his third paper, Professor Allen gave simple geometric methods for finding 16 tangents to an ellipse and their points of tangency, when only the semi-axes are known.

9. Professor McGaw discussed the problem of teaching mathematics with a view to reviving interest in its study. He suggested several means of attaining this interest, such as (*a*) the use of models, slide rules, paradoxes, etc., (*b*) the encouraging of mathematics clubs, and (*c*) the organization of a system of popular lectures designed to convince people that mathematics has a very vital and continuous contact with everyday affairs.

10. In his first paper Professor Reilly emphasized the fact that the isobaric differential equation of the first order and the first degree can be solved as easily as the corresponding homogeneous equation, and recommended that it be included in a first course in differential equations.

11. In his second paper Professor Reilly pointed out that for interpolating values of a function between two given values y_1 and y_2 there exists an infinite number of formulas dependent upon the type of curve assumed and upon additional conditions that may be imposed. Discretion in imposing conditions and simplicity of formula obtained were given as two criteria of a good interpolation formula. One such formula was developed.

12. In his first paper Professor Turner proved that if P and Q are any two points on a great circle passing through any point O on the sphere, and if PM and QN are great circle arcs drawn perpendicular to any great circle through O, then

$$\frac{\sin PQ}{\sin MN} = \frac{\sin OP}{\sin OM} \cos NQ.$$

As an application it was shown that if T be any point on the tangent at P to a spherical conic of which S is a focus, and if TM be drawn perpendicular to SP, and TN perpendicular to the directrix, then

$$\frac{\sin SM}{\sin TN} = \frac{e}{\cos TM},$$

where e is the eccentricity of the conic.

13. In his second paper Professor Turner showed that if AB is any chord of a circle, center O, and if P and Q are any pair of inverse points, then $AP \cdot BQ = AO \cdot QR$, where R is the symmetric point (reflection) of P with respect to AB. Some applications of the theorem were stated.

14. Professor Woods derived the form of the equation of the S_{n-1}, in a space of n dimensions S_n, on an S_{n-2} determined by two S_{n-1}'s, and on an S_0 determined by n S_{n-1}'s. He then found the harmonic conjugate of this S_{n-1}, and discussed the cases $n = 2$ and 3. As an application he showed how this theorem enables

one to write down easily the equation of the quadric as the locus of a line that always touches three given skew lines in space.

15. Professor Smith in his paper showed that a q-difference equation of the form

$$p_1(x)\varphi(x) + p_2(x)\varphi(qx) + p_3(x)\varphi(q^2x) = 0,$$

where $\varphi(x)$ is an unknown function, and $p_1(x)$, $p_2(x)$ and $p_3(x)$ are rational functions, may, with certain restrictions, be transformed by the substitution

$$f(x) = \varphi(x) \prod_{\nu=0}^{\infty} p_1(q^\nu x)$$

into the equation

$$f(x) = p_2(x)f(qx) + p_3(x)p_1(qx)f(q^2x) = 0,$$

which is satisfied by an infinite determinant $D(x)$. The convergence of $D(x)$ may be shown by elementary means and the method and results extended to similar equations of higher order.

16. If a linear point transformation is applied to the differential equation of Pearson's frequency distributions

$$\frac{dy}{dx} = \frac{(x + a)y}{b_0 + b_1x + b_2x^2},$$

under the usual condition of preservation of areas, Professor Snedecor showed that the form of the equation is invariant, together with the constants of the distribution. The relation of this transformation to the standardization of grades was discussed.

J. F. REILLY, *Secretary-Treasurer.*

THE MATHEMATICAL THEORY OF ECONOMICS.[1]

By G. C. EVANS, Houston, Texas.

1. Introduction. One interest in research in the mathematical theory of economics is that the necessary preparation for it either in mathematics or in economics is not so great as for theoretical research say in physics or chemistry, or even in biology. And yet even if one regards mathematics as the study of quantity merely, rather than of order in general, the fact that every financial transaction is controlled by numbers would indicate that if there is any theoretical science in economics, it must be a mathematical one. In fact it may well be that it is the lack of mathematical technique among economists which has prevented the theoretical side of the subject from developing as rapidly as the wealth of books and papers, devoted to it, would seem to indicate. On the other hand, if we turn to the trained mathematicians, we find them mainly

[1] Read at the annual meeting of the Association at Washington, D. C., Dec. 31, 1924.

engrossed in the more romantic fields of physics, chemistry and engineering, except in the case of the extensive analysis of statistics, where contact is made with kinetic theory on the one hand, and social and biological data on the other.

2. Monopoly and Competition. Augustine Cournot laid the foundations of this subject in 1837, and his contributions, although elementary in method, have dwarfed those of most other workers. The notable exception is Irving Fisher. In fact, the first requirements for study in our subject are to read Cournot's *Mathematical Researches into the Theory of Wealth,* and Irving Fisher's *The Purchasing Power of Money.* With the latter's investigation of the equation of exchange and its important practical consequences we have not time to deal here, but must confine ourselves to one or two special problems suggested by the former. Cournot discussed the problem of economic equilibrium, describing the steady states which, under various conditions of monopoly and competition, would make profit a maximum; inventing a theory of foreign exchange; and outlining a measure of community wealth.

If the cost of u units of a commodity is $q(u)$ and the demand is $u = \varphi(p)$, where p is the price, then in a state of equilibrium the cost is $q(\varphi(p))$, the profit is $\pi = up - q = p\varphi(p) - q(\varphi(p))$, and in order to make the profit a maximum, the equation

$$\frac{d}{dp}\{p\varphi(p) - q(\varphi(p))\} = 0 \tag{1}$$

must hold. Such is in brief Cournot's theory of monopoly. And the same sort of analysis, if there are several producers and several cost functions, will develop a theory of competition. Cournot points out the relations between the prices of equilibrium for these various kinds of production.

In an article in the MONTHLY (*1922,* 371–380), I showed that these theories could be made even simpler and more instructive by taking a simple approximate form of the cost function,

$$q = Au^2 + Bu + C, \tag{2}$$

where A, B, C are > 0, where u is the amount produced per unit time, and q the cost per unit time of producing that amount; and a simple approximate form,

$$y = ap + b, \qquad a < 0,\ b > 0, \tag{3}$$

for the amount that would be consumed in the market per unit time if the price were p. In this way the equation equivalent to (1) yields the value

$$p_m = \frac{b - 2Aab - Ba}{-a(2 - 2Aa)}.$$

If there are n producers, all with the same cost function $q(u_i) = Au_i^2 + Bu_i + C$, and the steady state is indicated by writing $y = u_1 + u_2 + \cdots + u_n$ in (3), the profit for each producer will be

$$\pi_i = pu_i - q(u_i).$$

Then under the hypothesis that each producer tries to make his profit a maximum by varying his production, independently of the others (which is Cournot's definition of competition), we get from the equation analogous to (1),

$$p_a = \frac{b - 2Aab - nBa}{-\, a(n + 1 - 2Aa)} \,.$$

On the other hand if each regulates his production so as to make the total profit a maximum, we have what may be called coöperation, with an equilibrium price

$$p_c = \frac{nb - 2Aab - nBa}{-\, a(2n - 2Aa)} \,,$$

while if each tries to regulate his production so as to make the biggest profit he can, assuming that the price will not depend on his own manipulation, there results the price

$$p_b = \frac{2Ab + nB}{n - 2Aa} \,.$$

It is easily verified that $p_m > p_c > p_a > p_b$. Rather than obtain these results again now, let me refer for them to the article in the MONTHLY, and turn to new details.

The statement is sometimes advanced that altruism is not a social benefit, since if each looks after the other it is the same as each looking after himself by a roundabout and inefficient process. However that may be, it is assumed to be of benefit to society to have the production of a typical commodity large, and of benefit to the producer to make his profit large. As a criterion of a somewhat altruistic society then, we might seek to make not π a maximum but, say, $\pi + k^2u$, where k^2 is some convenient constant. Equation (1) becomes

$$\frac{d}{dp}\{up - Au^2 - (B - k^2)u - C\} = 0,$$

so that (by changing B to $B - k^2$) we have

$$p_m = \frac{b - 2Aab - (B - k^2)a}{-\, a(2 - 2Aa)} \,, \tag{4}$$

a lower price than before, since $a < 0$. A calculation in terms of elementary algebra shows that the amount produced would be, by the quantity

$$\delta u = \frac{-\, ak^2}{2 - 2Aa} > 0,$$

more than before, the profit less than before by an amount

$$-\, \delta\pi = \frac{|k^4a|}{4(1 - Aa)}$$

and the profit plus k^2u greater than the original profit by

$$\frac{k^2}{4(1 - Aa)}\{2(b + Ba) - k^2a\}.$$

By means of this kind of analysis a satisfactory rate might be established for a public utility, public utilities being in fact monopolies in which cost functions may be accurately known, and demand functions established by a single change of rates. If we compare (4) with the formula (23) of the MONTHLY article already mentioned, which gives the price such that, when it is fixed, the monopolist will make his greatest profit by meeting the demand:

$$\frac{b - 2Aab - (B - k^2)a}{-\,a(2 - 2Aa)} = \frac{B + Ab}{1 - Aa},$$

we get a kind of maximum value for k^2:

$$k^2 = \frac{b + Ba}{-\,a}.$$

If p is given a value not less than that given by (4) with this value of k^2, it will be to his advantage to meet the demand.

A further investigation possible in terms of elementary algebra, besides its slight foundation in the calculus, would be the discussion of coöperation and competition in the case where the cost functions of the various producers are not all identical but have coefficients A_i, B_i, C_i which differ from A, B, C by infinitesimals δA_i, δB_i, δC_i. What are then the changes in price and production if the cost coefficients of a single producer are changed by amounts δA, δB, δC? In this way may be studied, as Cournot has pointed out, the influence on the commodity of special taxes and local conditions.

The general problem of taxation, even with respect to a particular commodity, is more difficult; for the general price level enters into the problem as well as the price of the particular commodity. The problem thus introduces the price index as an additional variable, with regard to which some hypotheses must be made as to the coefficients a, b of the demand function. But the investigation is not the less worth while. Although nothing is more certain than death and taxes, nothing is less certain than their incidence.

3. Monopoly and Competition. General Problems. An extensive mathematical development of the theory arises if we no longer assume that the situation is the "steady state." We are then led into problems such as the one which the author has tried to state in the simplest possible terms in an article in the MONTHLY (*1924*, 77–83) on the Dynamics of Monopoly: to make a maximum the total profit over the interval of time during which the prices are changing, assuming that the rate of production $u(t)$ is kept equal to the amount which would be absorbed by the market. In mathematical terms:

$$\pi = \int_{t_0}^{t_1} (up - q)dt \tag{5}$$

is to be a maximum. More general theories may well be considered, in which reserves of stock enter in, as in an article which the author has submitted to the *Proceedings of the National Academy*. But one great interest in this type of problem depends on what is taken as the law of demand. If for instance we write $u = y = ap + b$, the problem reduces to that of equilibrium already considered. If however we write

$$y = ap + b + h\frac{dp}{dt} = u, \qquad h > 0,\ b > 0,\ a < 0, \tag{6}$$

so that demand is greater if the rate of change of price is positive, other things being equal, the problem is most practical but quite different from the classical theories.

An advance, which would not involve further mathematical difficulties but would be again of considerable practical interest, would be to make use of a demand law of the form

$$y = u = ap + (b + b' \cos kt) + h\frac{dp}{dt}, \qquad b' < b,$$

which would enable us to consider seasonal changes. The reader will gain considerable familiarity with our problem if he will work out this case.

The most interesting type of law of demand has been suggested recently by Irving Fisher in a statistical investigation by means of which he establishes a correlation between the total volume of trade $T(t)$ at time t and the rate of change of price index at a previous time $t - C$, the interval of time being constant and about equal to two weeks. Professor Bray finds that the correlation is even considerably better for a formula of the type

$$T(t) = ap(t - c) + b + hp'(t - c).$$

Does not this suggest an interesting study of the integral (5)?

Another study would involve a still more general type of law of demand, say of the form

$$u(t) = ap(t) + \int_{-\infty}^{t} \varphi(t - T)p(T)dT. \tag{7}$$

This study has been initiated by C. F. Roos in an article to be published in the *American Journal of Mathematics*. As might be expected, a formula of this kind introduces integral equations into the analysis. Mr. Roos also considers a problem of competition which involves a treatment of several integrals like (5). It may be remarked that relations which arise from integral equations like (7) can often by differentiation be analyzed in terms of differential equations with simple coefficients. But we have devoted already space enough to this general problem.

4. Foreign Exchange. Cournot considered in some detail the problem of foreign exchange. For simplicity we suppose that each market makes prices with respect to a unit quantity of gold the same in all markets—say a dollar of 25 grains—because, of course, prices in other denominations can be evaluated in terms of a change of units; and we shall limit ourselves to three markets.

Let M_{ij} be the amount of money owed by market i to market j, and c_{ij} be the value for exchange purposes of a dollar of money in i in terms of credit at j. That is, suppose one dollar at i will buy c_{ij} dollars worth of credit at j. Cournot's hypothesis for the determination of rates of exchange c_{ij} is that credits shall be balanced without the transportation of money. That is, for the first market

$$M_{12} + M_{13} = c_{21}M_{21} + c_{31}M_{31}, \tag{8.1}$$

and similarly

$$M_{21} + M_{23} = c_{12}M_{12} + c_{32}M_{32}, \tag{8.2}$$

$$M_{31} + M_{32} = c_{13}M_{13} + c_{23}M_{23}. \tag{8.3}$$

Incidentally, the possibility of speculation maintains very approximately the equations

$$\begin{aligned} c_{ij}c_{ji} &= 1, \\ c_{ij}c_{jk} &= c_{ik}. \end{aligned} \tag{9}$$

By means of (9) all the rates of exchange may be expressed in terms of c_{21} and c_{31} so that the equations (8) become the following:

$$c_{21}M_{21} + c_{31}M_{31} = M_{12} + M_{13}, \tag{10.1}$$

$$c_{21}(M_{21} + M_{23}) - c_{31}M_{32} = M_{12}, \tag{10.2}$$

$$- c_{21}M_{23} + c_{31}(M_{31} + M_{32}) = M_{13}. \tag{10.3}$$

Here (10.1) $-$ (10.2) $=$ (10.3), so equation (10.3) may be disregarded; and there are left two equations to determine the two unknowns.

The mathematician, rather than the economist, will now stop to consider the exceptional cases where one or more two-order determinants vanish. Since the M_{ij} are essentially ≥ 0, these resolve themselves into cases where (a) one or more of the markets find themselves without credit with respect to others, or (b) the markets split into groups which have no dealings with each other, the latter case being that in which the augmented matrix of (10) is of rank < 2. Professor Bray in an article in the MONTHLY (*1923*, 365–371) has considered the corresponding situation for an arbitrary number of markets.

When the elementary problems have been considered, the practical nature of more extended mathematical investigation becomes manifest. Thus it is opportune to replace the equations (8) by differential relations, which emphasize the rate of increase of credit, or by integral relations, which emphasize the past history of credit. The persistence of equations (9) adds an unusual element to the mathematical problem, as one can see even in the simplest case of two markets.

5. General Points of View. There is no such measurable quantity as "value" or "utility" (with all due respect to Jevons, Walras and others) and there is no evaluation of "the greatest happiness for the greatest number"; or, more flatly,—there is no such thing. In a way, material happiness has to do with a maximum of production and a minimum of unpleasant labor; though again no such thing is realizable theoretically without an arbitrary definition of a composite function which is to take on a maximal value; and in the composition of this function the labor and profit of various classes of people enter capriciously. One might define a ratio of weighted production divided by weighted amounts of labor, according to classes, and study what sort of lash, economic or otherwise, would serve to impel Society towards this limit; but the choice of weights would depend essentially on whether the chooser is born a Bolshevik or a member of the Grand Old Party! Compromises carry us into the field of ethics.

That does not mean that such study is unprofitable. Far from it. How otherwise are we to evaluate the schemes of reformers and prophets, major and minor? Moreover the ground work of such studies must be laid well in advance, before there is any direct occasion for them; otherwise they will fail us when we do need them. There is not only an opportunity for mathematics in economics, but even a duty; and on mathematicians in an unusual degree lies the responsibility for the economic welfare of the world.

THE 21-POINT CUBIC.

By B. H. BROWN, Dartmouth College.

1. Introduction. It is the purpose of this paper to correlate certain portions of the geometry of the triangle by developing new properties of the 21-point cubic.[1] We shall give in paragraph 2 a new definition of this curve, and prove in paragraphs 2 and 3 that some 16 additional notable points lie on the curve. As the 21-point cubic is not the general circular cubic, our primary interest is not in circular cubics but in the geometry of the triangle. It is hoped that this paper will make a helpful contribution to this subject by exhibiting [2] a collection of 37 notable points (to which others may be added *ad lib.*) for which a large number of sets of 3 (4) points are readily seen to be collinear (concyclic) by means of one fundamental theorem on circular cubics.

Let us recall Neuberg's definition, and briefly enumerate his results.

Suppose that the sides opposite the vertices A_1, A_2, A_3 of a triangle have lengths a_1, a_2, a_3. Let a_4, a_5, a_6 represent the distances from any point P of the plane of the triangle to A_1, A_2, A_3 respectively. The 21-point cubic is the locus of points P for which

$$\begin{vmatrix} 1 & a_1^2 + a_4^2 & a_1^2a_4^2 \\ 1 & a_2^2 + a_5^2 & a_2^2a_5^2 \\ 1 & a_3^2 + a_6^2 & a_3^2a_6^2 \end{vmatrix} = 0. \tag{1}$$

[1] Neuberg, *Mémoire sur le tétraèdre.* Published as *Supplément V* to *Mathesis,* 1885, also *Mémoires couronnés,* etc., Belgium, 1886, pp. 1–72. See also Neuberg, *Bibliographie du triangle et du tétraèdre, Mathesis,* June, 1924, pp. 241 et seq.

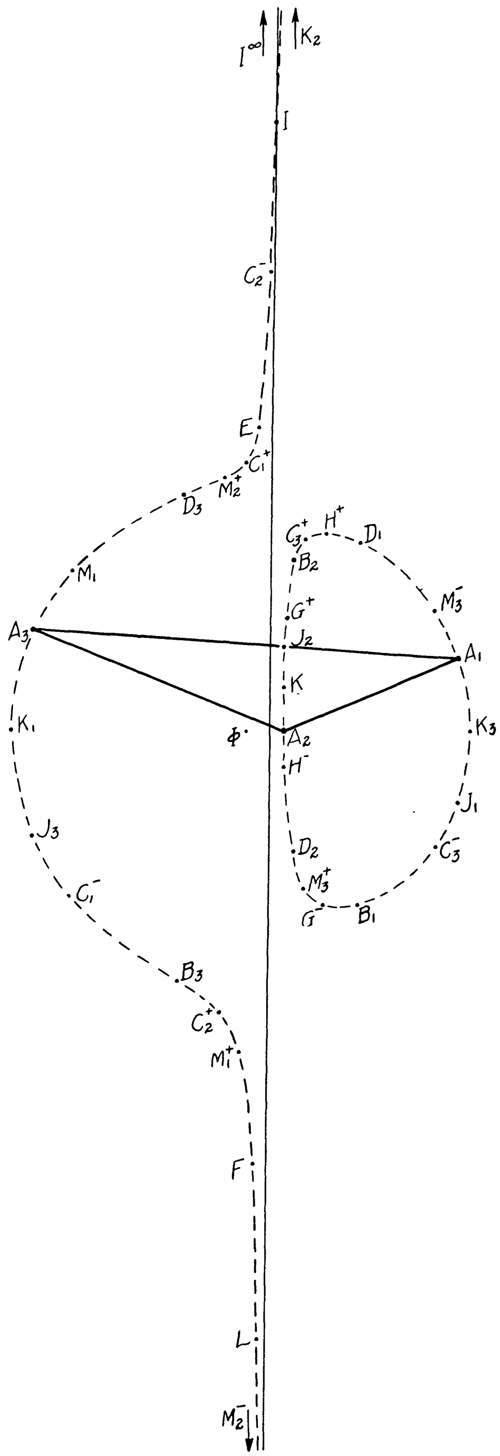
[2] I am indebted to Professor R. E. Langer for the construction of the figure, and to my pupil, Mr. L. S. Kennison, for the verification of certain computations.

There is little difficulty in showing from this equation that the following points lie on the locus, together with the circular points at infinity:

(*A*) the 3 vertices A_i,
(*B*) the 3 points B_i symmetric to A_i in a_i,
(*C*) the 6 vertices C_i^+, C_i^- of the equilateral triangles constructed on a_i,
(*D*) the 3 points D_i symmetric to A_i in the lines q_{jk}, q_{kj}, where q_{jk} is the point where the side a_j meets the perpendicular bisector of a_k,
(*E*) the circumcenter E,
(*F*) the orthocenter F,
(*G*) the 2 isodynamic centers G^+, G^- such that

$$A_1A_2\cdot A_3G = A_1A_3\cdot A_2G = A_1G\cdot A_2A_3,$$

(*H*) the 2 isogonal centers H^+, H^- from which a_i subtend angles of 60° or of 120°.



The reader is referred to the figure where all these points and others found later are shown. The details of the proofs together with the genesis of the curve may be found in either of Neuberg's articles to which reference was made. The author has recently (MONTHLY, *1924*, 371–375) published a new proof that the isogonal centers lie on the cubic. After a careful search no further published results have been found.[1]

2. New definition. Let us begin by considering the following 9 points: the 3 vertices A_i of a triangle, the 4 centers K and K_i of the in- and escribed

[1] But see Neuberg, *Bibliographie*, etc., "Notre documentation est incomplète, notamment pour la cubique des 21 points."

circles, and the 2 circular points at infinity. In general, 9 points determine a cubic. But since two cubics intersect in 9 points, through any such set of 9 points will pass a pencil of cubics. The set enumerated is of this nature. To show this, we note (1) $\angle K_iA_kK = \angle KA_jK_i = \pi/2$, hence K_i, A_j, K, A_k are concyclic, and (2) K_k, A_i, K_j are collinear. Hence the cubics composed of a circle (1) and line (2) contain the 9 points and there is therefore a pencil of cubics through these points. Our discussion for the present relates to any cubic of the pencil.

We recall the theorem that if a circle meet a circular cubic in R, S, T, U and if the line RS meet the cubic in V and TU in W (similarly for RT and SU, or RU and ST), then VW is parallel to the real asymptote.[1] This theorem is fundamental for the study of the set of points with which we concern ourselves.

For any cubic in the pencil, A_j, K_i, A_k, K are concyclic, and A_jK and also K_iA_k meet the cubic in K_j, hence

THEOREM: *The tangents at K_i (K) are parallel to the real asymptote.*

Let A_iA_j cut the cubic in J_k; since K_kK cuts the cubic in A_k, A_kJ_k is parallel to the real asymptote. Let the circumcircle of the triangle cut the cubic again in I. IA_i meets the cubic in, say, A_i'. A_jA_k meets the cubic in J_i. Hence $A_i'J_i$ must be parallel to the real asymptote, that is, $A_i' = A_i$, and

THEOREM: *IA_i is tangent to the cubic at A_i.*

Consider the three circular cubics formed by the circles $J_iJ_jA_k$ and the lines A_iA_j. These pass through 8 points of our cubic, hence all four cubics have a common ninth point L. Since A_i, J_j, J_k, L are concyclic and since A_iJ_j cuts the cubic in A_k, LJ_k cuts the cubic in J_k' such that A_kJ_k' is parallel to the real asymptote. Hence $J_k' = J_k$, and

THEOREM: *LJ_i is tangent to the cubic at J_i.*

These results are true for any cubic of the pencil. Let us now confine our attention to the cubic through C_3^+, the vertex of an equilateral triangle on A_1A_2 as base. Consider the circle $A_1A_2C_3^+$. A_1A_2 cuts the cubic at J_3. Hence $C_3^+A_3$ passes through the fourth intersection of this circle and cubic. But the intersection of this circle and line is the (positive) isogonal center H^+ which accordingly is on the cubic. Next consider the circle $A_2A_3H^+$; as A_2A_3 cuts the cubic in J_1, the fourth intersection of the circle and cubic is also on A_1H^+ and is consequently a point C_1^+, vertex of an equilateral triangle on A_2A_3. Similarly C_2^+ lies on the cubic.

We now have shown that 9 points, A_i, C_i^+, H^+, and the circular points at infinity are on the cubic, and since it can easily be shown in a special case that these 9 points determine a unique cubic, they do in general, and we can identify this cubic with Neuberg's cubic. The points I, J_i, and L are not constructible until we know the direction of the real asymptote. Our principal new result is then

THEOREM: *The in- and excenters lie on the 21-point cubic.*

[1] For theorems on circular cubics mentioned in this paper see Teixeira, *Traité des courbes spéciales remarquables*, Fr. ed., 1908, vol. 1, pp. 62–83.

3. Sixteen new notable points on the 21-point cubic. Let us now discard synthetic methods for the time being and grind out a few formulas and results. Taking a triangle with vertices $(a, 0)$, $(b, 0)$, $(0, c)$, on simplifying (1) we have for the equation of the cubic

$$
\begin{aligned}
&\{(-3ab - c^2)x + c(a+b)y\}(x^2+y^2) + 3ab(a+b)x^2 \\
&\quad + 2c(c^2 - a^2 + ab - b^2)xy + ab(a+b)y^2 + (a^2c^2 + b^2c^2 - 3a^2b^2 + abc^2)x \\
&\qquad\qquad - c^3(a+b)y - abc^2(a+b) = 0.
\end{aligned}
$$

The coördinates of the 21 notable points are given for the convenience of those who may wish to draw the cubic for other triangles. The case illustrated is for $a = 2$, $b = c = 1$. The case $a = 2$, $b = -1$, $c = 3$ is much simpler numerically but does not give quite so good a figure. A special case $a = c = 3$, $b = -1$, where the asymptote is parallel to A_1A_2, and passes through A_3, is of some interest.

(A) $(a, 0)$, $(b, 0)$, $(0, c)$;

$$
(B)\ \left(\frac{ab^2 - ac^2 + 2bc^2}{b^2+c^2}, \frac{(b-a)2bc}{b^2+c^2}\right),\quad \left(\frac{ba^2 - bc^2 + 2ac^2}{a^2+c^2}, \frac{(a-b)2ac}{a^2+c^2}\right), (0, -c);
$$

$$
(C)\ \left(\frac{b \pm \sqrt{3}c}{2}, \frac{c \pm \sqrt{3}b}{2}\right), \left(\frac{a \mp \sqrt{3}c}{2}, \frac{c \mp \sqrt{3}a}{2}\right),\quad \left(\frac{a+b}{2}, \pm\frac{\sqrt{3}}{2}(a-b)\right);
$$

$$
(D)\ \left(\frac{b}{a}\,\frac{2ac^2 + a^2b - bc^2}{b^2+c^2}, \frac{c(a-b)(ab+c^2)}{a(b^2+c^2)}\right),\quad \left(\frac{a}{b}\,\frac{2bc^2 + ab^2 - ac^2}{a^2+c^2}, \frac{c(b-a)(ab+c^2)}{b(a^2+c^2)}\right), \quad \left(\frac{c^2(a+b)}{ab+c^2}, \frac{2abc}{ab+c^2}\right);
$$

$$
(E)\ \left(\frac{a+b}{2}, \frac{ab+c^2}{2c}\right); \qquad (F)\ \left(0, -\frac{ab}{c}\right);
$$

(G) the intersections of $B_iC_i^{\pm}$; (H) the intersections of $A_iC_i^{\pm}$.

The real asymptote is found to be

$$
y - \frac{3ab + c^2}{c(a+b)}x + \frac{18a^2b^2c^2 + 9abc^4 + 9a^3b^3 + 2c^6 - 2a^2c^4 - 2b^2c^4 - 3a^3bc^2 - 3ab^3c^2}{c\cdot\Delta} = 0,
$$

where

$$
\Delta = c^4 + b^2c^2 + a^2c^2 + 8abc^2 + 9a^2b^2.
$$

Let us call the direction of the asymptote I^{∞}.

The Euler line (the line EF) is

$$y - \frac{3ab + c^2}{c(a + b)} x + \frac{ab}{c} = 0,$$

so that

THEOREM: *The real asymptote of the cubic is parallel to the Euler line.*

It is easy to verify that the asymptote is the Euler line if and only if the triangle is isosceles. In this case, and only in this case, the cubic degenerates into the Euler line and a circle. If the triangle be equilateral, neither the Euler line nor the Neuberg cubic is unique.

The following curious construction suffices to fix the asymptote by giving the point where it crosses the curve. It is a known theorem that the circles $A_i q_{jk} q_{kj}$ intersect in a point[1] φ on the circumcircle (not, in general, on the cubic). This point has coördinates

$$x = \frac{3a^3b^2 + 3a^2b^3 + a^3c^2 + b^3c^2 + a^2bc^2 + ab^2c^2 - ac^4 - bc^4}{\Delta},$$

$$y = \frac{4abc^3 + c^5 - 2a^3bc - a^2c^3 + 5a^2b^2c - 2ab^3c - b^2c^3}{\Delta};$$

and it is readily shown that

THEOREM: *The point φ is the singular focus* (that is, the intersection of the two imaginary asymptotes tangent to the cubic at the circular points at infinity).

The diametral point of φ in the circumscribed circle has the coördinates

$$x = \frac{2(c^2 + 3ab)(a + b)(c^2 + ab)}{\Delta},$$

$$y = \frac{2a^2c^4 + 2b^2c^4 + 5abc^4 + 3ab^3c^2 + 3a^3bc^2 + 12a^2b^2c^2 + 9a^3b^3}{c \cdot \Delta}.$$

With a little ingenuity in combination we may readily verify the

THEOREM: *The diametral point of the singular focus φ in the circumscribed circle is the point of intersection of the tangents at A_i, that is, the point we have called I, and is also the point of intersection of the cubic and asymptote.*

The point L was defined as the intersection of the circles $A_iJ_jJ_k$. We have for it the rather unexpected

THEOREM: *L lies on the tangent to the cubic at I.*

Proof. First let us note that L, J_i, A_i, I are concyclic, for let the circle through I, A_i, J_i cut the cubic in L'. Since IA_i cuts the cubic in A_i, $L'J_i$ must cut the cubic in J_i, hence $L' = L$. Now let LI cut the cubic again at I', and consider the circle LA_iI' cutting the cubic again at J_i'. LI' cuts the cubic at I which is on the real asymptote, hence either $J_i'A_i$ passes through I, or is parallel to the real asymptote. If the former be the case, $J_i' = A_i$; if the latter, $J_i' = J_i$ and consequently $I' = I$. In the numerical example exhibited, the latter case

[1] Neuberg, *Mémoire*, etc., p. 62.

holds. We proceed to prove by continuity that this is always so. There is surely no trouble unless A_i and J_i coincide, and when we recall that J_i is on A_jA_k we see that this is impossible. Hence L is on the tangent to the cubic at I.

Eckhardt's theorem [1] states that any line through I cuts the cubic in two points equidistant from the singular focus φ. Since LI is tangent to the cubic at I, $L\varphi = I\varphi =$ diameter of the circumscribed circle. This same theorem shows that for the points A_i, the distances from φ to the cubic are extrema, since the normals at A_i pass through φ.

Finally let us define $M_i^{\pm}$ as the intersection of $A_jC_k^{\pm}$ and a parallel through $C_i^{\pm}$ to the real asymptote. This gives 16 new points, I, I^{∞}, J_i, K, K_i, L, $M_i^{\pm}$, and for all of these we have elementary constructions starting from the base triangle. We may add other points (less and less "notable" as the complexity of definition increases) by repetition of the use of the fundamental theorem, by the method used to obtain the point L, and by Eckhardt's theorem.

The primary purpose of this paper is to exhibit this set of 37 points $A - M$ for the convenience of those interested in the geometry of the triangle. Starting with various sets of 9 or more points, it is interesting to try to show from our fundamental theorem, perhaps with an occasional resort to analytic methods, that the remaining points lie on the cubic. In such procedures some obvious collinearities must be assumed, but a surprising number of new ones will be found. The author has noted over 75. We have shown or may show that lines through the following pairs of points contain I^{∞}:

$$A_iJ_i; \qquad B_iD_i; \qquad C_i^{\pm}M_i^{\pm}; \qquad EF; \qquad G^{\pm}H^{\mp};$$

and the tangents at K and K_i. But it seems needless to catalog all the collinearities and concyclicities. Using our fundamental theorem, the student can read them more easily from the figure than from the printed page.

THE HUYGENS GOVERNOR.

By V. C. POOR, University of Michigan.

1. Introduction. The Huygens governor consists of a vertical axis, a horizontal arm and a simple pendulum attached to the end of the arm. The simple pendulum is constrained to swing in the plane of the arm and the axis. If we neglect all dissipating forces, we will have a conservative holonomic system with two degrees of freedom. The dynamical problem, then, is to study the motion of the bob subject to sets of somewhat general initial conditions. We shall also determine those positions of the bob for which the motion will be steady.

Let b and a be the length of the arm and the pendulum respectively, and let m be the mass of the bob. If we designate by ψ the angle that the pendulum makes with the arm b, and by φ the angle through which the arm swings about

[1] *Cf.* Durège, *Zeitschrift der Math. u. Phys.*, vol. 14, 1869.

the axis referred to a fixed plane of reference through the axis, then the Lagrangian function and the Lagrangian equations for the system may be written down.

The kinetic energy T and the potential energy W, if measured from the lowest point of the swing, are given by the equations

$$T = \frac{m}{2}[(b - a\cos\psi)^2\dot\varphi^2 + a^2\dot\psi^2],$$

$$W = mga(1 - \sin\psi).$$

The Lagrangian equations for a conservative system have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q},$$

where L is the Lagrangian function $T - W$. For our system there will be two such equations, one for each of the variables φ and ψ.

These equations are easily seen to be of the form

$$a^2\ddot\psi = (b - a\cos\psi)a\sin\psi\dot\varphi + ga\cos\psi, \tag{1}$$

$$\frac{d}{dt}(b - a\cos\psi)^2\dot\varphi = 0. \tag{2}$$

The second of these equations furnishes the integral

$$(b - a\cos\psi)^2\dot\varphi = c_2, \tag{3}$$

which could have been obtained from the principle of *angular momentum*, since the moment of the force (mg) about the axis is zero. If we eliminate $\dot\varphi$ from equation (1) by the use of equation (3) and integrate (1), we find that

$$a^2\dot\psi^2 + (b - a\cos\psi)^2\dot\varphi^2 - 2ga\sin\psi = c_1, \tag{4}$$

which expresses the fact that the total energy of the system is constant. This result could have been written down from the *principle of energy* for a conservative system. A further integral from (3) and (4) may easily be obtained in the form

$$t = a\int\frac{(b - a\cos\psi)d\psi}{\sqrt{c_1(b - a\cos\psi)^2 + 2ag\sin\psi(b - a\cos\psi)^2 - c_2}},$$

which would be useful in numerical cases. Reduced to algebraic form, this integral involves the square root of an octic and gives little explicit information about the motion.

2. Oscillatory and Rotational Motions. We will now impose the following initial conditions: we will start the bob from the point $\psi = \varphi = 0$ with the angular velocities $\dot\psi = \dot\psi_0$, $\dot\varphi = \dot\varphi_0$ and leave the system to itself. We then seek to

determine the conditions under which the bob will oscillate or make a complete revolution. The above initial conditions substituted in equations (3) and (4) give

$$c_1 = a^2\dot{\psi}_0^2 + (b - a)^2\dot{\varphi}_0^2, \tag{5}$$

$$c_2 = (b - a)^2\dot{\varphi}_0. \tag{6}$$

At the points where the bob comes to rest in its oscillations, $\dot{\psi}$ will be equal to zero. Equation (4) furnishes this condition in the form

$$(b - a\cos\psi)^2\dot{\varphi}^2 - 2ag\sin\psi - c_1 = 0$$

or

$$\frac{c_2^2}{(b - a\cos\psi)^2} - 2ag\sin\psi - c_1 = 0. \tag{7}$$

If we replace $b - a\cos\psi$ by r, this last equation takes the form

$$4g^2r^6 - 8g^2br^5 + [4g^2(b^2 - a^2) + c_1^2]r^4 - 2c_1c_2^2r^2 + c_2^4 = 0. \tag{8}$$

If the bob comes to rest at the topmost point, *i.e.*, if the initial velocity $\dot{\psi}_0$ is just sufficient to carry the bob three fourths of the way around or to the point $\psi = 3\pi/2$, r will just equal b. In this case equation (8) furnishes the relation $c_1b^2 - c_2^2 = \pm 2gab^2$. Expressed in terms of the initial conditions, this becomes

$$a^2\dot{\psi}_0^2 = 2a[\pm g - B\dot{\varphi}_0^2], \tag{9}$$

where

$$B = \left(1 - \frac{a}{b}\right)^2\left(b - \frac{a}{2}\right).$$

We may well distinguish three cases: $b > a/2$; $b = a/2$; and $b < a/2$.

Case I. $b > a/2, b \neq a$. In this case B is always positive, so that for $B\dot{\varphi}_0^2 < g$ there exist two values of the initial velocity $a\dot{\psi}_0$, equal in magnitude but of opposite sense, which will bring the bob to rest at the point $\psi = 3\pi/2$. For a velocity numerically smaller than $a\dot{\psi}_0$, the bob will oscillate and for a numerically larger velocity it will make a complete revolution. In general, if a negative g is used in determining the velocity $a\dot{\psi}_0$, the velocity so determined would bring the bob to rest at the point $\psi = \pi/2$, or the lowest point in the swing. In this case evidently no such velocity exists.

If $B\dot{\varphi}_0^2 > g$, $a^2\dot{\psi}_0^2$ would have to be less than a negative quantity for oscillatory motion. Since in this case there exists no initial velocity that will bring the bob to rest at the highest point, the bob will revolve for every $a\dot{\psi}_0$.

For g equal to $B\dot{\varphi}_0^2$ the critical velocity is zero. There is nothing to determine which way the bob will swing, but if it starts from rest in either sense it will come to rest at the highest point, $\psi = 3\pi/2$.

Case II. $b = a$ or $b = a/2$. Here the critical velocity $a\dot{\psi}_0$ is independent of the initial angular velocity $\dot{\varphi}_0$ about the axis. We have the same situation we had in Case I, but with the critical velocity $a\dot{\psi}_0 = \pm \sqrt{2ag}$.

Case III. $b < a/2$. This condition makes B always negative, so that for every $\dot\varphi_0$ there are two equal and opposite values of $a\dot\psi_0$ which will bring the bob to rest at the point $\psi = 3\pi/2$. For $a\dot\psi_0$ less or greater in magnitude than the magnitude of these critical values, the bob will oscillate or make a complete revolution, respectively.

For $-B\dot\varphi_0^2 > g$ there are two equal and opposite values for $a\dot\psi_0$ which will bring the bob to rest at the lowest point, $\psi = \pi/2$. For values of $a\dot\psi_0$ numerically greater than the critical values but smaller than the critical value carrying the bob to $\psi = 3\pi/2$, the bob will oscillate through the point $\psi = \pi/2$, but for $a\dot\psi_0$ less in magnitude the bob will oscillate but never reach the lowest point, $\psi = \pi/2$. If the bob returns to its original position in any of these cases, it will assume its initial velocities. This fact may readily be seen by replacing ψ by $2n\pi$ in equations (3) and (4), n being any positive or negative integer or zero. This shows that if the initial velocities cause the bob to oscillate or make a complete revolution, the same type of motion will persist ever after.

3. Equivalent Swings. We are also able to answer the question as to whether the in-swings and the out-swings are equal. In fact if $a \sin\psi$ be replaced by y in equation (7) and the resulting equation rationalized, we have a sextic. If the bob reaches the same height in its in-swing as in its out-swing, this sextic will have a double root. This root will appear also in its first derived equation. From this we conclude that the coefficient of y and the absolute term of our sextic must have a common factor. The ratio of these two coefficients is

$$\frac{[c_2^2 - c_1(b^2 + a^2)]^2 - 4b^2a^2c_1^2}{-4g(b^2 + a^2)[c_2^2 - c_1(b^2 + a^2)] - 16ga^2b^2c_1}.$$

After the values of c_1 and c_2 have been substituted, a careful inspection will show that there is no common factor present for general values of a, b, $\dot\psi_0$ and $\dot\varphi_0$. The rest points are therefore at unequal distances from the horizontal plane through b.

For $\dot\varphi_0 = 0$ the ratio reduces to

$$\frac{(b^2 + a^2)a^4\dot\psi_0^4 - 4a^6b^2\dot\psi_0^4}{4g(b^2 + a^2)a^2\dot\psi_0^2 - 16ga^4b^2\dot\psi_0^2}.$$

We find in this case the common factor $-a^2\dot\psi_0^2$. If in addition to this we select the factor $1/2g$ furnished by the coefficient of the highest power, we may choose $y = -(a^2\dot\psi_0^2/2g)$ which gives exactly the rest points in the swing of a simple pendulum. In fact if we substitute the condition $\dot\varphi_0 = 0$ in equation (7), c_2^2 vanishes and our sextic reduces to $y = -(a^2\dot\psi_0^2/2g)$.

Choosing b equal to zero, we have the spherical pendulum. Our ratio in this case reduces to $a^8\dot\psi_0^4/4ga^4\dot\psi_0^2$. The fact that equal factors appear here has no bearing on the previous argument. It is well known that the heights of the points where $\dot\psi$ is zero for the spherical pendulum are in general unequal. In fact

any factor we might choose would have to satisfy the cubic

$$2gy^3 + c_1y^2 - 2ga^2y + c_2^2 - c_1a^2 = 0,$$

as follows from (7) by putting b equal to zero.

The result obtained in the general case should be expected since the instantaneous centrifugal force acting on the bob is $m(b - a\cos\psi)\dot{\varphi}^2$. This force assists the out-swing and opposes the in-swing.

4. Oscillations about a State of Steady Motion. For steady motion the angular velocity $\dot{\varphi}$ must be constant. But if $\dot{\varphi}$ is constant, the quantity $(b - a\cos\psi)$, according to (3), must be constant. Therefore ψ itself must have a constant value.

Let $\dot{\varphi} = \omega$, $\psi = \alpha$, where ω is constant and α is the corresponding constant value of ψ. Substituting these values in (3), we have

$$(b - a\cos\psi)^2\dot{\varphi} = (b - a\cos\alpha)^2\omega. \tag{10}$$

If we substitute $\dot{\varphi}$ from (10) into equation (1), we get

$$a\ddot{\psi} - \frac{(b - a\cos\alpha)^4 \sin\psi\cdot\omega^2}{(b - a\cos\psi)^3} - g\cos\psi = 0. \tag{11}$$

Since ψ is constant for steady motion, $\ddot{\psi}$ must vanish. Putting ψ equal to the constant α in equation (11), the first term drops out and we have

$$\omega^2 = \frac{-g\cos\alpha}{\sin\alpha(b - a\cos\alpha)}\,. \tag{12}$$

If the bob be given the angular velocity ω about the axis and if its inclination to the horizontal is the corresponding α, then the motion of the system if left to itself will be steady. The bob will then move on a circle, radius $(b - a\cos\alpha)$, with the constant angular velocity ω. Suppose that we start the bob with the velocity ω about the axis but at an inclination $\alpha + \xi$ differing very little from the angle α, the defining equation (11) will then become

$$a\ddot{\xi} - \frac{\omega^2(b - a\cos\alpha)^4 \sin(\alpha + \xi)}{[b - a\cos(\alpha + \xi)]^3} - g\cos(\alpha + \xi) = 0. \tag{13}$$

We take ξ so small that the second and higher order terms are negligible; the effect on ω due to this small change in α will also be neglected in our first approximation, *i.e.*, ω will retain its constant value as defined by equation (12). If we expand those functions occurring in equation (13) into a power series in ξ and neglect all terms beyond the first degree and replace ω by its value defined by equation (12), we find that

$$a\ddot{\xi} + \frac{g[b - a\cos\alpha - 3a\cos\alpha\sin^2\alpha]}{a\sin\alpha(b/a - \cos\alpha)}\xi = 0, \tag{14}$$

which we will briefly write in the form

$$\ddot{\xi} + k^2\xi = 0. \tag{15}$$

We will designate the second factor $b - a \cos \alpha - 3a \cos \alpha \sin^2 \alpha$ by v. If then we replace $\cos \alpha$ by u, we shall have the following equations:

$$\begin{aligned} v &= 3au^3 - 4au + b, \\ k^2 &= \frac{+ gv}{\pm a\sqrt{1 - u^2}(b/a - u)}, \\ \omega^2 &= \frac{- gu}{\pm a\sqrt{1 - u^2}(b/a - u)}. \end{aligned} \tag{16}$$

Geometrically, the uv curve defined by the first of (16) has its maximum point at $u = -2/3$ and its minimum point at $u = 2/3$. The ordinate is always positive for $-1 < u \leqq 0$. Its minimum value at $u = 2/3$ is $v = b - 16a/9$. For b equal to a this curve cuts the positive axis at $u = (\sqrt{21} - 3)/6$ and $u = 1$. Thus for $b < a$ the curve will cut the positive axis at two points, u_1 and u_2, where $u_1 < (\sqrt{21} - 3)/6$ and $u_2 > 1$. For $a < b < 16a/9$, there will still be two intercepts such that $(\sqrt{21} - 3)/6 < u_1 < 2/3$ and $2/3 < u_2 < 1$. So for $b = 16a/9$, v will be zero and for $b > 16a/9$, v will always be positive.

It should be observed that k^2 positive implies, according to equation (15), small oscillations about a steady state. When the deviation from the steady state is slight, the motion is termed *stable motion*. For k^2 negative there will be a great deviation from the steady state, or the motion will be *unstable*. Wherever the motion is stable a steady state is possible. For real stable motion ω^2 must of course be positive.

There are several cases to consider. In the second quadrant steady motion is always possible for every value of a and b. For here $-1 < u < 0$ in equations (16) gives $v > 0$, $k^2 > 0$, $\omega^2 > 0$, or the motion is stable.

In the third quadrant, on the other hand, whatever the values of a and b, steady motion is never possible. For in this quadrant the radical will carry the negative sign, thus reversing the signs of k^2 and ω^2. The motion is thus unstable.

Results not so physically evident may be obtained from the first and fourth quadrants. Here again the radicals in the first and fourth quadrants carry opposite signs. We can thus draw immediate conclusions for the fourth quadrant as soon as we have treated the cases arising in the first quadrant, for the signs of k^2 and ω^2 will be just reversed as before. This fact justifies our conclusions for the fourth quadrant in the following table based on equations (16), the signs of v, k^2, and ω^2 being given for the first quadrant only.

5. Summary. The results of this study for the first and fourth quadrants are given in the following table from which it appears that steady motion is possible only in limited regions of these quadrants and for restricted values of a and b.

TABLE.

CONDITIONS.		CONCLUSIONS.			MOTION.	
a, b	u	v	k^2	w^2	1st Quadrant.	4th Quadrant.
$b > \frac{16}{9}a$	$0 < u < 1$	+	+	−	Unstable	Unstable
$b = \frac{16}{9}a$	$0 < u < 1$					
	$u \neq \frac{2}{3}$	+	+	−	"	"
	$u = \frac{2}{3}$	0	0	−	"	"
$a \leqq b < \frac{16}{9}a$	$0 < u < u_1$	+	+	−	"	"
	$u_1 < u < u_2$	−	−	−	"	Stable
	$u_2 < u < 1$	+	+	−	"	Unstable
	$u = u_1, u_2$	0	0	−	"	"
$b < a$						
$b/a < u_1$	$0 < u < b/a$	+	+	−	Unstable	"
	$b/a < u < u_1$	+	−	+	"	"
	$u_1 < u < 1$	−	+	+	Stable	"
$u_1 < b/a < 1$	$0 < u < u_1$	+	+	−	Unstable	"
	$u_1 < u < b/a$	−	−	−	"	Stable
	$b/a < u < 1$	−	+	+	Stable	Unstable
$b < a$	$u = b/a$	−	∞	∞	Impossible	
	$u = 0$	−	$\pm$	0	Rest	
	$u = \pm 1$		∞	∞	Impossible	

THE CURVE WHOSE CURVATURE IS EVERYWHERE THE SAME AS THAT OF ITS PEDÁL.

By A. R. WILLIAMS, University of California.

The locus of the foot of the perpendicular from a fixed point, or "pole," on the tangent to a curve is called the pedal of that curve.

A fairly comprehensive account of the general properties of pedal curves is given by Gino Loria in Arts. 270–274 of his *Spezielle Algebraische und Transcendente Ebene Kurven.* The German translation is by Prof. Schütte and is published by Teubner. The results given fall generally into two classes: those that are concerned with the relation between the class, order, and singularities of a given curve and those of its pedal and antipedal, and those that have reference to the area of the pedal of an oval with respect to a variable pole, or which exhibit the relation between the area of the pedal of the oval and that of the roulette traced by the pole, when the oval, carrying the pole, rolls on a fixed curve. In this connection reference may be made to the *Integral Calculus* of Joseph Edwards, Art. 422 *et seq.*, Art. 444 *et seq.*, and Arts. 665 and 672 *et seq.*

It is worth noting, though no use will be made of it in this paper, that the operation of finding the pedal is equivalent to polarization with respect to the

unit circle followed by geometric inversion with respect to the same circle. As these operations are of period two, we find the antipedal by performing them in the opposite order. With the aid of this fact and the Plücker equations the characteristics of the pedal and the antipedal of an algebraic curve may be easily deduced from those of the given curve.

1. It is the purpose of this paper to find a curve whose curvature is the same as that of its pedal at corresponding points. A direct formulation of the problem in Cartesian or polar coördinates is of course simple, but leads to an unpromising differential equation of the second order. This difficulty is avoided by expressing the curvature directly in terms of the radius vector and the perpendicular on the tangent. Thus in a system of polar coördinates, let r denote the radius vector and p the perpendicular from the origin on the tangent. Then, if ϕ and ψ are the angles which the tangent makes with the fixed line and the radius vector respectively, we have

$$p = r \sin \psi; \qquad dp = dr \sin \psi + r \cos \psi d\psi = dr \frac{rd\theta}{ds} + r \frac{dr}{ds}(d\phi - d\theta).$$

Hence

$$\frac{dp}{dr} = r\frac{d\phi}{ds} \qquad \text{or} \qquad \frac{ds}{d\phi} = r\frac{dr}{dp} = \rho,$$

where ρ is the radius of curvature. Now in figure 1 let P be a general point of a given curve, P_1 the corresponding point of the pedal with respect to the origin, and P_2 the foot of the perpendicular from the origin on the tangent to the pedal at P_1. It may be easily shown that this tangent is simply the tangent at P_1 to the circle OPP_1. Hence the triangles OPP_1 and OP_1P_2 are similar, and $OP_2 \cdot OP = OP_1^2$; or $p_1 r = p^2$, (1), where $OP_2 = p_1$, $OP_1 = p$, and $OP = r$. The angle ψ between radius vector and tangent is the same for the original curve and its pedal. OP_1, or p (which we shall call later r_1), is the radius vector of the pedal and is equal to $r \sin \psi$. The corresponding vectorial angle P_1OX we shall call θ_1. We have $\theta_1 = \theta + \psi - (\pi/2)$. These are the fundamental relations that hold between any curve and its pedal. Moreover we have found that $\rho = r(dr/dp)$ is true of any curve. Hence we have at the corresponding point on the pedal

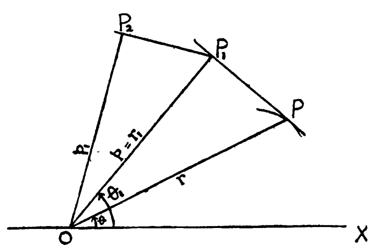


FIG. 1.

$$\rho_1 = p\frac{dp}{dp_1} = \frac{r^2}{2r - p\dfrac{dr}{dp}}$$

by means of the relation $p_1 r = p^2$. Hence the curve we seek is characterized by the relation

$$\frac{r^2}{2r - p\dfrac{dr}{dp}} = r\frac{dr}{dp}, \qquad \text{or} \qquad p\left(\frac{dr}{dp}\right)^2 - 2r\frac{dr}{dp} + r = 0. \tag{2}$$

An obvious solution, corresponding to the case of a circle, is $p = r$. The general solution is

$$4cr = (cp + 1)^2, \tag{3}$$

where c is the constant of integration. From (3)

$$r\frac{dr}{dp} = \frac{(cp+1)^3}{8c} = r\sqrt{cr} = \frac{1}{c}(cr)^{3/2}. \tag{4}$$

Now (3) is the relation between p and r on our curve. To obtain the corresponding polar equation we write

$$cp = cr\sin\psi = cr^2\frac{d\theta}{ds} = \frac{cr^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = 2\sqrt{cr} - 1,$$

whence we obtain

$$\pm\frac{dr}{d\theta} = \frac{r}{2\sqrt{cr}-1}\sqrt{c^2r^2 - (2\sqrt{cr}-1)^2} = \frac{(r\sqrt{cr}-1)}{2\sqrt{cr}-1}\sqrt{cr + 2\sqrt{cr} - 1}.$$

Separating the variables, and making the obvious substitution $\sqrt{cr} = z$, we are able to integrate. Returning to the variable r, we have

$$\pm\,\theta + k = 2\sin^{-1}\frac{\sqrt{cr}-1}{\sqrt{2cr}} - \sqrt{2}\log\frac{\sqrt{2cr} + \sqrt{cr + 2\sqrt{cr} - 1}}{\sqrt{cr}-1}\,. \tag{5}$$

The form of the curve is certainly not apparent. We may, however, omit the constant k, which only rotates the curve about the origin, and we note that

$$2\sin^{-1}\frac{\sqrt{cr}-1}{\sqrt{2cr}} = \cos^{-1}\frac{2\sqrt{cr}-1}{cr} = \cos^{-1}\frac{p}{r} \text{ by } (3);$$

that is, the angle between p and r. Now the polar equation of the pedal is simply the relation between r_1 and θ_1 where r_1 is p and θ_1 is the angle made by r_1 (*i.e.*, p) with OX, and is equal to $\theta \pm \cos^{-1} p/r$. So we infer that the equation of the *pedal* can be got directly from (4) by replacing θ by θ_1, omitting the inverse function, and expressing r in terms of r_1 (*i.e.*, p) as given by (3). Thus we get for the pedal

$$\pm\,\theta = -\sqrt{2}\log\frac{\sqrt{2}(cr_1+1) + \sqrt{c^2r_1^2 + 6cr_1 + 1}}{cr_1 - 1}\,. \tag{6}$$

We may easily verify this. From the fundamental relation $p_1 r = p^2$ which holds between any curve and its pedal, and from (3) which characterizes our curve, we have, by replacing p by r_1,

$$p_1 = \frac{4cr_1^2}{(cr_1+1)^2}, \tag{7}$$

which we may call the p and r equation of the pedal. As a check we note that

from the last relation we have on the pedal

$$\rho_1 = r_1\frac{dr_1}{dp_1} = \frac{(cr_1+1)^3}{8c} = \frac{(cp+1)^3}{8c} = r\sqrt{cr} = \rho$$

as given by (4). That is, the pedal and the original curve have the same curvature at corresponding points. To obtain from (7) the corresponding polar equation (of the pedal) in r_1 and θ_1, we write

$$p_1 = r_1 \sin \psi_1 = r_1^2\frac{d\theta_1}{ds_1} = \frac{r_1^2}{\sqrt{r_1^2 + \left(\dfrac{dr_1}{d\theta_1}\right)^2}} = \frac{4cr_1^2}{(cr_1+1)^2}\,.$$

Removing the factor r_1^2, and integrating as before, we obtain (6) as the polar equation of the pedal.

2. If we can determine the form of the pedal, we can of course easily pass back to the original curve. Equation (6) is at least free of an inverse trigonometric function, and fortunately a very marked further simplification is possible. From (6), using the lower sign before θ_1, we get

$$(cr_1 - 1)e^{(\theta_1/\sqrt{2})} - \sqrt{2}(cr_1+1) = \sqrt{c^2r_1^2 + 6cr_1 + 1}.$$

Squaring, and removing the factor $cr_1 - 1$, which evidently corresponds to a circle, we have

$$cr_1 = \frac{e^{\sqrt{2}\theta_1} + 2\sqrt{2}e^{(\theta_1/\sqrt{2})} + 1}{e^{\sqrt{2}\theta_1} - 2\sqrt{2}e^{(\theta_1/\sqrt{2})} + 1} = \frac{e^{(\theta_1/\sqrt{2})} + e^{(-\theta_1/\sqrt{2})} + 2\sqrt{2}}{e^{(\theta_1/\sqrt{2})} + e^{(-\theta_1/\sqrt{2})} - 2\sqrt{2}} = \frac{\cosh(\theta_1/\sqrt{2}) + \sqrt{2}}{\cosh(\theta_1/\sqrt{2}) - \sqrt{2}}. \tag{8}$$

Since this last is unchanged when we replace θ_1 by $-\theta_1$, we see, as we would expect, that we do not need to consider the double sign before θ_1 in (5). From (8) we can easily see the form of the pedal curve. As remarked it is symmetric with respect to OX. cr_1 becomes infinite when $\cosh(\theta_1/\sqrt{2}) = \sqrt{2}$, *i.e.*, when $\theta_1 = \pm 1.246$ radians. For those values of θ_1 the polar subtangent, $r_1^2(d\theta_1/dr_1)$, is finite, and equal to $\mp 4/c$. Hence the curve has asymptotes, parallel to $\theta_1 = \pm 1.246$, and at a distance $4/c$ from the origin. As θ_1 becomes infinite, cr_1 approaches 1; that is, the curve winds about the circle $cr_1 = 1$. We see from (3), (5) and (8) that the constant c is simply equivalent to a change of unit of length in r and r_1; that is, we may put $c = 1$ and still get all the curves by varying the unit of length. Taking $c = 1$ we note that when $\theta_1 = 0$, $r_1 = -5.83$. As θ_1 increases (positively),

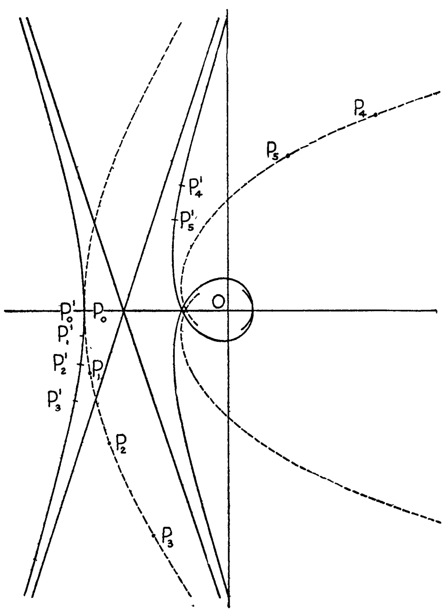


FIG. 2.

r_1 increases (negatively), becomes infinite for $\theta_1 = 1.246$, then changes sign, decreases very rapidly, and winds upon the unit circle. We have the points $(11.53, \pi/2)$, $(1.87, \pi)$, $(1.224, 3\pi/2)$, and $(1.068, 2\pi)$.

The form of the original curve, or antipedal of the pedal curve, is now easily obtained. Draw the radius vector OP_1 to any point on the pedal, and OP_2, the perpendicular from O on the tangent to the pedal at P_1. At P_1 draw the perpendicular to OP_1. Then OP and OP_2 make equal angles with OP_1 on the opposite sides of OP_1 (see Fig. 1). In our case we know that $OP = (OP_1 + 1)^2/4$, and that θ differs from θ_1 by $\cos^{-1}(OP_1/OP) = \cos^{-1}[4OP_1/(OP_1 + 1)^2]$ which rapidly approaches 0 as OP_1 (*i.e.*, r_1 or p) approaches 1. The original curve is of course symmetric with respect to OX. r becomes infinite when r_1 does in a direction perpendicular to the corresponding direction of r_1; that is, the line at infinity is twice tangent to the curve, and there are no proper asymptotes. In the graph the pedal is drawn solid; its antipedal, or the original curve, is the dotted line. $P_0'\,P_0$, $P_1'\,P_1$, $P_2'\,P_2$, etc., are pairs of corresponding points on the two curves. The original curve goes off to infinity along P_1P_2 and returns along P_3P_4. Both curves wind asymptotically around the unit circle and soon become indistinguishable. Recalling from (4) that on the original curve $\rho = r\sqrt{cr}$, and that a change of c is simply equivalent to a change of the linear unit, we may state our result in the following

THEOREM. *The curve whose curvature is the same as that of its pedal at corresponding points is a transcendental spiral, having an axis of symmetry, winding asymptotically about a circle whose center is the pole of pedalization, and whose radius of curvature, in terms of the radius of the asymptotic circle as unity, is everywhere equal to the three halves power of the radius vector drawn from the pole.*

Similarly a curve for which $\rho = (r + 1)^3/8$ has the same curvature as its *antipedal.* It is easily shown that the pedal of a logarithmic spiral is another logarithmic spiral, which, however, does not have the same curvature at corresponding points. As a check on these results we may note that it is easy, starting from the equation of the pedal curve (8), to derive the equation of the original curve from the relations $\theta_1 = \theta + \psi - (\pi/2)$ and $r = r_1 \csc \psi_1$, and show that the two have the same curvature. Also it is possible on our curve by integrating $ds = \sec\psi dr$, and using the relation $\rho = r\sqrt{cr}$, to obtain a direct relation between s and ρ, *i.e.*, an *intrinsic* equation. We can do the same for the pedal curve, but the result is not illuminating in either case.

Finally, we note that we have an illustration of the fact that two curves may be in 1 : 1 correspondence, and have the same curvature at corresponding points, without being congruent, or even similar. The ratio of ds on the one to ds on the other is variable. Thus, denoting the elements of the pedal by subscripts we have

$$ds = \frac{ds}{dr}dr = \sec\psi dr = \sec\psi_1 dr = \frac{ds_1}{dr_1}dr = \frac{dr}{dr_1}ds_1 = \sqrt{cr}ds_1 = \frac{cr_1 + 1}{2}ds_1.$$

The effect of this appears on comparing in the graph the arc lengths between pairs of corresponding points, *e.g.*, $P_1'\,P_2'$ with $P_1\,P_2$, or $P_3'\,P_4'$ with $P_3\,P_4$.

THE SOLUTION OF CERTAIN PROBLEMS IN FINANCE BY THE METHOD OF ITERATION.

By C. H. FORSYTH, Dartmouth College.

1. Introduction. The use of the method of iteration[1] in finding approximations to roots of rather complicated equations is far from new and the writer has been surprised to find no mention of it in any of the textbooks on finance. The writer has tried various methods (interpolation, binomial expansions, Newton's method, etc.) in his courses in finance, but both he and his students have consistently preferred the method of iteration. In this paper it will be applied to the determination of the investment rate in annuities and bonds in order to call the attention of readers of this journal to its peculiar fitness in such problems.

2. Application to Annuities. Since we are concerned primarily with the method we shall restrict our consideration first to the type of annuity which occurs most frequently: the one whose interval of payment coincides with the interval of conversion of interest. The equations of value to be considered resolve themselves then essentially to one of the two forms:

$$S = \frac{(1+i)^n - 1}{i} \qquad \text{and} \qquad A = \frac{1-(1+i)^{-n}}{i},$$

where i is the rate of interest.

Obviously neither of these equations can be solved explicitly for i. If, however, we distinguish between each of the two i's in each equation, each equation can be expressed in two ways as follows:

$$i = \frac{(1+i)^n - 1}{S}, \quad (A) \qquad i = (1 - iA)^{-1/n} - 1, \quad (A')$$

$$i = (1 + iS)^{1/n} - 1, \quad (B) \qquad i = \frac{1-(1+i)^{-n}}{A}. \quad (B')$$

Supposing now that i is the sole unknown, we can proceed to substitute a reasonably close trial value on the right-hand side to obtain a new value of i. It will be found that if equations (B) and (B') are used this new value of i will always prove to be a closer approximation to the value sought than the value tried, regardless of whether the trial value is too large or too small; and that if equations (A) and (A') are employed the values obtained will always prove divergent. If the proper equation is employed, then it merely remains to repeat or *iterate* the process—trying each new approximation in turn—until a value is obtained which reproduces itself.

[1] See Whittaker and Robinson, *Calculus of Observations*, pp. 79–94, for a general account of the method and various modifications of the method; also a good bibliography.

In iteration in general a given equation is broken up into two parts, of which one is quite frequently, but *not necessarily,* an expression of the first degree, and the process consists in substituting successive approximations alternately in the two equations but *beginning* with the one whose graph has the slope of lesser magnitude. The scheme suggested here for the solutions of annuities is equivalent to substituting first in the equation $y = f(i)$, where $f(i)$ denotes either (B) or (B'), and then (in all cases) in the equation $y = i$. In such applications then this secondary graph is always a straight line and the same straight line—the bisector of the first quadrant. It so happens, however, that the slopes of the graphs of (B) and (B') are positive throughout (as the derivative will show) but less than unity, the reverse is true in each case of the graphs of (A) and (A'). The essential forms of these graphs are shown below. There should be no difficulty in recognizing the so-called *staircase* representation of the process—convergent toward the point of intersection (or the desired root) in the case of equations (B) and (B'), and divergent in the other case.

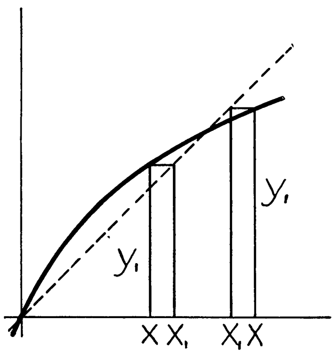


FIG. 1.

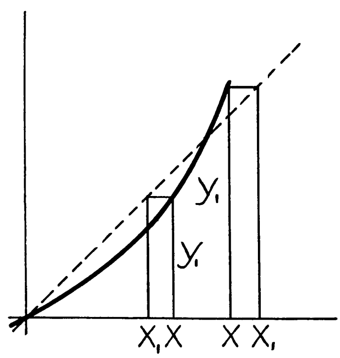


FIG. 2.

In most courses in finance the calculus would not be available but there still remain at least three methods of identifying the proper equation to be used: (1) by plotting the graph using values taken from the tables; (2) by verifying by actual application; and as a special case of the latter method, (3) by trial of a value which is both absurd and yet susceptible of mental substitution. The latter method is used rather to disqualify an equation. As an example, the substitution of the absurd value 9 for i in the right in equation (A) will obviously yield a value of iS of n digits! The same substitution in equation (A') leads to either imaginary or negative results.

3. Application to Bonds. The scheme suggested here works as well for determining the investment rate of a bond of given price but in this case we must choose between three equations and the graphs are a little more interesting or complicated—according to the point of view.

The formula for the premium or discount per dollar of the ordinary bond is

$$k = a_{\overline{n|}}(g - i) \qquad \text{(at rate } i\text{)},$$

where, in case the dividend (at rate g) and the interest (at rate i) on the investment are to be converted m times a year, we need merely to change the unit of time from a year to the conversion period (making mn such periods), and the essential form of the equation given above is unaltered.

As i appears in three separate places in the above formula, the latter may be written in either of the three forms:

$$i = g - \frac{k}{a_{\overline{n}|}} = g - \frac{ki}{1 - (1 + i)^{-n}}, \tag{1}$$

$$i = \{1 - (1 + i)^{-n}\} \frac{g - i}{k}, \tag{2}$$

$$i = \left\{ (1 + k) - \frac{gk}{g - i} \right\}^{n} - 1. \tag{3}$$

The important parts of the graphs of these equations (and of $y = i$) would appear essentially as follows:

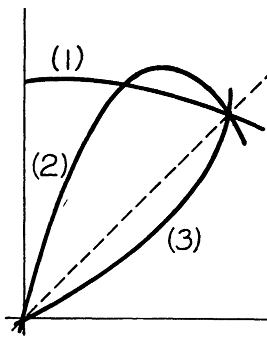


FIG. 3.

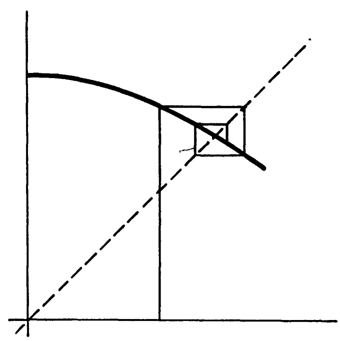


FIG. 4.

The graphs of both equations (2) and (3) have slopes which exceed unity (the slope of $y = i$). Hence these equations can be eliminated at once for purposes of iteration. Equation (1) proves to be satisfactory.

It is well known that when the slopes of the graphs of the two equations used in the process of iteration have opposite signs the staircase representation is replaced by a so-called *spiral* representation, as shown in Fig. 4, where, of course, the substitutions begin, as before, with the equation whose slope is of lesser magnitude.

The method of iteration could, no doubt, be modified to advantage wherever the derivative could be easily obtained, and in the case of annuities no great difficulty is met; but in the case of bonds such modifications lead to serious difficulties due to the necessity of taking derivatives of quotients. As a general method in finance then the writer prefers that outlined here to any other—and especially to the irksome interpolations and binomial expansions employed in current textbooks on finance. The writer's preference is influenced greatly also

by the advantages of (*a*) directness of method, (*b*) the ultimate check, and (*c*) the effective drill in the use of logarithms.

One of the most important steps in the practical application of the process of iteration is the utilization of the successive possibilities of anticipation of closer approximations. After all, the process insures merely successive closer approximations which ultimately converge to the desired value, and as we all know, such convergence is frequently exasperatingly slow. In any case, the use of an arithmetic mean of two approximations will frequently shorten the process.

We shall conclude with a simple example of an annuity to illustrate both the possibilities of shortening the process and also a suggested arrangement of the successive approximations which will help to center the attention upon the location of the desired value.

Example: Given that \$12,102 is the present value of an annuity of \$1,000 per year for twenty years; to find the corresponding rate of interest.

\$12.102 is therefore the present value of an annuity of \$1 per year for twenty years and the use of the tables shows that the rate is between 0.05 and 0.06. Employing equation (B') and using arrows to indicate substitutions, the work can be outlined as follows:

$$.05 \rightarrow .05149 \qquad\qquad .06 \rightarrow .05687$$
$$\text{Average} = .05418,$$
$$.053 \rightarrow .05322 \qquad\qquad .054 \rightarrow .05377$$
$$\text{Average} = .0535 \rightarrow .05349 \rightarrow .05349.$$

And the process could be carried on to give additional places if the table of logarithms justifies.

HENRY BRIGGS AND HIS WORK ON LOGARITHMS.

By A. J. THOMPSON, London, England.
(Communicated by KARL PEARSON, University of London.)

The year 1924 was the tercentenary of the publication of the first great table of logarithms, the *Arithmetica Logarithmica*, by Henry Briggs. It contained the logarithms, to fourteen decimal places, of all numbers from 1 to 20,000, and from 90,000 to 100,000. The Biometric Laboratory of University College, University of London, is celebrating the tercentenary by commencing the issue of *Logarithmetica Britannica*, a standard table of logarithms, which, when completed, will give the logarithms, to twenty decimal places, of all numbers up to 100,000. The first portion of this table,[1] which contains the logarithms of

[1] *Logarithmetica Britannica*, being a Standard Table of Logarithms to 20 decimal places, by Alexander John Thompson. Part IX (the first to be published), Numbers 90,000 to 100,000. Cambridge University Press, 1924. The work is intended to be completed in nine parts. Subscriptions are invited at the rate of 10 shillings per part, payable on publication. Intending subscribers should send their names to Professor Karl Pearson, F.R.S., University College, London, W. C. 1.

numbers from 90,000 to 100,000 with second and fourth differences, has already been published.

Henry Briggs was born about 1556 at Warley Wood, a hamlet near Halifax. He showed signs of mathematical ability at an early age, and proceeded to St. Johns College, Cambridge, in 1579. He took the degree of B.A. in 1581, of M.A. in 1585, and became a Fellow of his College in 1588. He remained there until 1596, when he became the first Reader in Geometry at Gresham College in London.

In 1614, Napier's first work on logarithms was published, the *Mirifici Logarithmorum Canonis Descriptio.* This book came into Briggs's hands soon afterwards. He began to read it with interest, but, by the time he had finished, his interest was changed into enthusiasm. The book was his constant companion: he carried it with him when he went abroad; he conversed about it with his friends; and he expounded it to the pupils who attended his lectures. He soon perceived that the system of logarithms which would be described, in modern language, as having 10 as base, would be much more convenient than Napier's. He described this improvement to his classes, and as soon as his courses of lectures for the session of 1614–15 were ended, he travelled to Edinburgh and discussed it (among many other subjects) with Napier. Briggs remained for a month as Napier's guest, and, on his return to London, busied himself in calculating logarithms according to the new plan. In 1616 he again visited Napier, taking with him the calculations he had made. The results of these calculations were printed in 1617, for the benefit of his personal friends, as *Logarithmorum Chilias Prima.* In this rare brochure were given the logarithms of the first 1,000 numbers, to 14 decimal places. Specimen pages are reproduced as a frontispiece to the table now being published.

In 1619, Briggs became the first Savilian Professor of Geometry at Oxford. He settled at Merton College, and resided there for the remainder of his life. He continued to carry on his computing and, in 1624, after a labor extending over a period of about eight years, produced his *Arithmetica Logarithmica.* This work contains the logarithms of 30,000 numbers, with first differences. It is accompanied by a masterly introduction, in which the construction of tables, interpolation by means of differences, and other matters of the greatest importance were dealt with for the first time. Although this work is now very rare and costly, it is said that the edition of the tabular portion was too large and that surplus copies were hawked in the streets of London at eighteenpence each. After the publication of this great book, Briggs set to work, with the help of a few friends, to fill up the large gap of 70,000 logarithms which had been left. He had almost completed them when, in 1628, Adrian Vlacq, a Dutchman, published the logarithms of the first 100,000 numbers to 10 decimal places in a book which he also called *Arithmetica Logarithmica.* Although Vlacq had only copied 30,000 logarithms from Briggs's book (cutting them down from 14 to 10 decimal places) and had calculated 70,000 himself, he described his work merely as a second edition. Briggs may have felt some disappointment at the way in

which he had been forestalled by Vlacq; he seems, however, to have cherished no resentment, but on the contrary to have felt relieved that the burden of printing 70,000 logarithms had been removed from his shoulders. In such circumstances, most men would have given up computing; but not so Briggs. At the age of 70 or thereabouts he hastened to commence another great work, the logarithms of the trigonometrical functions, and had almost completed it at the time of his death. Vlacq printed these logarithms at his own expense and published them in 1633 under the title of *Trigonometria Britannica.*

Henry Briggs died on the 26th of January, 1631. He was buried with great pomp in the chapel of Merton College, but his name only was inscribed on his tomb-stone.

No complete reprint of Briggs's great table of the logarithms of numbers has ever been made; and, up to the present year, only two 10-figure tables have been published since Vlacq produced his table in 1628. The need for an extended table has long been felt, and the present work is intended to meet this need.

No more fitting tribute than this could be paid to the memory of a great and good man on the tercentenary of his greatest work.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. GENERALIZED CONVERGENCE WITH BINARY RELATIONS.

By A. A. BENNETT, University of Texas.

The term "limit" does not of course apply to a single independent variable, but requires a correspondence between an independent and a dependent variable for it to have a meaning. The two variables do not figure in the same manner, although they enter almost analogously. Thus the notation $\lim\limits_{x\to a} y = b$ could be written in the somewhat more symmetrical if less convenient form as $y\,|_{x\to a} \to b$. It might at first appear that at least the existence of the a is an essential feature of the relation. On the other hand, the expression $\lim\limits_{n\to\infty} c_n = c$ reminds us that when the limit is a sequential one the independent variable is a natural number while the value approached by this variable in determining the limit is denoted by ∞ but is not a natural number. The formulation in this case can be so made that ∞ is never mentioned. Generalizing this and other more interesting types of limits, Prof. E. H. Moore has developed "a general theory of limits," of which a thoroughgoing discussion has been published under this title.[1] In this treatment the dependent variable is numerical throughout. For the independent variable there is introduced a binary relation of a very general sort

[1] E. H. Moore and H. L. Smith, *Am. Journ. Math.* (44), 1922 (102).

which in particular renders the existence of and even the reference to the a mentioned above unnecessary. In the paper referred to all the more important classical theorems on limits are stated and proved for this generalized definition of limit. Had the range of the dependent variable also been so generalized, some of the familiar theorems would have been rendered meaningless or incapable of expression, which is ample reason for confining attention to numerical functions. However the same process could obviously be applied for the dependent variable also. In fact the study of expressions "converging toward infinity" and of improper limits in general would suggest such a procedure. A fairly inclusive generalization is here made.

Let A be a set of elements $[a]$. Let R be a *binary* relation, that is, a relation between elements of an ordered pair selected from A. Thus given two elements, a_1 and a_2, the question is assumed to be definite and its answer determined completely by the two elements chosen in order, namely, "Does a_1 have the relation R to a_2?" In case that this relation does hold, we may use as a proposition the expression a_1Ra_2 (read "a_1 is in the R-relation to a_2"). We shall further suppose that although R applies among elements of A, it is also more extensive and there may be for example such an element as e not in A which is such that for some a, eRa.

The relation R will be required to be *transitive* in A, that is, if a_1, a_2 and a_3 are elements of A such that a_1Ra_2 and a_2Ra_3, then a_1Ra_3. Among the familiar transitive binary relations for numbers may be mentioned those denoted by $=, >, <, \geq, \leq$, while $\neq$ is not transitive.

The relation R will be required to be *antireflexive* in A, that is, it will be assumed that there is no element a of A for which aRa. This apparently stringent condition is frequently satisfied, or can be made to be satisfied by a minor modification of A or R. By a more detailed statement of the condition of convergence to be discussed this limitation could be avoided.

The relation R will be required to be *compositional*,[1] that is, for every two (not necessarily distinct) elements a_1 and a_2 of A, it is required that there exist an element a_3, of A, such that simultaneously a_3Ra_1 and a_3Ra_2. It is to be noted that while the other properties mentioned for R hold equally for the inverse relation in each case, the compositional property is unsymmetrical with respect to righthand and lefthand members.

We shall assume further the existence of a set B and of a relation S for which statements analogous to those for A and R hold throughout.

Before proceeding further we shall show that A cannot contain a fixed "limit element" a', such that for every element a, of A, $a'Ra$. For if possible suppose that such an element a' existed. On account of the compositional property (using a' twice) there must be an element of A, say a'', such that $a''Ra'$. Now a'' must be distinct from a' by the antireflexive condition, and hence by the hypothesis $a'Ra''$. By the transitive property it follows from $a'Ra''$ and $a''Ra'$

[1] This term is here used in the sense defined for the composition property in the article referred to above. The terms "binary," "transitive" and "reflexive" are familiar in logical discussions.

that $a'Ra'$, which is impossible by the antireflexive condition. However there may or may not be an element a' outside of A such that, for every a of A, $a'Ra$.

Let X be a set including A as a subset, and let x be a variable of X. We shall take x to be the independent variable in the present discussion. Similarly let Y be a set including B as a subset. Let $y(x)$ be a function of x, all of the values of $y(x)$ being in Y. The function $y(x)$ is said to *converge* with respect to the range A, and the relations R and S, if and only if the following condition [1] is satisfied: *Given any b of B, there exists an a of A, such that for each x in A for which xRa, it is true that $y(x)Sb$.* It is to be noted that $y(x)$ is not necessarily confined to B. If we insist upon inquiring as to what it is that $y(x)$ converges toward, we may answer that it converges toward a limit element d in Y but not in B, and which is such that for each b of B, dSb.

For the most familiar case of $\lim_{x \to c} y = d$, the set A is the set of all (finite) numbers (real or complex as the case may be) exclusive of c, and B is the similar set where d is excluded. Here xRa means $|x - c| < |a - c|$ and ySb means similarly that $|y - d| < |b - d|$. Since B does not include d, $|b - d|$ is greater than zero and may be called ϵ if desired, while δ may be similarly defined as the positive quantity $|a - c|$. The condition then reads in the familiar fashion: For every ϵ, > 0, there exists a δ, > 0, such that for every x, $\neq c$, for which $|x - c| < \delta$, it is true that $|y - d| < \epsilon$. The relative economy of the generalized statement may be appreciated.

That the formulation of the convergence condition in such a manner that the independent variable need not be numerical and so that the compositional property may be used in its fulness is not without significance can be observed in the problem of defining the Riemann integral. Suppose for a given interval of the real axis a one-valued real function has been defined. For each particular subdivision of the interval suppose an approximating polygon defined. We are interested in the area of the polygon as a function of the subdividing of the interval. For certain special methods of successive subdivision, the area becomes a function of a finite ordinal number indicating the particular step in the process; in the general case, however, there is no assigned sequence in the process of subdivision. If a_1 denotes one subdividing of the interval, and a_2 another, we use the fact that there is a subdividing a_3 among whose points of subdivision occur all those of a_1 and also all those of a_2, the simplest choice of this a_3 being obtained by mere superposition of the interval as subdivided in one way upon the interval as subdivided in the other.

That there is also elementary use for a non-numerical function is seen by reference to the question of uniform convergence, which we will take with respect to a scale function. Consider a set of functions $f_n(x)$ where for our purposes the independent variable is n. Assume a limit function $F(x)$ and a scale func-

[1] Only the usual fundamental convergence criterion is here discussed, and Cauchy's and other such forms will not be mentioned further except to note here that while in many of these other forms no explicit mention of the limit of the dependent variable is made, the use here to be made of the binary relations avoids explicit use of limits even in this fundamental case.

tion $s(x)$. If for every positive ϵ there is a natural number m_ϵ so large that $|f_n(x) - F(x)| < \epsilon s(x)$, for all values of x on the interval in question, whenever n exceeds m_ϵ, then the set of functions is said to converge uniformly with respect to the scale function. This comes directly under our theory with the dependent variable as a correspondence and not as a number.

A discussion of further technical features and the enumeration of other examples will not be appropriate here.

II. A Method for Solving the Cubic.

By Orrin Frink, Jr., Brooklyn.

Consider the equation

$$x^3 + px + q = 0. \tag{1}$$

To complete the cube, write (1) in the form

$$x^3/8 + (3/2)xy^2 = - q/2, \tag{2}$$

where

$$y^2 = x^2/4 + p/3. \tag{3}$$

Let

$$\tfrac{3}{4}x^2y + y^3 = R. \tag{4}$$

By adding (2) and (4) and taking the cube roots,

$$x/2 + y = \sqrt[3]{- q/2 + R} \quad (1, \omega, \omega^2), \tag{5}$$

and by subtracting (2) and (4),

$$x/2 - y = \sqrt[3]{- q/2 - R} \quad (1, \omega, \omega^2). \tag{6}$$

Addition of (5) and (6) now gives the unknown x in terms of q and R; while multiplication of (5) and (6) shows, by means of (3), that $- p/3 = \sqrt[3]{q^2/4 - R^2}$, or

$$R = \sqrt{q^2/4 + p^3/27}.$$

This solution can be used conversely to extract the cube root of such an expression as $9 + 25\sqrt{-2}$ whenever the root can be exactly taken.

Thus, with a view to (6), let $x/2 - y = \sqrt[3]{9 + 25\sqrt{-2}} = \sqrt[3]{- q/2 - R}$. Here $q = - 18$, $- p/3 = \sqrt[3]{9^2 - (25\sqrt{-2})^2} = \sqrt[3]{1331} = 11$.

We can now form equation (1), since we know p and q, obtaining

$$x^3 - 33x - 18 = 0.$$

We find that this has the integral root 6, so that $y = \sqrt{x^2/4 + p/3} = \sqrt{9 - 11} = \sqrt{-2}$; and $x/2 - y = 3 - \sqrt{-2}$.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Die Geburt der Modernen Mathematik. I. Analytische Geometrie. By HEINRICH WIELEITNER. Karlsruhe in Baden, G. Braun, 1924. 60 pages. Price 1 Goldmark.

This little book is No. 12 of the valuable series of monographs issued under the attractive and significant title of "Wissen und Wirken." The purpose of the series is to popularize knowledge through inexpensive booklets written by men of high standing as scholars and possessed of the happy faculty of presenting serious subjects in the everyday language that people of fair education can understand and enjoy reading. It is a series of particular value to teachers and students, and for those who read German the entire set would form a valuable nucleus for a working library.

As to Dr. Wieleitner's contribution, it should first be said that he is one of the best-known historians of mathematics in Germany at the present time. He is well equipped on the side of scholarship and is able to present his subject in a way that will appeal to the average reader who cares to know the origin of the various branches of mathematical science.

The work is divided into seven chapters relating, respectively, to (1) the birth of modern science, (2) the essential features of analytic geometry, (3) the Greek mathematics, (4) the development of algebra, (5) analytic geometry as developed by Fermat, (6) the contribution of Descartes, and (7) the general outlook.

The presentation being professedly and necessarily merely a sketch of the historical development of the subject, it is hardly necessary to speak further of the details. The reader of the monograph will be interested to learn that the first use of the term "analytic geometry" in the present sense is much later than he would expect, being first found in a work by Lacroix at the close of the 18th century. The most valuable chapter for most readers, however, will be the one on the contribution of Fermat, particularly for those who think of Descartes as the sole inventor of this branch of mathematics.

It would interest this reviewer to know Dr. Wieleitner's authority for pronouncing Oughtred's name as if it were spelled Utred. The nearest approach to historical certainty in the matter would seem to lie in the fact that some of his contemporaries spelled the name Owtred, from which we may reasonably infer the pronunciation.

It is to be hoped that similar popular series, issued at a nominal price, may soon become possible in this country.

The effect of making available for our teachers a set of booklets in paper covers, at twenty-five cents a volume, and written by those who know their subjects and can tell their stories in an interesting fashion, would be most salutary.

DAVID EUGENE SMITH.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, no. 8, October, 1924: "Invariance of the Poincaré numbers of a discrete group" by O. Veblen, 405–406; "Analytic functions and periodicity" by J. F. Ritt, 406–409; "A convergence proof for simple and multiple Fourier series" by M. G. Carman, 410–416; "On certain topics in the mathematical theory of statistics" by H. L. Rietz, 417–453.

PHILOSOPHICAL MAGAZINE, volume 48, no. 288, December, 1924: "The principle of equivalence in the theory of relativity" by T. Y. Thomas, 1056–1068.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 11, November, 1924: "Concerning sets of segments which cover a point set in the Vitali sense" by R. L. Moore, 464–466.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 9, December, 1924: "The supplementary project in mathematics" by C. A. Stone, 905–912; "A new number system" by H. C. Christofferson, 913–916; "Shall we mathematize or demathematize high school physics?" by J. M. Hughes, 916–921; "Probabilities" by J. M. Synnerdahl, 922–935; "Geometry, a laboratory science" by J. A. Nyberg, 948–953, by F. H. Sutton, 954–956, by J. O. Pyle, 956–957.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

MU THETA EPSILON OF THE UNIVERSITY OF CALIFORNIA, Berkeley, California.

Mu Theta Epsilon, women's mathematics honor society, was organized in April, 1920. The purposes of the society are: to stimulate interest in mathematics, and to encourage better work in the same; to discuss mathematical topics; and to promote congeniality among students and closer coöperation between students and professors. Members are elected by the society from the upper-division and graduate women majoring in mathematics, who have been approved by professors of the department. The professors are honorary members except that we have chosen Dr. Pauline Sperry as our one active faculty member. The average attendance during the past semester was twenty members.

The officers for the present college year are: Dorothy Godward '24, pres.; Dorothy Scott '24, vice-pres.; Veronica Satorius '24, sec.; and Mary Sweeney '24, treas.

This semester we inaugurated the plan of holding a second meeting each month, to be devoted to open discussion of a question or topic which has been announced at a previous meeting. The plan has hardly had a fair trial but promises to be a success.

The programs for the past two years and up to date are as follows:

Sept. 6, 1922. "Digit method among the ancients"—Miss Muriel Wilkinson. "Practical mathematics in surveying"—Miss Agusta Wellman.

Oct. 4, 1922. "Magic squares"—Miss Isabel Smith. The new members gave five minute speeches as follows: "Correlation of mathematics to foreign language"—Dorothy Godward. "Famous monkey problem"—Helen Growe. "Contributions of mathematics to world problems"—Florence Breed.

Nov. 1, 1922. "An interesting fourth century table"—Miss Thelma Baker. "Line coördinates"—Dr. Elsie McFarland.

Jan. 17, 1923. "The Einstein theory"—Miss Ruth van Pelt. "Time"—Miss Mary Shafer.

Feb. 7, 1923. "Applied mathematics in physics"—Mrs. Aylesworth.

Mar. 7, 1923. "Proof of the binomial theorem"—Miss Mary Scoffield. "Report of research on Huntington's first set of postulates for fields"—Miss Verna Jeffery.
April 4, 1923. "History of definitions in mathematics"—Miss Josephine Brubaker. A review of the current AMERICAN MATHEMATICAL MONTHLY—Miss Ruth Pearson.
Sept. 5, 1923. Review of the MONTHLY for April–May—Miss Igerna Hurd. "Mathematical instruction in the secondary schools"—Miss Helen Growe.
Oct. 3, 1923. "The value of mathematics in the development of the reasoning powers of the individual"—Margaret Harper. "Amicable numbers"—Edith Henderson. "Relation of mathematics to fine arts"—Gwenyth Springsteen. "An interesting problem"—Elizabeth Lange. These five minute speeches by the new members were followed by "A mathematical romance"—Dr. Elsie McFarland, "Astronomy and mathematics"—Miss Muriel Wilkinson, and "The history and origin of Mu Theta Epsilon Society"—Mrs. Aylesworth.
Nov. 7, 1923. Problems from Ball's *Mathematical Recreations*—Miss Veronica Satorius. "Theory of factoring numbers"—Miss Florence Raphael.
Feb. 1, 1924. Initiation and Banquet.
Feb. 6, 1924. "The relation of vector analysis to quaternions"—Miss Dorothy Godward.
Mar. 5, 1924. "Relation of quaternions and vectors"—Miss Ethel Barnabey.
Mar. 15, 1924. Informal entertainment for the mathematics faculty and all senior and graduate students majoring in mathematics.
April 2, 1924. "The psychology of mathematics"—Miss Beatrice Conley. "On the porism of four tangents to a twisted cubic"—Miss Vesta Sanger.
Sept. 3, 1924. "The fundamental assumptions of geometry"—Miss Margaret Harper.
Oct. 1, 1924. "Conformal mapping and its applications"—Miss Ethel Neily. "The solution of a problem in digits"—Mrs. Esther Bell. Amusing numbers on the program were: "A mathematical romance"—Dr. Elsie McFarland, and "Possibilities in teaching mathematics"—Miss Mary Shafer.
Oct. 22, 1924. Open discussion of the practical applications of mathematics.
Nov. 12, 1924. "Finite geometry"—Miss Edith Rockwell. "General displacements in space"—Miss Mary Sweeney.

(Report by Miss Neily.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3120. Proposed by V. M. SPUNAR, Chicago, Illinois.

If ρ_1 and ρ_2 be the greatest and least radii of curvature of a curved surface at a given point, φ and ψ, the angles which the normal to the surface at the given point makes with the axes of x and y, show that

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{d}{dx}(\cos\varphi)+\frac{d}{dx}(\cos\psi).$$

3121. Proposed by A. A. BENNETT, University of Texas.

Find the smallest positive integer x, not a multiple of 31, such that x^2+1 shall have two factors whose difference is divisible by 31.

3122. Proposed by A. A. BENNETT, University of Texas.

Let $P(n)/Q(n)$ denote the fraction reduced to its lowest terms which represents the minimum value of $\varphi(m)/m$, for $0 < m \leqq n$, where $\varphi(m)$ is the indicator or totient of m. Show that $P(n)$ is of the form $2^a 3^b$ for every $n \leqq 2 \cdot 10^{11}$.

3123. Proposed by B. F. FINKEL, Drury College.

A circular hole, radius r, in the bottom of a flat-bottomed water-tank is covered with a weightless spherical rubber shell, radius R. Water is then poured into the tank to the depth h. What is the ratio of R to r when the shell is just on the point of rising?

3124. Proposed by M. B. PORTER, University of Texas.

$f(x)$ and $\varphi(x)$ are polynomials and all the zeros of $\varphi(x)$ are real. Let $P(x) = \varphi(x)[f^{(i)}(x)]^r$ where $f^{(i)}$ stands for the ith derivative of $f(x)$.

Prove that $[P'(0)]^2 < P''(0)P(0)$ (1) is a sufficient condition that $f(x)$ has imaginary zeros. Newton's test [1] $C_i^2 \leqq C_{i-1}C_{i+1}$ where C_i is the coefficient of x^i in $f(x)$ is a special case of (1).

SOLUTIONS.

3067 [1924, 148]. Proposed by J. H. MURPHY, Pittsburgh, Pa.

On the base of a right triangle whose altitude, a, is greater than the base, b, is constructed a triangle whose vertex angle is α. What are the lengths of the two variable sides of this triangle when that part of its area outside of the right triangle is a maximum?

SOLUTION BY HARRY LANGMAN, New York City.

Let CA be the base of the triangle ABC, right-angled at C. We have given $CB = a$, $CA = b < a$. Take $AB = c$, and put $b = ea$. Hence $e < 1$.

Let P be the third vertex of the triangle to be constructed. It is evident then that the point P must be within the right angle ACB; *i.e.*, CP must intersect the hypotenuse AB—suppose in Q.

Let angle $ACP = \theta$, and $z = \cot \theta$. Then

$$AP = \frac{b \sin \theta}{\sin \alpha}, \qquad CP = \frac{b \sin (\theta + \alpha)}{\sin \alpha} = b(\cos \theta + \cot \alpha \sin \theta),$$

$$CQ = \frac{b \sin A}{\sin (A + \theta)} = \frac{b}{\cos \theta + \cot A \sin \theta}.$$

The area of that part of the triangle APC outside of the triangle ABC is $s = \frac{1}{2}AP \cdot QP \sin \alpha$,

$$\frac{2s}{b^2} = \sin \theta \left[\cos \theta + \cot \alpha \sin \theta - \frac{1}{\cos \theta + e \sin \theta} \right].$$

Hence

$$\frac{2s}{b^2} = \frac{z + \cot \alpha}{1 + z^2} - \frac{1}{e + z}$$

is to be a maximum. Thus

$$-(1 + z^2)^2(e + z)^2 \frac{d}{dz}\left(\frac{2s}{b^2}\right) \equiv \phi(z) \equiv 2(e + \cot \alpha)z^3$$
$$+ (e^2 + 4e \cot \alpha - 3)z^2 + 2e(e \cot \alpha - 1)z - (e^2 + 1) = 0.$$

It can be shown as follows that this equation has only one real root:

$$\phi'(z) \equiv 2\{(e + \cot \alpha)z + (e \cot \alpha - 1)\}\{3z + e\}.$$

By substitution we find that ϕ is negative for both values of z which make $\phi'(z) = 0$, *i.e.*, that both the maximum and the minimum values of ϕ are negative; hence $\phi(z) = 0$ has only one real root. Since, for values of z sufficiently large, $\phi(z)$ is positive, and, for values of z sufficiently small algebraically, $\phi(z)$ is negative, it follows that the real root of $\phi(z) = 0$ makes s a maximum. In a numerical problem this root can be found approximately by Horner's process, or its expression in terms of e and α can be found by Cardan's process.

Also solved by A. G. CLARK.

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[1] *Cf.* Netto: *Vorlesungen über Algebra*, vol. I, p. 234, edition of 1896.

3071 [1924, 206]. Proposed by E. L. POST, Cornell University.

Prove that, when $|h| < 1/e$,

$$1 + \frac{(x+h)}{1!} + \frac{(x+2h)^2}{2!} + \cdots + \frac{(x+nh)^n}{n!} + \cdots = ce^{mx},$$

where both c and m are independent of x, and the latter satisfies the equation $e^{mh} = m$. Note: Under the given conditions both members of the first equation considered as functions of x directly satisfy the mixed difference equation $f'(x) = f(x+h)$. The identification of these two equations is therefore of some interest.

SOLUTION BY OTTO DUNKEL, Washington University.

With the condition $|h| < 1/e$ the series of the problem is convergent for all values of x. For replace in it x by its absolute value X and h by its absolute value H, $H > 0$. Then in the resulting series the ratio of the $(n+2)$th term to the $(n+1)$th is

$$\left(1 + \frac{1}{n}\right)^n \frac{\left[1 + \dfrac{X}{(n+1)H}\right]^{n+1}}{\left[1 + \dfrac{X}{nH}\right]^n} H. \tag{1}$$

For a given X and H the limit of this ratio as n becomes infinite is $eH < 1$. Hence the series represents an analytic function of x; also its terms may be rearranged in any order. It will now be shown that, if we denote this function by $f(x)$ for a given h, the ratio $f(x+h)/f(x)$ is a constant m. This is obviously true if $h = 0$ and in this case $m = 1$. It will be convenient to suppose at first that $0 < h < 1/e$. If, with such a value of h, $f(x)$ vanishes, it must be for a negative value of x; and, if x_0 is the largest root, we shall suppose that $x > x_0$. Set $x = ht$, then

$$f(x) = \varphi(t) = \sum_{n=0}^{\infty} \frac{(t+n)^n h^n}{n!}, \qquad \frac{f(x+h)}{f(x)} = \frac{\varphi(t+1)}{\varphi(t)}. \tag{2}$$

For a given t the last expression in (2) may be written as a power series in h,

$$\frac{\varphi(t+1)}{\varphi(t)} = \sum_{i=0}^{\infty} \frac{a_i h^i}{i!}, \tag{3}$$

and this series is convergent for $h < 1/e$. We shall show that the a_i's do not depend on t. The equations for the determination of these coefficients are

$$(t+1+k)^k = \sum_{j=0}^{j=k} (t+j)^j \, {}_kC_j a_{k-j}, \qquad {}_kC_j = \frac{k!}{j!\,(k-j)!}. \tag{4}$$

It is readily seen that $a_0 = a_1 = 1$. Suppose then that all the a's up to and including a_k are independent of t. Then (4) is an identity in t, and on integrating both sides with respect to t, we find

$$(t+1+k)^{k+1} = A_{k+1} + \sum_{j=0}^{j=k} (t+j)^{j+1} \, {}_{k+1}C_{j+1} a_{k-j}, \tag{5}$$

where A_{k+1} is determined by setting $t = 0$. Now replace t by $t+1$ and j by $j-1$, then (5) becomes

$$(t+2+k)^{k+1} = A_{k+1} + \sum_{j=1}^{j=k+1} (t+j)^j \, {}_{k+1}C_j a_{k+1-j}. \tag{5'}$$

A comparison of this result with (4) shows that $a_{k+1} = A_{k+1}$, and hence it is independent of t. Thus by mathematical induction we see that no a will depend on t, and our theorem is proved. Therefore

$$\frac{f(x+h)}{f(x)} = \frac{f(h)}{f(0)} = \sum_{i=0}^{\infty} \frac{a_i h^i}{i!} = m. \tag{6}$$

Since $f(x)$ satisfies the equations

$$f(x+h) = mf(x), \qquad f'(x) = f(x+h), \tag{7}$$

we have $f'(x) = mf(x)$. Hence it follows that

$$f(x) = ce^{mx}, \tag{8}$$

where c is a constant. Since $f(x)$ must also satisfy the second equation in (7), we find that

$$e^{mh} = m, \tag{8'}$$

and, as h approaches zero, m must approach unity as we saw above.

An examination of the equation (8′) shows that for $0 < h < 1/e$ there are two real values of m, one such that $1 < m < e$, and the other such that $m > e$. Each of these furnish a solution of the second equation of (7), but only the first gives a solution developable in the form of the series of the problem, since in the second as h approaches zero m becomes infinite. We now see that $f(x)$ does not vanish for any value of x. If h is negative, the equation (8′) has one solution for m, $0 < m < 1$. It is easily seen now from (6) that the results obtained may be extended to the case of h negative provided that $|h| < 1/e$. The equation (6) or (3) gives a solution of (8′) expressed in an infinite variety of forms since any real number may be substituted for x or t. The second equation of (7) has an infinite number of other real solutions, which oscillate. It is interesting to compare these results with problem **3103** **[1924, 498]** and the solution of **3046** **[1924, 403]**.

Also solved by the PROPOSER.

3072 [1924, 206]. Proposed by W. H. RASCHE, Virginia Polytechnic Institute.

If the four vertices, A, B, C, and D (taken in cyclic order), of a simple quadrangle are four points in a plane rigid body undergoing complanar motion, and if two opposite sides, as AB and DC, represent vectorially the accelerations of A and D respectively, then Clifford's point of the quadrangle is the center of acceleration of the rigid body.

Note: The Clifford point associated with four complanar straight lines is the point common to the circumcircles of the four triangles formed by omitting the lines in turn.

SOLUTION BY THE PROPOSER.

Definition.—Center of Acceleration. The center of acceleration of a rigid body undergoing complanar motion is a point so located in the plane of motion that—

1. The ratio of its distances from any two points of the rigid body which are in the plane of motion is equal to the ratio of the accelerations of these points.

2. The angles which the lines joining it to any two points of the rigid body which are in the plane of motion make with the directions of the accelerations of these points are equal.

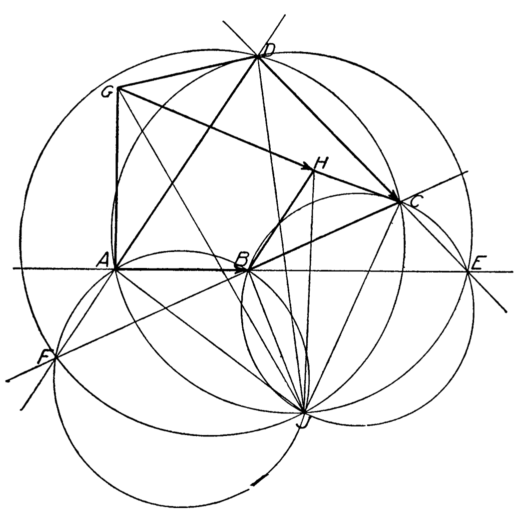

FIG. 1.

Let A and D, Fig. 1, be two points in a rigid body undergoing complanar motion, and let the vectors AB and DC represent, respectively, the accelerations of these two points. Produce the sides of the quadrangle, $ABCD$, and about each of the four triangles thus formed circumscribe a circle. Then it is to be shown that the common point, J (Clifford's point), of these four circles is the center of acceleration of the rigid body.

Proof.—I. First, consider the triangles JDC and JAB. For these triangles we have the relations: $\angle JDC = \angle JAB$, since each of these angles is measured by $\frac{1}{2}$ arc JE; $\angle BJA = \angle BFA$, because each of these angles is measured by $\frac{1}{2}$ arc AB; but $\angle BFA = \angle CJD$, for each of these angles is measured by $\frac{1}{2}$ arc DC; therefore $\angle CJD = \angle BJA$, and triangles JDC and JAB are similar.

Hence we conclude that $DC/DJ = AB/AJ$. And thus it follows that (*a*) the accelerations of these two special[1] points A and D make equal angles with the lines which join these points to J; and (*b*) that these accelerations are proportional to their distances from J.

II. Secondly, take *any* third point G in the rigid body, and on BC construct the triangle BHC similar to the triangle AGD and similarly placed. Then it is a well-established kinematic fact that the line GH represents vectorially the acceleration of point G. (See, for example, Burmester's *Lehrbuch der Kinematik*, p. 805.)

The foregoing lemma granted, it is easy to show that—

$$\text{A.} \quad \angle JGH = \angle JDC = \angle JAB,$$

$$\text{B.} \quad \frac{GH}{JG} = \frac{DC}{JD} = \frac{AB}{JA},$$

and thus that the position of point J fulfills the requirements of the definition of the center of acceleration of the rigid body.

First, consider the triangles JBC and JAD. In these triangles $\angle CJB = \angle DJA$, for $\angle CJB = \angle CJD + \angle DJB$, and $\angle DJA = \angle BJA + \angle DJB$; but $\angle CJD = \angle BJA$. Therefore, triangles JBC and JAD are similar, and hence, also, the quadrangles $JBHC$ and $JAGD$. But if these quadrangles be similar, then the triangles HJC and GJD are also similar; accordingly $CJ/HJ = DJ/GJ$, and $\angle CJH = \angle DJG$. Moreover, therefore, $\angle CJD = \angle HJG$, for $\angle CJD = \angle CJH + \angle HJD$, and $\angle HJG = \angle DJG + \angle HJD$.

Thus it follows that triangles CJD and HJG are similar; whence

$$\frac{GH}{JG} = \frac{DC}{JD} = \frac{AB}{JA},$$

and

$$\angle JGH = \angle JDC = \angle JAB, \qquad Q. E. D.$$

III. In exactly the same manner, evidently, taking the other pair of opposite sides (DA and CB) of the quadrangle as the accelerations of the two points D and C of the rigid body, it can be shown that the same point J is the center of acceleration of the body; thus the proposition as stated is true.

3078 [1924, 254]. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Find the curve having the property that the tangent and the normal at any point determine on a fixed line two points conjugate in a given involution.

Solution by C. K. Robbins, Purdue University.

Let the fixed line be the x-axis, with the origin at the center of the involution. The involution can then be defined by $XX' = m^2$, where X and X' are coördinates of corresponding points of the involution (m may be real or pure imaginary). If the point on the curve is (x, y), then

$$X = \frac{px - y}{p}, \qquad X' = x + py \quad \left(p = \frac{dy}{dx}\right)$$

and we have at once the differential equation

$$xyp^2 + (x^2 - y^2 - m^2)p - xy = 0.$$

This may be solved by the parametric method. Let $xp - y = v$. Solve this with the differential equation for x and y, getting

$$x = \frac{p(v^2 + m^2)}{v(p^2 + 1)}; \qquad y = \frac{p^2m^2 - v^2}{v(p^2 + 1)}.$$

The fact that $p = dy/dx$ now gives a differential equation in p and v as follows:

$$\frac{vdv}{v^2 + m^2} - \frac{pdp}{p^2 + 1} = 0,$$

the integral of which is $v^2 + m^2 = c^2(p^2 + 1)$.

[1] These points are special because they are involved in the given data of the problem.

The problem reduces to eliminating v and p, using this equation and the equations expressing x and y in terms of v and p. The result is

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - m^2} = 1.$$

This is a system of confocal conics having foci at the two fixed points of the given involution. Since a conic has no singular points, the tangent and normal at points corresponding to the fixed points of the involution must coincide. Such points are necessarily imaginary. They turn out to be the four points $(\pm c^2/m, \pm [(c^2 - m^2)i]/m)$. Note that for a real conic these are imaginary whether m is real or pure imaginary.

The singular solution is $(x^2 - y^2 - m^2)^2 + 4x^2y^2 = 0$, which is a pair of conics having $(\pm m, 0)$ as their only real points in case m is real, and $(0, \pm mi)$ as their only real points in case m is pure imaginary.

Also solved by THEODORE BENNETT.

3086 [1924, 305]. Proposed by A. S. WIENER, Cornell University.

Prove that the determinant of the nth order

$$\begin{vmatrix} x_1^2 + \lambda & x_1x_2 & x_1x_3 & x_1x_4 & \cdots & x_1x_n \\ x_1x_2 & x_2^2 + \lambda & x_2x_3 & x_2x_4 & \cdots & x_2x_n \\ x_1x_3 & x_2x_3 & x_3^2 + \lambda & x_3x_4 & \cdots & x_3x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1x_n & x_2x_n & x_3x_n & x_4x_n & \cdots & x_n^2 + \lambda \end{vmatrix}$$

is divisible by λ^{n-1} and find the other factor.

SOLUTION BY J. A. BULLARD, U. S. Naval Academy.

Let A denote the matrix whose constituents are

$$(A)_{ij} = x_ix_j, \quad (i, j = 1, 2, \cdots, n).$$

Expanding the given determinant in powers of λ, we have

$$|A + \lambda| = \lambda^n + \psi_1\lambda^{n-1} + \cdots + \psi_{n-1}\lambda + \psi_n,$$

where the ψ's are functions of the x's.

If $S_1, S_2, S_3, \cdots, S_n$ denote the sums of the powers of the roots of the equation $|A - \lambda| = 0$, then

$$S_1 = \sum_{i=1}^{n} (A)_{ii} = \sum_{i=1}^{n} x_i^2,$$

$$S_2 = \sum_{i=1}^{n} (A^2)_{ii} = \sum_{i,j=1}^{n} (A)_{ij}(A)_{ji} = \sum_{i,j=1}^{n} x_i^2x_j^2 = S_1^2,$$

$$S_3 = \sum_{i=1}^{n} (A^3)_{ii} = \sum_{i,j=1}^{n} (A)_{ij}(A^2)_{ji} = \sum_{i,j,h=1}^{n} x_i^2x_j^2x_h^2 = S_1^3,$$

and similarly in general for $k = 1, 2, \cdots, n$

$$S_k = \sum_{i=1}^{n} (A^k)_{ii} = \sum_{i,j=1}^{n} (A)_{ij}(A^{k-1})_{ji} = \cdots = \sum_{h_1, h_2 \cdots h_k=1}^{n} x_{h_1}^2x_{h_2}^2 \cdots x_{h_k}^2 = S_1^k.$$

Substituting these values in Newton's formulæ connecting the sums of the powers of the roots with the coefficients, we see that $\psi_1 = S_1$, $\psi_2 = \psi_3 = \cdots = \psi_n = 0$.

Therefore,

$$|A+\lambda| = \lambda^n + \psi_1\lambda^{n-1} = \lambda^{n-1}(\lambda + S_1) = \lambda^{n-1}(\lambda + \sum_{i=1}^{n} x_i^2).$$

Also solved by THEODORE BENNETT, E. P. BUGDANOFF, E. H. CLARKE, P. J. DA CUNHA, F. HENROTEAU, J. J. NASSAU, HAZEL E. SCHOONMAKER, and the PROPOSER.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Miss BERTHA K. DUNCAN, of the University of Texas, has been appointed professor and head of the department of mathematics at Grenada College (Miss.).

Dr. G. W. HESS has been appointed professor of mathematics and head of the department at Union University, Jackson, Tenn.

Dr. W. P. OTT, of Vanderbilt University, has been appointed head of the department of mathematics at the University of Alabama.

Dr. R. L. WILDER, of the University of Texas, has been appointed assistant professor of mathematics at Ohio State University.

Mr. P. L. REA, of Marietta College, has been promoted to an assistant professorship of mathematics.

Miss INEZ MARTIN has been appointed assistant professor of mathematics at the Indiana State Normal School, Terre Haute.

Assistant Professor L. J. COMRIE, of Swarthmore College, has accepted a position in the observatory at Northwestern University.

Professor J. M. MELCHIOR, of Rockhurst College, Kansas City, Mo., has been appointed head of the department of mathematics at Campion College, Prairie du Chien, Wis.

Mr. W. H. LYONS has been appointed assistant professor of mathematics at the Kansas State Agricultural College.

Mr. G. F. ALRICH has been appointed assistant professor of mathematics at Des Moines University.

Mr. J. H. SYNNERDAHL, of Lake Forest College, has been appointed assistant professor of mathematics at Park College, Parkville, Mo.

At Washington University, St. Louis, Assistant Professor F. W. BUBB is on leave of absence for the academic year 1924–25, and is studying at the University of Chicago as a Research Council fellow. Mr. H. R. GRUMANN has been appointed instructor in mathematics and Mr. E. H. LUND instructor in applied mathematics.

At Galloway College, Searcy, Ark., Miss BERD R. ALLEN, of the Kansas State Teachers' College, has been appointed professor of mathematics and head of the department and Mrs. GEORGIA M. WILLIAMS instructor.

At Southwestern University, Georgetown, Texas, Miss VELMA TISDALE has been promoted to an assistant professorship and Mr. J. G. CHANEY has been appointed instructor.

At the University of Texas, Mr. C. A. RUPP, of Hamline University, has been appointed adjunct professor of mathematics, and Miss ELIZABETH T. STAFFORD, of Brown University, instructor.

The following appointments to instructorships have been announced:

University of Maine, Mr. E. H. HADLOCK;
Brown University, Mr. F. C. JONAH;
University of Buffalo, Mr. H. D. HIGHET;
Swarthmore College, Mr. D. B. MCLAUGHLIN;
Wofford College, Mr. W. C. HERBERT;
Athens College, Athens, Ga., Miss NANCY L. MOOREFIELD;
Mercer University, Messrs. J. L. TALLEY and C. H. BERRYMAN;
Case School of Applied Science, Mr. J. E. MERRILL;
University of Kansas, Mr. R. H. MARQUIS;
Union University, L. W. HUSSEY, E. E. STIVERT;
State College of Washington, Miss ELEANOR E. BOYD.

Professor E. L. LARKIN, director of Mt. Lowe Observatory, died October 11, 1924, at the age of seventy-seven years.

Professor P. A. Lambert, head of the department of mathematics and astronomy at Lehigh University, died on February 15, 1925.

A committee of the Society for the Promotion of Engineering Education has begun a general investigation of engineering education under a three-year grant by the Carnegie Corporation. The declared objectives of the study are (1) to clarify the educational functions and responsibilities of the colleges of engineering; (2) to establish guiding principles for the content and arrangement of curricula and the improvement of teaching; (3) to consider in what ways problems relating to engineering students, graduates and teachers may be dealt with more effectively; (4) to examine the practicability and possible benefits of closer group relationships among the colleges and with the professional organization of engineers; (5) to make an analytical comparison of the organization and practices of engineering education in Europe and America. A statement of the present status of the study is given in the *Journal of Engineering Education*, January, 1925.

Members of the Association who do not plan to keep files of the MONTHLY are urged to send back numbers of all issues to the Association library. These numbers do not have any appreciable market value when sold by a private individual to a bookseller; however the Association is able to sell complete volumes occasionally. The Association is willing to pay express charges on any numbers which members may be willing to donate. Reference to the occasional notices in the advertising columns or inquiry of the Secretary will inform any one of the issues which are specially desired.

A DEVELOPMENT OF FIFTY YEARS IN DESIGN AND QUALITY

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 7–8, 1925;
Tenth Annual Meeting, Kansas City, Mo., December, 1925.
The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Peoria, May 8–9, 1925
INDIANA, Purdue Univ., April
IOWA, State College, Ames, May 2–3, 1924
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA–MISSISSIPPI, Baton Rouge, March 1, 1924
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Baltimore, December 22, 1924
MICHIGAN, Ann Arbor, April 3, 1924
MINNESOTA, Hamline Univ., St. Paul, May 24, 1924
MISSOURI, Kansas City, November 15, 1924
NEBRASKA, Creighton Univ., Omaha, May 2, 1924
OHIO, Ohio State Univ., Columbus, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 21
TEXAS, Dallas, November 27–28, 1925

DESIRED COPIES OF THE MONTHLY

THROUGH an oversight the Association has not reserved a sufficient number of copies of the MONTHLY for October, 1920, and for August, September, and October, 1921, also May, September, and October, 1924. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary,

W. D. Cairns, Oberlin, Ohio

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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NINTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The ninth annual meeting of the Association was held at George Washington University on Wednesday, December 31, 1924, and Thursday, January 1, 1925, following and in conjunction with the annual meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. 268 were present at the various sessions, including the following 167 members of the Association:

C. R. Adams, Brown University.
O. S. Adams, U. S. Coast and Geodetic Survey.
J. W. Alexander, Princeton University.
R. C. Archibald, Brown University.
G. N. Armstrong, Ohio Wesleyan University.
Katherine S. Arnold, Amer. Association of University Women.
C. S. Atchison, Washington and Jefferson University.
Clara L. Bacon, Goucher College.
Ida Barney, New Haven, Conn.
L. A. Bauer, Dept. of Terrestrial Magnetism.
Sarah Beall, U. S. Coast and Geodetic Survey.
W. J. Berry, Brooklyn Polytechnic Institute.
O. F. H. Bert, Washington and Jefferson College.
W. W. Bigelow, U. S. Coast and Geodetic Survey.
G. D. Birkhoff, Harvard University.
W. H. Bixby, Brig. Gen., U. S. Army, Retired, Washington.
G. A. Bliss, University of Chicago.
R. F. Borden, George Washington University.
E. W. Brown, Yale University.
H. S. Brown, Hamilton College.
R. E. Bruce, Boston University.
J. A. Bullard, U. S. Naval Academy.
W. G. Bullard, Syracuse University.
W. D. Cairns, Oberlin College.
Florian Cajori, University of California.
Minnie W. Caldwell, Chowan College.
R. D. Carmichael, University of Illinois.
E. W. Chittenden, University of Iowa.
G. R. Clements, U. S. Naval Academy.
Abraham Cohen, Johns Hopkins University.
J. L. Coolidge, Harvard University.
C. H. Currier, Brown University.
J. E. Davis, Drexel Institute.
W. W. Denton, University of Michigan.
Alexander Dillingham, U. S. Naval Academy.
C. N. Dickinson, Hollins College.
Arnold Dresden, University of Wisconsin.
J. A. Eiesland, West Virginia University.
Harry English, High Schools, Washington.
J. T. Erwin, George Washington University.
H. S. Everett, Bucknell University.
Fay Farnum, Graduate School, Cornell University.
B. F. Finkel, Drury College.
W. B. Fite, Columbia University.
J. A. Foberg, Pennsylvania State Dept. of Public Instruction.
T. M. Focke, Case School of Applied Science.
J. G. Fowlkes, University of Wisconsin.
J. F. Fox, U. S. Bureau of Standards.
T. C. Fry, Western Electric Co., New York.
W. V. N. Garretson, Rutgers College.
Leslie J. Gaylord, Agnes Scott College.
H. M. Gehman, University of Pennsylvania.
F. J. Gerst, St. Louis University.
O. E. Glenn, University of Pennsylvania.
Alice A. Grant, Oldsfield School, Glencoe, Md.
T. E. Gravatt, Pennsylvania State College.
W. M. Hamilton, U. S. Nautical Almanac Office.
Harris Hancock, University of Cincinnati.
J. G. Hardy, Williams College.
Isabel Harris, University of Richmond.
G. A. Harter, Delaware College.
W. E. Heal, U. S. Coast and Geodetic Survey.
E. R. Hedrick, Southern Branch, University of California.
P. E. Hemke, Langley Aeronautical Laboratory, Langley Field, Va.
Archibald Henderson, University of North Carolina.
Robert Henderson, Equitable Life Assur. Co., New York.
C. L. Herron, Hillsdale College.
A. O. Hickson, Brown University.
T. H. Hildebrandt, University of Michigan.
C. E. Hille, Princeton University.
A. A. Himwich, New York.
H. L. Hodgkins, George Washington University.

Jewell C. Hughes, University of Arkansas.
L. S. Hulburt, Johns Hopkins University.
I. W. Huntzberger, Central High School, Washington.
W. A. Hurwitz, Cornell University.
Nelle L. Ingels, Interstate Commerce Commission.
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J. R. Kline, University of Pennsylvania.
E. H. Koch, Jr., Brooklyn Technical High School.
W. D. Lambert, U. S. Coast & Geodetic Survey.
A. E. Landry, Catholic University of America.
C. G. Latimer, Tulane University.
Solomon Lefschetz, University of Kansas.
Anna D. Lewis, Lake Erie College.
Florence P. Lewis, Goucher College.
Zung-nyi Loh, Graduate School, Cornell University.
A. C. Lunn, University of Chicago.
W. D. MacMillan, University of Chicago.
E. S. Manson, Ohio State University.
Sister Mariola, Rosary College.
L. C. Mathewson, Dartmouth College.
J. F. Messick, Emory University.
Bessie I. Miller, Rockford College.
J. A. Miller, Swarthmore College.
G. R. Mirick, Lincoln School.
C. N. Moore, University of Cincinnati.
R. L. Moore, University of Texas.
C. C. Morris, Ohio State University.
Marston Morse, Cornell University.
F. R. Moulton, University of Chicago.
Frank Morley, Johns Hopkins University.
Paul Muehlman, Loyola University.
F. D. Murnaghan, Johns Hopkins University.
F. H. Murray, Dalhousie University.
J. R. Musselman, Johns Hopkins University.
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C. J. Rees, University of Delaware.
B. L. Remick, Kansas State Agricultural College.
J. N. Rice, Catholic University.
R. G. D. Richardson, Brown University.
H. L. Rietz, University of Iowa.
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T. H. Taliaferro, University of Maryland.
J. S. Taylor, University of Pittsburgh.
R. W. Thomas, Washington & Jefferson College.
W. A. Titsworth, Alfred University.
Bird M. Turner, West Virginia University.
H. W. Tyler, Massachusetts Institute of Technology.
H. S. Uhler, Yale University.
C. E. Van Orstrand, U. S. Geological Survey.
Oswald Veblen, Princeton University.
W. J. Wallis, Central High School, Washington.
J. L. Walsh, Harvard University.
J. M. West, Pennsylvania State College.
Willis Whited, Pennsylvania State Dept. of Highways.
A. H. Wilson, Haverford College.
Elizabeth W. Wilson, Central High School, Washington.
D. W. Woodard, Howard University.
E. W. Woolard, U. S. Weather Bureau.
J. W. Young, Dartmouth College.

The meetings of the American Association for the Advancement of Science began with the formal opening Monday evening at Memorial Continental Hall where Dr. C. D. Walcott, secretary of the Smithsonian Institution, gave the

address of the retiring president on "Science and Service" (*Science*, January 2, 1925) and the Honorable CHARLES E. HUGHES, Secretary of State, spoke on "Some Aspects of International Coöperation" (*Science*, January 9, 1925), saying that scientists can and do aid the promotion of peace greatly by developing new and enlarged conceptions of national and international interests, by thinking in terms of peoples and not simply of governments, by urging insistently that international coöperation in scientific research is not only desirable but absolutely necessary. This meeting was followed by a reception given at the new National Museum under the auspices of the Washington members of the American Association. The President of the United States addressed the visiting scientists at the White House on Wednesday noon (*Science*, January 9, 1925), expressing the great dependence which the government places on science, paying tribute to the scientists as the wonder workers of all the ages, and recognizing that they are animated by a profound purpose to better the estate of men. Among the other general sessions were a public lecture on Thursday afternoon by Professor A. E. DOUGLAS, director of the Observatory of the University of Arizona, on the University's eclipse expedition of September, 1923, and a showing on Thursday evening of motion pictures taken on the western trip that followed the Toronto meeting of the British Association, by Doctor E. E. SLOSSON, director of Science Service.

Professor M. I. PUPIN of Columbia University was elected president of the American Association for 1925, Dr. W. J. HUMPHREYS of the U. S. Weather Bureau was elected general secretary, and Professors E. B. WILSON, B. M. DUGGAR and VERNON KELLOGG were elected members of the Executive Committee. On nomination by the section committee for Section A, Professor W. H. ROEVER was chosen vice-president, Professor R. C. ARCHIBALD, secretary, and Professor E. W. CHITTENDEN member of the section committee.

A large exhibition of scientific apparatus, books, products and research methods was held, chiefly in the gymnasium of George Washington University, in easy access from the lecture halls where many of the science meetings were held. These exhibits included, among others, the publications of the American Mathematical Society, many science texts by various publishers, Professor W. H. Roever's mechanism for illustrating systems of lines of force, a display of graphical and other material by the University of Missouri chapter of Pi Mu Epsilon, extensive collections of instruments and maps from the U. S. Coast and Geodetic Survey, the Bureau of Standards, the Metric Association, the U. S. Department of Agriculture and various manufacturers of scientific apparatus. A new and very novel exhibit was the latest development of the representation of the earth's surface as adopted by the Corps of Engineers, U. S. Army, obtained by building a model from the topographic map to show the hills and valleys in miniature form, photographing the illuminated model so as to obtain relief in pictorial form, the lighting producing a deepness of shade graded according to the steepness of slopes, and finally "overprinting" this relief upon the topographic map. Professor W. H. Roever was in charge of the mathematical portion of the exhibition.

On Tuesday evening the mathematicians and visiting scientists were entertained at a reception and dance given at the Washington Club by the Columbian Women of George Washington University; Miss Elizabeth W. Wilson, a member of the Mathematical Association, is president of the Columbian Women. Numerous facilities were afforded by George Washington University under the chairmanship of Dean H. L. Hodgkins, as well as by the local committees, for the comfort and convenience of its guests and this was recognized in a vote of thanks which was offered at the closing session by Professor H. S. Everett and adopted heartily.

The joint dinner of the mathematicians with 125 persons present was held on Thursday evening at Franklin Square Hotel which was the headquarters of this group. After a witty introduction by the toastmaster, Professor Coolidge, speeches were made by Professor Cajori, who spoke of the solid worth and reward which came to those who attended these meetings even in the face of various difficulties; by Professor Hedrick, who explained the continuing need of printing American mathematical journals in Germany because of the large increase in printing costs since the war and urged strongly the necessity of giving American mathematics its proper place in the world through feasible facilities for printing books prepared by our first-rank mathematicians, citing as a notable forward step the example of Mrs. Carus in providing so generously for the publication of the Carus Mathematical Monographs, the first of which is just off the press; by Professor Moore who urged that the teachers of mathematics and our mathematical societies should be less reluctant to maintain before the world the importance of mathematics; and by President Olds who, a teacher for forty-five years, told of the glorious work of the teacher, the great history of mathematics in this country and the fine spirit of unity exemplified by such gatherings as this.

The American Mathematical Society held its thirty-first annual meeting with sessions for the reading of papers Monday afternoon and Tuesday morning and afternoon. Fifty-four papers were read at these sessions. The second Bôcher prize was divided equally between Professor E. T. Bell of the University of Washington for his memoir "Arithmetical Paraphrases" (*Transactions*, Volume 22) and Professor Solomon Lefschetz for his memoir "On certain numerical invariants of algebraic varieties with applications to Abelian varieties" (*Transactions*, Volume 22). The second Josiah Willard Gibbs lecture under the auspices of the Society was delivered by Mr. Robert Henderson, Vice-President of the Equitable Life Assurance Society, on Tuesday evening at the auditorium of the Department of the Interior Building. The title of the lecture was "Life insurance as a social service and as a mathematical problem." It will be printed in the *Bulletin* of the Society.

The mathematical fraternity, Pi Mu Epsilon, met for a business session and dinner on Monday with delegates from the chapters in various colleges and universities.

The History of Science Society met in conjunction with Section L on Wednesday and Thursday. Professor L. C. Karpinski, Vice-President of Section L,

presided at the session, and among the other papers were those by Professor Cajori on "Leibniz, the master builder of mathematical notations," and by Professor Archibald on "Benjamin Peirce." Because of these two papers the Thursday session of the Association was postponed until a later hour of the afternoon.

The sessions of the Mathematical Association consisted of joint sessions on Wednesday and Thursday mornings and separate sessions on the two afternoons. The program was prepared under the following Committee on Program: F. D. Murnaghan, chairman; A. A. Bennett and T. McN. Simpson. Abstracts of most of the papers are given, numbered in accordance with the numbers of the papers.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION.

(1) "Remarks on the foundations of geometry" by Professor OSWALD VEBLEN, Princeton University, retiring president of the Society.

(2) "The foundations of the theory of algebraic numbers" by Professor HARRIS HANCOCK, University of Cincinnati, retiring vice-president of Section A.

1. Professor Veblen's address will appear in the *Bulletin* of the American Mathematical Society.

2. Professor Hancock's address was printed in *Science* for January 2 and 9, 1925. After making some general observations on the ultimate recourse to mathematics in the formulation of physical theories, he introduces by simple examples the notion of number realms. He then points out that in the extension of those realms, by the introduction of new numbers, it becomes necessary to make certain modifications in order that the usual theorems of arithmetic hold in the more general realms; examples were given illustrating this point. The ideal factors of Kummer and Dedekind were then mentioned and other points in the theory illustrated.

Professor J. C. Fields of the University of Toronto, chairman of Section A, who was prevented by illness from attending the meeting, sent a telegram of greetings; in his absence President Rietz of the Mathematical Association presided at this session.

FIRST SESSION OF THE ASSOCIATION.

(3) "Outlines of the fields of research: The mathematics of economics" by Professor G. C. EVANS, Rice Institute.

(4) "Outlines of the fields of research: General analysis" by Professor T. H. HILDEBRANDT, University of Michigan.

(5) "On the empirical representation of certain production curves" by C. E. VAN ORSTRAND, geophysicist in the United States Geological Survey.

(6) "Preliminary report of the Committee on Standard Departments of Mathematics" by Professor R. D. CARMICHAEL, University of Illinois, Chairman.

3. Professor Evans's paper appeared in full in the MONTHLY for March.

4. Professor Hildebrandt's paper, which considered general analysis from the standpoints of the theory of abstract sets and of functionals in a particular class or the function field, as exemplified by the work of Fréchet and E. H. Moore, respectively, and which gave suggestions as to tasks that are worth doing, or at least attempting, in these fields, will appear in an early issue of the MONTHLY. In the discussion which followed, Professor Chittenden expressed the judgment, contrary to a suggestion made in the paper, that the idea of neighborhood cannot well be dispensed with as a fundamental notion.

5. Mr. Van Orstrand's paper appeared in the *Journal of the Washington Academy of Sciences* for January 19, 1925. This abstract, like the address, is given with the permission of the Director of the Geological Survey.

Production data representing the yearly output from individual mines or from groups of mines included within areas of the order of magnitude of states or nations can generally be represented by a curve which has a zero value at the origin of time—the curve rises irregularly and oftentimes quite abruptly to a maximum value and then declines rather slowly, presumably to a zero value, in an interval of time which may be assumed to be finite, or infinite.

Ten functions satisfying the mathematical conditions were defined by means of theorems of the theory of functions. The list includes the seven types of the frequency distribution deduced by Pearson and three additional functions, the most important of which is the equation,

$$y = ax^m e^{-bx^n}.$$

The value, $n = 1$, does not minimize the sum of the squares of the residuals, but it is a sufficiently close approximation to the true value for most practical purposes. The remaining constants are easily evaluated by the method of least squares. The observation equations are written in the logarithmic form, and the theoretical weight, y^2, is assigned to each equation. An approximate evaluation of the constants of Pearson's types I and III was made, using the method of moments, but this method, in its present state of development, is not easily applied to an incomplete set of observations.

6. The report made by Professor Carmichael for the committee appointed at the Cincinnati meeting was purely preliminary. He stated that it appeared quite difficult to determine just what should be treated in such a report. Thus, as an example, when it was proposed to draw up as a standard a graded series of the number of hours per week which should be taught by college teachers of mathematics, judgments of extreme approbation and of extreme mistrust as to the wisdom of the Association's presuming to set a standard were expressed. A view was supported by some members of the committee that a very extended study should be made of the questions given to the committee, and the chairman added that it now appeared that any brief analysis such as at first contemplated would be insufficient to enable the committee to reach a definite formulation.

Professor Cairns remarked that the very fact that such divergent views were held made more evident the importance for the American colleges of a full study

of the questions and that it is very desirable that the committee should through the Association's financial backing be enabled to meet for face-to-face consideration and debate on the general questions of what constitute a good college department of mathematics. Professor Berry said that many institutions emphasize strong instructional work and that such cases should be distinguished from the large institutions which take as their distinctive work emphasis on research. Professor Armstrong desired information as to how certain courses should be grouped, *e.g.*, the teaching of mathematics in the department of mathematics or of education, surveying in the department of mathematics or of physics.

Joint Session of the Association with Sections A, B and D of the American Association and the Mathematical Society.

(7) "Stellar evolution" by Professor H. N. Russell, Princeton University, representing the Astronomical Society.

(8) "Is the universe finite?" by Professor Archibald Henderson, University of North Carolina, representing the Mathematical Association.

An audience of 250 was present at this joint session. It was presided over by Professor W. F. G. Swann of Yale University, who was relieved later by Professor J. A. Miller of Swarthmore College.

7. Professor Russell gave a classification of stars according to brightness and color. He pointed out how the plotting of the surface temperatures against the amount of light emitted gives definite clusterings on the diagram, running in what Eddington calls the main sequence from hot white stars to cooler red stars but with a branch composed of the giant stars for which brighter light is accompanied by lower temperature. It was explained how the consideration of inner temperature, rate of radiation of heat, and radio-active changes, with the convertibility of mass and energy, are at present used to account for the evolution of the stars which is implied in the diagram above referred to.

8. Professor Henderson's paper will appear in full in the Monthly.

The discussion was shared in by Dr. Silberstein and Professors Veblen, MacMillan and Russell.

Second Session of the Association.

(9) "Application of Ritz's method to practical problems in engineering" by Willis Whited, Pennsylvania State Department of Highways.

(10) "Browse: A course in scientific literature" by Professor Bessie I. Miller, Rockford College.

(11) "New conformal world maps derived from elliptic functions" by Dr. O. S. Adams, United States Coast and Geodetic Survey.

9. In 1908 Walter Ritz published in *Crelle's Journal*, vol. 135, page 1, a method whereby a large number of practical problems in elasticity can be solved with ample accuracy for all practical purposes and with little difficulty. Many of these problems involve partial differential equations of a high order, the solution

of which is hardly practicable except in a few special cases. This method does not seem to be understood in this country so widely as its value would justify. The present paper was written to explain and illustrate the method by determining the stresses in a concrete roadway paving slab, a subject which is of vast importance to American engineers at the present time.

The essence of this method, as applied to determining the stresses in a body subjected to flexure, consists essentially of formulating an expression for the elastic deflection of the body with unknown coefficients. The equations of internal and of external work must then be deduced. By Castigliano's principle of least work, these must reduce to minima. These equations of work are then differentiated with respect to the various unknown coefficients and their values determined by substituting in the following equations: Let the equation of internal work be $L_1 = F(a_1, a_2, \cdots a_n, x', y', z')$ and that of external work be $L_0 = G(a_1, a_2, \cdots a_n, x', y', z')$, then $L_1 = L_0$,

$$\frac{dF}{da_1}\cdot\frac{dG}{da_2} = \frac{dF}{da_2}\cdot\frac{dG}{da_1},$$

$$\cdots\cdots\cdots$$

$$\frac{dF}{da_1}\cdot\frac{dG}{da_n} = \frac{dF}{da_n}\cdot\frac{dG}{da_1}.$$

These equations will be sufficient to determine the values of the unknown coefficients $a_1, a_2, \cdots a_n$, from which the stresses can be determined by the ordinary principles of elastics.

In the case of the flat plate of uniform thickness and isotropic material, the internal work of flexure is

$$L_1 = \frac{mEh^3}{24(m+1)}\int\int\left\{\frac{m}{m-1}\left[\left(\frac{d^2z}{dx^2}\right)^2 + \left(\frac{d^2z}{dy^2}\right)^2\right] + \frac{2}{m-1}\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2} + 2\left(\frac{d^2z}{dxdy}\right)^2\right\}dxdy,$$

in which m = Poisson's ratio = 3, E = modulus of elasticity of the plate = 2,000,000, h = thickness of the plate = 6″, assuming a load of one pound and all dimensions in inches.

The problem of the flat roadway slab of indefinite extent resting on an elastic subgrade and loaded in the middle is comparable to that of a sheet of ice resting on water. This problem was solved by Hertz in 1884 (*Wiedemann's Annalen*, vol. 22, p. 449) and a more useful solution is given by Föppl in *Vorlesungen über Technische Mechanik*, vol. 5, p. 103 ff., 1922.

If the load is applied at the edge of the slab, we can place

$$z = Ce^{-a(x+y)}\cos ay\,(\sin ax + \cos ax),$$

the origin being at the loaded point. The work of compression of the soil,

considered elastic, is $90z^2$ per square inch, the coefficient of elasticity of the soil being assumed as 180. Treating these expressions as indicated above, we obtain $a = .03217$, $C = .00000511$, and the extreme fiber stress in the slab—longitudinally under the load—equals .0714 lb. Placing $x = y = 0$ gives $z = C$, whence the external work $= \frac{1}{2}C + \int 90z^2 dxdy$.

If the slab is warped up at the edge, as it frequently is on a cool night, so that the edges are not supported by the soil, we can place

$$z = Ce^{-ay}(\sin ay + \cos ay)\left(1 - \frac{3x}{2l} + \frac{x^3}{2l^3}\right),$$

in which $l =$ width of the unsupported portion; and, proceeding as before, we find $C = .00000000396l^2$ and $a = \dfrac{1.1646}{l}$, the extreme fiber stress, longitudinally under the load, $= .0724$ lb. per sq. inch, and the extreme fiber stress, transversely at the line of support opposite the loaded point, $= .08014$ lb. per sq. inch. These stresses are independent of the width of the overhanging portion of the slab, but, of course, do not include the stress due to the weight of the slab itself, nor that due to a neighboring wheel load. These results may seem absurd, but a little reflection will show that the greater the width of the overhanging portion, the wider the distribution of the stress.

If the slab is warped up in the middle, as is frequently the case on a warm day, so that it is supported only at the edges, we can place

$$z = Ce^{-ay}(\sin ay + \cos ay)\left(1 - \frac{3x^2}{2l^2} + \frac{x^3}{2l^3}\right),$$

in which $2l =$ unsupported width, and proceeding as before we find $a = .9473/l$, $C = .000000001573l^2$ and the extreme transverse fiber stress—under the load, which is in the middle of the length as well as of the width of the slab—equals .03787 and the longitudinal fiber stress at the same point $= .02867$ lb. per sq. inch, both independent of the width of the unsupported portion.

If the slab is warped up in the center, and loaded at the end of the warped-up portion, *e.g.*, at an expansion joint or a transverse crack, we can place

$$z = Ce^{-ay} \cos ay\left(1 - \frac{3x^2}{2l^2} + \frac{x^3}{2l^3}\right)$$

and, proceeding as before, we find $C = .000000005316l^2$ and $a = .8563/l$ and the extreme fiber stress transversely under the load $= .1076$ lb. per sq. inch, also independent of the width of the warped-up portion.

The soil supporting the corner of a slab may be removed by erosion or settlement, or it may be warped up as mentioned above. In this case a one-pound load on the corner will produce an extreme fiber stress of $3/h^2$ at all points of the unsupported portion of the slab. If the slab is 6″ thick, the stress will be $= .0833$.

If, however, the slab is supported by an elastic subgrade, the stresses can be obtained by the solution of the following differential equation:

$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}+\frac{12E_0}{Eh^3}y = 0,$$

in which E_0 = modulus of elasticity of the subgrade and x = the distance from the corner, measured in the line bisecting the corner angle which is 90°. If we place $12E_0/Eh^3 = k$, the solution of the equation is as follows:

$$\begin{aligned} y = {} & A\left(1-\frac{kx^4}{2\cdot 3^2\cdot 4}+\frac{k^2x^8}{2\cdot 3^2\cdot 4\cdot 6\cdot 7^2\cdot 8}-\cdots\right) \\ & +(B+B'\log_e x)\left(x-\frac{kx^5}{3\cdot 4^2\cdot 5}+\frac{k^2x^9}{3\cdot 4^2\cdot 5\cdot 7\cdot 8^2\cdot 9}-\cdots\right) \\ & +B'\left[x+\frac{kx^5}{3\cdot 4^2\cdot 5}\left(-1+\frac{1}{3}+\frac{2}{4}+\frac{1}{5}\right)\right. \\ & \left.-\frac{k^2x^9}{3\cdot 4^2\cdot 5\cdot 7\cdot 8^2\cdot 9}\left(-1+\frac{1}{3}+\frac{2}{4}+\frac{1}{5}+\frac{1}{7}+\frac{2}{8}+\frac{1}{9}\right)+\cdots\right] \\ & +C\left(x^2-\frac{kx^6}{4\cdot 5^2\cdot 6}+\frac{k^2x^{10}}{4\cdot 5^2\cdot 6\cdot 8\cdot 9^2\cdot 10}-\cdots\right). \end{aligned}$$

This is a harmonic function and will give the same stresses adjacent to the corner as those just given, but they will diminish as x increases, becoming alternately positive and negative, approaching zero as an asymptote.

Professor Murnaghan reported that a discussion of Ritz's method is to be found in a recent number of the *Bulletin* of the French mathematical society.

10. The paper called "Browse" gave the results of a course in scientific literature. Most of the students enrolled were among the best in college and the results obtained were therefore of primary interest in connection with the teaching of "best" students.

11. This paper gave a short review of the work heretofore done on world maps and then showed a series of fifteen slides that illustrated both what had already been done and what advances have recently been made by the author. In all nine new projections were shown, four being for a single hemisphere and five for the complete sphere. Seven of the nine new projections belong to the rhombic class, since they are associated with a rhombus which may become a square.

It is a somewhat remarkable fact that of the eleven projections of this kind that have been computed and constructed for geographic purposes, ten of them have been produced by the U. S. Coast and Geodetic Survey. The first projection of this class was computed and constructed by C. S. Peirce, at that time an assistant in the Coast and Geodetic Survey, and published by him in the Superintendent's Report for 1877. This projection which Peirce called the quincuncial projection, together with the nine new examples shown in connection with this paper, makes up the number contributed by the Coast and Geodetic Survey.

The most important projections shown for practical purposes are the three rhombic projections of the western hemisphere including the one in the square; the rhombic projection of the whole world in a regular hexagon; that of the whole world in the 60°, 120° rhombus; the whole world in a six-pointed star; and finally, the whole world within an ellipse. Some of these maps compare very favorably with other maps in use, many of which are not conformal and consequently such that the meridians and parallels often do not intersect at right angles. The complete theory and discussion of the various projections will be given in a forthcoming publication of the U. S. Coast and Geodetic Survey.

Meeting of the Board of Trustees of the Association.

Nine members of the Board were present at the two sessions.

The following fifty-five persons and three institutions were elected to membership on applications duly certified:

To Individual Membership.

C. S. Allen, M.S. in E.E. (Lafayette). Asst. Prof., Physics, Muhlenberg Coll., Allentown, Pa.
E. E. Allen, A.M. (Park Coll.). Prof., Occidental Coll., Los Angeles, Calif.
C. F. Barr, M.S. (Chicago). Instr., Purdue Univ., W. LaFayette, Ind.
May M. Beenken, Ed.B. (So. Branch, California). Asso., So. Branch, Univ. of California, Los Angeles, Calif.
R. L. Cary, A.M. (Haverford). Woodbrook, Govans, Baltimore, Md.
Mae E. Conn, A.M. (So. California). Instr., Univ. of Southern California, Los Angeles, Calif.
Hollis Cooley, A.M. (Middlebury). Teacher, High School, Hackensack, N. J.
Jessie B. Edmondson, A.B. (Wilson). Teacher, Western High School, Washington, D. C.
Carolyn Eisele, A.B. (Hunter). Instr., Hunter Coll., New York, N. Y.
G. G. Entz, B.S. in Archit. Eng. (Columbia). 1352 Poinsettia Pl., Hollywood, Calif.
Bess M. Eversull, Ph.D. (Cincinnati). Instr., Smith Coll., Northampton, Mass.
Irving Fisher, Ph.D. (Yale). Prof., Political Econ., Yale Univ., New Haven, Conn.
S. L. Fitch, 512 W. Sullivan St., Olean, N. Y.
F. C. Hall, A.M. (Columbia). Teacher, Union High School, Redondo, Calif.
F. C. Hartwick, Licentiate in Math. and Sc. (Fribourg, Switz.). Prof., Univ. of Dayton, Dayton, Ohio.
C. M. Hobart, A.M. (Illinois). Prof., Northern Ill. State Teachers Coll., De Kalb, Ill.
G. B. Horlocker, A.B. (South Dakota). Instr., High School, Huron, S. Dak.
G. K. Howe, B.S. (Worcester Poly. Inst.). Prof., Atlanta Univ., Atlanta, Ga.
R. S. Hoyt, M.S. (Princeton). Engr., Dept. of Development and Research, Amer. Tel. and Tel. Co., New York, N. Y.
I. W. Huntzberger, Ph.D. (American Univ.). Instr., Central High School, Washington, D. C.
Olive M. Jones, A.M. (Columbia). Head of Dept., Queens Coll., Charlotte, N. C.
P. C. Jones, B.S. (Mass. Inst. of Tech.). Elec. Engr., Goodyear Tire and Rubber Co., Akron, Ohio.
Dora E. Kearney, A.M. (Minnesota). Supervisor, Teaching of Math. in Jr. and Sr. High School, State Teachers Coll., Valley City, S. Dak.
B. F. Kimball, A.M. (Harvard). Instr., Cornell Univ., Ithaca, N. Y.
P. A. Knedler, A.B. (Muhlenberg). Instr., High School, Cranford, N. J.
Martha M. Knepper, A.M. (Missouri). Prof., State Teachers Coll., Cape Girardeau, Mo.
E. J. Larkin, B.S. (Cooper Union). Instr., Cooper Union, New York, N. Y.
Sophia H. Levy, Ph.D. (California). Asst. Prof., Univ. of California, Berkeley, Calif.
Viola Lindberg, A.B. (Bethany). Instr., Bethany Coll., Lindsborg, Kans.

DECA LODWICK, Ph.B. (Iowa). Head of Dept., High School, Long Beach, Calif.
(Miss) ZUNG-NYI LOH, A.B. (Wellesley). Grad. Student, Cornell Univ., Ithaca, N. Y.
ADA A. MCCLELLAN, B.S. (Chicago). Teacher, High School, Long Beach, Calif.
MARY E. MARRS, Grad. N. Tex. Normal Coll. Asst. Prof., John Tarleton Agric. Coll., Stephenville, Tex.
W. E. MASON, M.S.E. (Michigan). Instr., Appl. Math., So. Branch, Univ. of Calif., Los Angeles, Calif.
E. B. MORRIS, Ph.B. (Sheffield Sc. School). Actuary, Life Dept., The Travelers Ins. Co., Hartford, Conn.
ANNA M. MULLIKIN, Ph.D. (Penna.). Instr., Germantown High School, Germantown, Pa.
W. C. MYERS, A.M. (American Univ.). Head Teacher, McKinley Tech. High School, D. C. *Riverdale, Md.*
HARRY OKEAN, B.S. (Alfred Univ.). Instr., Physics and Alg., High School, Lambertville, N. J.
B. C. PATTERSON, A.B. (Washington & Jefferson). Grad. Student and Student Asst., Johns Hopkins Univ., Baltimore, Md.
A. W. PHILIPS, A.M. (Chicago). Head of Dept., State Teachers Coll., Emporia, Kans.
BORIS PODOLSKY, B.S. in E.E. (So. California). 715 W. Court St., Los Angeles, Calif.
G. T. PUGH, Ph.D. (Vanderbilt). Head of Dept., Winthrop Coll., Rock Hill, S. C.
PATRICK QUILTY, C.E. Instr., Cooper Union, New York, N. Y.
G. EDNA ROBINSON, A.M. (Missouri). Head of Dept., Shorter Coll., Rome, Ga.
S. A. ROWLAND, A.B. (Ouachita). Grad. Student, Univ. of Chicago, Chicago, Ill.
RAFAEL SANCHEZ, B.S. (Coll. of A. & M. Arts, Porto Rico). Instr., High School., Mayaguez, P. R.
J. H. SCHAD, A.M. (Johns Hopkins). Instr., Arts and Sc. Coll., Univ. of Maryland, College Park, Md.
Y. S. SETO, B.S. (Huron Coll., S. Dak.). Grad. Student, Columbia Univ., New York, N. Y.
J. T. SPANN, B.S. (Mississippi). Asst. Prof., Univ. of Maryland, College Park, Md.
T. H. TALIAFERRO, Ph.D. (Johns Hopkins). Prof., Univ. of Maryland, College Park, Md.
Sister MARY THECLA, Ph.D. (Fordham). Teacher, St. Agnes Sem., Brooklyn, N. Y.
G. S. VAN FLEET, B.S. (Montana State Coll.). Instr., Univ. of Michigan, Ann Arbor, Mich.
E. E. WALDEN, A.B. (Hendrix Coll.). Prof., Lambuth Coll., Jackson, Tenn.
W. J. WALLIS, A.M. (Columbia Univ.), LL.B. (George Washington). Head of Dept., Washington High School; Teacher, George Washington Univ., Washington, D. C.
R. M. WINGER, Ph.D. (Johns Hopkins). Asso. Prof., Univ. of Wash., Seattle, Washington.

To Institutional Membership.

POMONA COLLEGE, Claremont, Calif.
UNIVERSITY OF WESTERN ONTARIO, London, Ontario, Canada.
UPSALA COLLEGE, East Orange, N. J.

The following have been appointed associate editors of the MONTHLY for the year 1925:

N. H. ANNING	H. J. ETTLINGER	C. N. MILLS
R. W. BURGESS	B. F. FINKEL	F. D. MURNAGHAN
W. B. CARVER	TOMLINSON FORT	D. E. SMITH
OTTO DUNKEL	C. F. GUMMER	V. H. WELLS.

The Secretary and Dean T. M. FOCKE of Case School of Applied Science were re-appointed as the representatives of the Association on the Council of the American Association for 1925. It was voted to hold the annual meetings in December 1925 and 1926 at Kansas City, Mo., and Philadelphia, Pa., respectively, in affiliation with the American Association; it was felt by the Trustees that the general program of the American Association meetings and the reduced railroad rates obtainable through this large gathering will attract a large number

of our members in the far West to the Kansas City meetings. The Trustees approved the organization of a Southern California Section, the sixteenth section of the Association, in response to a request of about forty members who were present at a meeting in Los Angeles, November 22, 1924. By the instructions of the Trustees the Secretary cabled the greetings of the Association to the Third Pan-American Scientific Congress then in session at Lima, Peru; the Association's representatives to this Congress were Professors E. V. HUNTINGTON and D. N. LEHMER.

Recognizing that the function of the National Committee on Mathematical Requirements had been fulfilled, this committee was formally discharged at the request of the chairman of the committee; it will be recalled that appropriate recognition of the work of the committee was made at the Cincinnati meeting as follows:

> "The Mathematical Association of America in session at Cincinnati desires to place on record its sense of high appreciation of the splendid piece of work brought to happy conclusion by the National Committee on Mathematical Requirements. The devotion displayed by the various members of the Committee in articulating and organizing the thought of the country on a wide range of important problems has made possible the accomplishment of a task which might well have been deemed of formidable magnitude.
>
> "Especially does the Association desire to recognize the great ability, tact, and energy with which the undertaking has been prosecuted by the Chairman of the Committee, JOHN WESLEY YOUNG, who has to a notable degree placed the stamp of his own personality on one of the greatest achievements of coöperative effort in the history of mathematical teaching."

It was voted to express appreciation of the strenuous and able efforts put forth by the Joint Committee on Membership of the Society and the Association and especially by Professor CLARA E. SMITH who has been actively in charge of that work. Letters have been sent to 3,500 teachers of mathematics in the colleges and universities of the United States and Canada, both members and non-members, appealing for additional memberships. On the Association side, over fifty have become members through this agency, and beside this immediate result there has been a useful campaign of educating the rank and file of our teachers to the importance of enlivening alliances with active societies like these. It is a startling fact that over half of our teachers, including many in junior colleges and the stronger normal schools, belong to neither organization; here is missionary work close at hand for all of our members!

A formal report for the Carus Publication Committee was made by Professor Slaught, who presented a proposed working agreement concurred in by Mrs. Carus, whereby the Open Court Publishing Company undertakes full responsibility for the publication and distribution of the Monographs. This plan involves a very liberal increase in expense above that contemplated in the original deed of gift and to this Mrs. Carus has acceded with the most cordial good will.

The Trustees by unanimous action adopted the recommendations of the

Committee ratifying the agreement and directed the Secretary to send to Mrs. Carus their hearty thanks for her generous spirit and to express to her their deep appreciation of her interest in the Association and its endeavor to serve the cause of mathematics through this monograph series.

An arrangement is being made whereby Chancellor A. B. CHACE of Brown University will publish under the auspices of the Association his scholarly work on the Ahmes Papyrus. It is to be in two volumes. The first will contain the free translation, commentary and notes; the second will contain over one hundred large plates on which may be traced the correspondence of the hieratic text with the hieroglyphic translation, with the English transliteration, and with the literal English translation. The terms of issue will be unusually favorable to the Association and to its members. The Trustees expressed their strong appreciation of Dr. Chace's generosity, and appointed as a committee to act for the Association, Professor R. C. ARCHIBALD, chairman, and Professor DAVID EUGENE SMITH. Detailed announcement will be made later in the year.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership. He reported also the death of the following members:

C. B. AUSTIN, Professor of mathematics, Ohio Wesleyan University (September 9, 1924).
J. E. HODGSON, Professor of mathematics, West Virginia University (April 11, 1924).
JOSEPH LIPKA, Assistant professor of mathematics, Massachusetts Institute of Technology (January 15, 1924).
C. S. SPERRY, Professor of engineering mathematics, University of Colorado (August 10, 1924).
R. E. WILSON, Dean of men, Northwestern University (December 30, 1923).
R. S. WOODWARD, former president of the Carnegie Institution (June 29, 1924).

The election of officers for the year 1925 was conducted by mail and in person at this meeting, the tellers, Professors Messick and Simpson, reporting the result of the balloting as follows:

For President: J. L. Coolidge, 188 votes; J. W. Young, 177 votes.
For Vice-Presidents: A. A. Bennett, 196 votes; Elizabeth B. Cowley, 164 votes; Dunham Jackson, 228 votes; R. E. Moritz, 136 votes.
For additional members of the Board of Trustees (to serve until January 1928): R. C. Archibald, 249 votes; L. P. Eisenhart, 195 votes; E. V. Huntington, 199 votes; D. N. Lehmer, 166 votes; G. A. Miller, 179 votes; C. N. Moore, 133 votes; F. D. Murnaghan, 98 votes; H. L. Rietz, 228 votes.

The following were accordingly declared elected:

President: J. L. COOLIDGE, Harvard University.
Vice-Presidents: A. A. BENNETT, University of Texas; DUNHAM JACKSON, University of Minnesota.

Additional members of the Board of Trustees: R. C. ARCHIBALD, Brown University; L. P. EISENHART, Princeton University; E. V. HUNTINGTON, Harvard University; H. L. RIETZ, University of Iowa.

Professor Slaught made a report for the Committee on the Carus Monographs and a hearty vote of thanks was given to Mrs. Carus for her continued and increased generosity.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 16, 1924.

RECEIPTS.

Balance Dec. 4, 1923		$5,527.56
1923 indiv. dues	$ 432.76	
1923 instit. dues	9.50	
1923 subscriptions	10.00	
1924 indiv. dues	5,707.85	
1924 instit. dues	629.20	
1924 subscriptions	833.09	
Contributions to 1924 expenses	13.00	
Initiation fees	357.00	
Life membership fees	144.52	
Sale copies MONTHLY	71.54	
Advertising	630.00	
For Monograph	3.75	
For *Annals* subscription	1.50	
Interest Oberlin Savgs. Bank	106.87	
Interest Peoples Bkg. Co.	72.59	
Interest Treasury Note.	32.81	
Interest Liberty Loan	21.25	
Interest Hardy Fund	120.00	
Total 1924 receipts		9,197.33
Total assets to the end of 1924 business		$14,724.79
Total expenditures		8,565.31
Balance to the end of 1924 business		6,159.48
Received on 1925–26 business		687.87
Book balance Dec. 16, 1924		$6,847.35

EXPENDITURES.

Publishers' bills (Nov. '23–Sept. '24)		$5,448.28
Insurance on reserve copies of MONTHLY		6.40
Manager's office		23.38
Editor-in-chief's office		633.36
Committee on Membership		70.50
Joint Committee on Membership		91.95
Part expense Register		108.40
Secretary-Treasurer's office:		
Postage	$272.40	
Bond	5.00	
Safety deposit	4.00	
Office supplies	24.95	
Express, tel., etc.	37.10	
Clerical work	793.95	
Printing	374.05	
Cincinnati meeting	50.00	
Paid copies of MONTHLY	1.75	
Refund on subscriptions	33.05	
Library expense	58.79	
Paid for *Annals* subscription	1.50	
		1,656.54
Annals subvention		250.00
Paid to sections from initiation fees		156.50
Paid to B. F. Finkel int. Hardy Fund		120.00
Total expenditures		$8,565.31
Cash on hand		39.17
Checking account		1,142.61
Oberlin Savings Bk. account		2,098.53
Peoples Bkg. Co. account		1,567.04
Liberty Bond		500.00
U. S. Treasury Note		500.00
Treas. Sav. Cert.		1,000.00
Bank balance Dec. 16, 1924		$6,847.35

Of the funds on hand, $213.36 is held in a separate Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1925, and $1,000 is held, as the beginning of a permanent endowment fund, in the form of U. S. Treasury Savings Certificates, the present value being estimated at $1,035.

Aside from the above-mentioned funds on hand, the first payment of \$1,200 on the Carus Fund amounts with the interest accruing at 4 per cent. compounded quarterly to \$1,250.01.

When the accounts were closed December 16, 1924, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1924 the following items:

Bills Receivable.	
1924 individual dues	\$200.00
Advertising	150.00
	\$350.00

Bills Payable.	
(Either paid in December or estimated.)	
Publishers' bills (Oct.–Dec. 1924)	\$2,100.00
Register	650.00
Manager's office	25.00
Editor-in-chief's office	80.00
Other editors' postage	20.00
Committee on Membership (Joint campaign)	150.00
Secretary-Treasurer's office	80.00
Initiation fees due to sections	300.00
Printing annual ballots, programs, etc.	200.00
Life Membership Fund	213.00
	\$4,018.00

If to the balance on 1924 business shown in this report, \$6,159.48, there be added the bills receivable, \$350, and there be subtracted the estimated amount of bills payable, \$4,018, there results an estimated final balance on 1924 business of approximately \$2,500. The corresponding estimated final balance one year ago on 1923 business was \$2,400; the Association continues thus to have a narrow margin of profit.

W. D. Cairns, *Secretary-Treasurer.*

ANNUAL MEETING OF THE TEXAS SECTION.

The fourth annual meeting of the Texas section of the Mathematical Association of America was held in San Antonio, Texas, November 28, 1924. In the morning the section held a joint session with the mathematical section of the Texas State Teachers Association. These meetings were held in the primary room of the First Presbyterian Church.

In the morning Professor J. M. Bledsoe of the East Texas State Teachers College, Commerce, Texas, presided as Chairman of the mathematical section of the Texas State Teachers Association. The afternoon session was presided over by Professor L. R. Ford of Rice Institute, Houston, Texas, Chairman of the Texas section of the Association. There were seventy-five (75) present, including the following members of the Association:

W. L. Ayres, A. A. Bennett, H. E. Bray, J. E. Burnam, E. Dice, H. J. Ettlinger, L. R. Ford, H. Halperin, A. S. Hathaway, G. P. Horton, G. K. Mayne, C. A. Rupp, G. S. Smith, G. R. West, W. M. Whyburn, C. N. Wunder.

The following papers were presented at the joint meeting in the morning:

(1) "Junior high school mathematics" by J. F. HOWARD, San Antonio Junior High Schools.

(2) "The viewpoint governing mathematical instruction" by Miss ELIZABETH DICE, North Dallas High School.

(3) "A theorem concerning parallelograms" by J. E. BURNAM, Simmons College.

(4) "Economic value of the study of mathematics" by H. J. ETTLINGER, University of Texas.

(5) "Reorganization of secondary education in the light of survey findings" by F. J. KELLY, University of Minnesota.

(6) "Why young women should include one year of mathematics in their college training" by Miss EDNA GRAHAM, West Texas State Teachers College.

(7) "The beauties of geometry" by J. W. CALHOUN, University of Texas.

In the afternoon the following papers were presented:

(1) "The solution of equations by successive approximation" by L. R. FORD, Rice Institute.

A study of a very general method of solving numerical equations of all kinds; particular reference was made to quadratic equations in illustrating the method.

(2) "Darboux integrals" by G. P. HORTON, University of Texas.

It was shown in this paper that the Riemann theory of integration follows from a study of the relation of the Darboux integrals to the Lebesgue integral of the maximum function.

(3) "Some properties of the annulus" by J. E. BURNAM, Simmons College.

(4) "Geometrography" by A. A. BENNETT, University of Texas.

Geometrography is the science of analyzing geometrical constructions into elementary steps, classifying and weighting these elementary steps, and thus weighting geometrical constructions.

(5) "Anharmonic ratios" by A. S. HATHAWAY, Boerne, Texas.

(6) "Fourier series" by H. J. ETTLINGER, University of Texas.

This paper dealt with the convergence of a Fourier expansion of a general class of functions and the Gibbs phenomenon of the series thus obtained.

(7) "Statistical correlation and linear independence" by H. E. BRAY, Rice Institute.

The purpose of the paper was to show how the formula for multiple correlation can be derived by means of a useful theorem on determinants.

(8) "Approximation formulæ" by A. J. MARIA, Rice Institute (by invitation).

This paper dealt with a method of constructing simple formulæ to replace complicated ones; the derivation of a simple formula for the length of an ellipse is used to exemplify the method.

(9) "On explosion transformations" by C. A. RUPP, University of Texas.

The paper dealt with questions attendant upon the formation of envelopes by replacing the points of a curve by circles whose radii vary in accordance with some known law.

(10) "On an extension of a definition of a Green's function" by W. M. WHYBURN, Texas A. and M. College.

This paper appeared in full in the Sept.–Dec. (1924) number of the *Annals of Mathematics*, 2d series, vol. 26, pp. 125–130.

The following officers were elected to serve for the next year: A. A. BENNETT, University of Texas, chairman; W. M. WHYBURN, Texas A. and M. College, vice-chairman. The term of the present Secretary-Treasurer continues for three (3) more years.

A very pleasant feature of the meeting was a dinner in honor of the Texas section at Our Lady of the Lake College, a first rank senior college for girls, conducted by the Sisters of Divine Providence. Father Constantineau, President of the college, welcomed the guests in behalf of the Sisters of Divine Providence, who were the hosts of the occasion. Dr. A. A. Bennett, Chairman elect, responded for the section and expressed the appreciation of the section for the hospitality extended to them. Dr. L. R. Ford, retiring chairman, also made a few remarks.

H. J. ETTLINGER, *Secretary-Treasurer.*

AN ALGEBRAICALLY REDUCIBLE SOLUTION OF THE CUBIC EQUATION.[1]

By GLENN JAMES, Southern Branch, University of California.

The essential steps in this treatment of the cubic equation are deriving a fundamental theorem on interpolation, isolating a root and solving for this root by means of the interpolation theorem. The irreducible case is given separate treatment, partly to introduce the general theory but mainly because the separate treatment gives formulæ for the roots that are simpler than those immediately deducible from the general solution.

1. Interpolation Theorem: *If $f(x)$ is a polynomial such that*

$$0 < f(x_1)/[f(x_1) - f(x_n)] \overline{\gtrless} 1 \qquad (A)$$

and

$$\lim_{x_n \to r} f(x_1)/[f(x_1) - f(x_n)] = 1, \qquad (B)$$

then r is a root of the equation

$$f(x) = 0. \qquad (C)$$

PROOF: The first hypothesis implies that $f(x_1)$ and $f(x_n)$ have different signs for the denominator must have the same sign as the numerator. Hence, clearing of fractions and subtracting $f(x_1)$ from both members of the inequality gives [2]

$$-f(x_n) \overline{\gtrless} 0 \text{ whenever } f(x_1) > 0$$

[1] Presented to the American Mathematical Society, February 28th, 1925, under the title, "A Complete Solution of the Cubic Equation."

[2] Assumption (A) excludes the trivial case in which $f(0) = 0$.

and

$$-f(x_n) \leqq 0 \text{ whenever } f(x_1) < 0.$$

Thus a root of (c) is either x_n or lies in the interval bounded by x_1 and x_n. Hypothesis (B) implies that

$$\lim_{x_n \to r} [f(x_1) - f(x_n)] = f(x_1),$$

from which it follows that

$$\lim_{x_n \to r} f(x_n) = 0.$$

Whence,[1]

$$f(r) = 0.$$

2. Solution of the Irreducible Cubic Equation. Consider the cubic equation in the form

$$x^3 + px + q = 0, \tag{1}$$

where

$$q^2/4 + p^3/27 \leqq 0, \qquad q \neq 0 \tag{2}$$

and the signs of the roots have been changed, if necessary, to make the constant term positive. The condition (2) can be written in the form

$$|q| \leqq 2\sqrt{-p^3/27} = (2|p|/3)\sqrt{-p/3} - e^2, \tag{3}$$

where e^2 depends upon p and q. Substituting this form of q in the left member of (1), giving x the value $\sqrt{-p/3}$ and simplifying, we get $-e^2$. Hence one root, at least, lies in the closed interval bounded by 0 and $\sqrt{-p/3}$. Now divide[2] the roots of (1) by $\sqrt{-p/3}$, that is, substitute $x = x'\sqrt{-p/3}$. Whence (1) takes the form

$$x^3 - 3x + c = 0, \tag{4}$$

where

$$0 < c = |q|\sqrt{27/-p^3} \leqq 2, \tag{5}$$

the second inequality resulting from (2).

Let r_1, r_2, r_3 be the roots of (4), written in ascending order of magnitude. Then from the above transformation and from consideration of $f(-1)$, $f(-2)$, $f(1)$ and $f(2)$ we see that[3]

$$-1 \geqq r_1 \geqq -2, \qquad 1 \geqq r_2 > 0, \qquad 2 \geqq r_3 \geqq 1.$$

[1] The restriction that $f(x)$ be a polynomial is more stringent than necessary to make this step valid. It would suffice to assume that $f(r)$ is defined.

Throughout this paper the notion of limit approaching shall include, "vacuously," the case in which the limit is actually reached.

[2] Since the roots are real p cannot be zero.

[3] Smaller intervals can be found, for instance $\sqrt[3]{c/2} \geqq r_2 > 0$, but the evaluation of this radical makes more labor in the evaluation of the root than does the use of a larger interval.

Solution for r_2. Consider the linear interpolation formula

$$x_{n+1}/x_n = f(0)/[f(0) - f(x_n)]. \tag{6}$$

Letting $f(x) = x^3 - 3x + c$, this becomes

$$x_{n+1}/x_n = c/[c - (x_n^3 - 3x_n + c)] = c/[3x_n - x_n^2], \tag{7}$$

whence [1]

$$x_{n+1} = c/[3 - x_n^2]. \tag{8}$$

The hypotheses of the interpolation theorem will be satisfied by (6) when we have proved that the terms of the sequence $x_1, x_2, \ldots x_n \ldots$ never increase and are positive, for then

$$0 < f(0)/[f(0) - f(x_{n+1})] \leqq 1$$

and x_n will approach a limit, different from zero, by 8; it follows that

$$\lim_{n\to\infty} x_n/x_{n-1} = \lim_{n\to r_2} f(0)/[f(0) - f(x_n)] = 1.$$

By (8) we have

$$x_2 = c/[3 - x_1^2] = c/[3 - 1] = c/2, \tag{9}$$

whence

$$|x_2| \leqq |x_1| = 1. \tag{10}$$

Now

$$|c/[3 - x_{n+1}^2]| \leqq |c/[3 - x_n^2]|, \tag{11}$$

that is,

$$|x_{n+2}| \leqq |x_{n+1}| \tag{12}$$

provided

$$|x_{n+1}| \leqq |x_n| < 3. \tag{13}$$

But (10) is a case in which (13) holds, which completes the induction. Obviously x_n is positive since

$$x_n = c/[3 - x_{n-1}^2] \qquad \text{and} \qquad |x_{n-1}| \leqq c/2 \leqq 1.$$

We have now proved that r_2 is a root of (3), r_2 being the limit of the function built up by means of the recursion formula

$$x_n = c/[3 - x_{n-1}^2], \qquad x_1 = 1. \tag{14}$$

In practice the approximation of this root can well be made by the following rule:

Subtract unity from three, divide the difference into c, square the quotient, subtract the result from three, divide the difference into c, etc.

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[1] Geometrically this is equivalent to holding fixed the point whose abscissa is zero and moving the point whose abscissa is x_n along the curve toward the x-axis.

We will write r_2 in the following form:

$$r_2 = c/3^-[c/3^-[c/3^- \dots [c/3^-[c/3^-[1]^2]^2]^2 \dots]^2]^2. \tag{15}$$

Since a change in the sign of c merely changes the sign of r_2, we can disregard the restriction that the constant term of (1) must be positive and say that the above expression multiplied by $\sqrt{[-p/3]}$ is a root of (1) whatever be the sign of the constant term. Obviously the formula holds when $c = 0$.

Solutions[1] *for* r_1 *and* r_3. In order to secure r_1, add 2 to the roots of (4), obtaining

$$x^3 - 6x^2 + 9x - 2 + c = 0. \tag{16}$$

The root of this equation which is equal to $r_1 + 2$ we will denote by r_1'. Then $0 \,\overline{\gtrless}\, r_1' \,\underline{\gtrless}\, 1$. Applying formula (6), temporarily assuming that $c - 2 \neq 0$, we have

$$\begin{aligned} x_{n+1}/x_n = f(0)/[f(0) - f(x_n)] &= [-2 + c]/[-x_n^3 + 6x_n^2 - 9x] \\ &= [2 - c]/[x_n(x_n - 3)^2], \end{aligned} \tag{17}$$

whence

$$x_{n+1} = [2 - c]/(3 - x_n)^2,$$

and essentially the same procedure as we used in securing r_2 gives[2]

$$r_1' = (2-c)/0^+[3 - [(2-c)/0^+ \dots [3 - [(2-c)/0^+[3 - 1]^2]^2]^2 \dots]^2]^2. \tag{18}$$

We will denote this function of c by $I(2 - c, 0, 3, 1)$. Then

$$r_1 = I(2 - c, 0, 3, 1) - 2. \tag{19}$$

Interpreted literally this formula means: *Subtract unity from three, square the difference and divide the result into two minus c, subtract the result from three, square and divide into two minus c, etc., and subtract two from the last approximation.*

In order to solve for r_3, subtract[3] 2 from the roots of equation (4), change the signs of the roots in the new equation, then proceed as when solving for r_1'. This gives

$$\begin{aligned} r_3' &= (2+c)/0^+[3 - [(2+c)/0^+ \dots [3 - [(2+c)/0^+[3 - 1]^2]^2]^2 \dots]^2]^2, \\ &= I(2 + c, 0, 3, 1) = -r_3 + 2. \end{aligned}$$

Whence

$$r_3 = -I(2 + c, 0, 3, 1) + 2. \tag{20}$$

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[1] We could of course divide out $x - r_2$, then solve the resulting quadratic for the other two roots, but this is more tedious than using the direct solutions, when the extreme roots alone are desired.

[2] This notation conforms with the general notation used later.

[3] This transformation, rather than reducing the roots by unity, is necessary for if we start the solution by the latter plan it is immediately seen that the interpolation theorems fails.

Observing that

$$I(2 - c, 0, 3, 1) = - I(c - 2, 0, 3, 1),$$

we can write the two roots, r_1 and r_3, in the combined form

$$\mp 2 - I(c \mp 2, 0, 3, 1). \tag{21}$$

In order to secure the extreme roots of (1) from this formula, we note that to change the sign of c merely interchanges the pair (21) and changes their signs, which is as it should be, since changing the sign of c merely changes the signs of the roots of (4). Thus, whatever be the sign of q, the extreme roots of (1) are

$$\pm 2\sqrt{-p/3} + \sqrt{-p/3}\, I(c \pm 2, 0, 3, 1), \tag{22}$$

where

$$c = q/\sqrt{-p/3}.$$

3. The Solution of Any Cubic Equation. Consider the equation

$$x^3 + px + q = 0. \tag{23}$$

If q be negative, change the sign of the roots. That is, we first solve the supplementary equation

$$x^3 + px + |q| = 0. \tag{24}$$

The first step is to locate one root between zero and some other limit. This is done by considering limiting values for p and q. Let r_1 be the root we are seeking, and x_1 be the other limit of this root. Since $f(0) = |q|$, x_1 must be such that $f(x_1) < 0$. Hence, if q be zero, we must have

$$x_1^3 + px_1 = x_1(x_1^2 + p) < 0.$$

Whence, if p is positive, x_1 must be negative, while if p is negative, x_1 must be equal to or less than $\sqrt{|p|}$. Hence $-\sqrt{|p|}$ is the largest value that x_1 can have for all values of p, and q equal to zero. Now let p be zero. Then x_1 must be equal to or less than $-\sqrt[3]{|q|}$. This suggests the limit $-\sqrt{|p|} - \sqrt[3]{|q|}$. And actual substitution in (24) gives

$$-(|p| + p)\sqrt{|p|} - 4|p|\sqrt[3]{|q|} - 3\sqrt{|p|}\sqrt[3]{|q|^2}$$

which is always negative. Hence [1]

$$-(\sqrt{|p|} + \sqrt[3]{|q|}) < r_1 \gtreqless 0.$$

[1] Not more than one root lies in this interval for (24) cannot have three negative roots. However, this fact is not used in the following solution.

See "An inequality for the roots of an algebraic equation" by J. L. Walsh, *Annals of Mathematics*, vol. 25, no. 3, p. 285, in which he determined the limits for the roots, in the complex plane.

We exclude the case $f(-\sqrt{|p|} - \sqrt{|q|}) = 0$, for in this case the hypotheses of our interpolation theorem do not hold. See (27). However, the formula that will be obtained will be seen to give the root in this case also.

Now add $\sqrt{|p|} + \sqrt[3]{|q|}$ to the roots of (24). The new equation is

$$(x - k)^3 + px - pk + |q| = 0, \tag{25}$$

where

$$k = \sqrt{|p|} + \sqrt[3]{|q|} \geqq 0. \tag{26}$$

We seek the root r_1' such that $r_1 = r_1' - k$. Now write the linear interpolation formula

$$\begin{aligned} x_{n+1}/x_n &= f(0)/[f(0) - f(x_n)] \\ &= [k^3 + pk - |q|]/[x_n^3 - 3x_n^2k + 3x_nk^2 + px_n]. \end{aligned} \tag{27}$$

Multiplying both members by x_n and completing the square in the denominator on the right gives

$$x_{n+1} = [k^3 + pk - |c|]/[p + 3k^2/4 + (3k/2 - x_n)^2]. \tag{28}$$

We will need to compare this identity with that obtained by substituting $n - 1$ for n, namely

$$x_n = [k^3 + pk - |q|]/[p + 3k^2/4 + (3k/2 - x_{n-1})^2]. \tag{29}$$

In order to prove that the interpolation theorem holds, we need only show that, for all values of n, x_n is positive and never increases. For x_n then approaches a limit $\neq 0$ and hence x_n/x_{n-1} approaches unity. Obviously x_n is positive if $x_{n-1} \leqq k$, which is true if x_n never increases, for $x_1 = k$. We will prove by induction that x_n never increases. From (28) and (29) we see that, if

$$|x_n| \leqq |x_{n-1}| \leqq 3k/2, \tag{30}$$

then

$$|x_{n+1}| \leqq |x_n|. \tag{31}$$

The induction will then be complete when we shall have verified (30) for a particular value of n. Since the upper limit of the root is k, we start with k ($= x_1$). Putting $n = 2$ and $x_1 = k$ in (29) gives

$$\begin{aligned} x_2 &= [k^3 + pk - |q|]/[p + 3k^2/4 + (3k/2 - k)^2] \\ &= [k^3 + pk - |q|]/[p + k^2]. \end{aligned} \tag{32}$$

Whence

$$|x_2| < |x_1|,$$

for the right member of (32) is easily seen to be less than k.

Using (29) as a recursion formula, we write

$$r_1^1 = c_1/c_2{}^+[c_3 - [c_1/c_2{}^+[c_3 - \ldots [c_1/c_2{}^+[c_3 - [c_1/c_2{}^+[c_3 - k]^2]^2]^2]^2 \ldots]^2]^2]^2, \tag{33}$$

where

$$c_1 = k^3 + pk - |q|, \tag{34}$$
$$c_2 = p + 3k^2/4, \tag{35}$$
$$c_3 = 3k/2, \tag{36}$$
$$k = \sqrt{|p|} + \sqrt[3]{|q|}. \tag{37}$$

We will denote the second member of (33) by

$$I(c_1, c_2, c_3, k). \tag{38}$$

Then one root of (24), namely r_1, is $I(c_1, c_2, c_3, k) - k$ Now a change of sign in the constant term of (23) merely changes the signs of the roots. Therefore, in all cases, one root of (23) is [1]

$$[q/|q|][I(c_1, c_2, c_3, k) - k]. \tag{39}$$

Factoring x minus this root out of (23), solving the resulting quadratic and simplifying the results, gives for the other two roots

$$-[q/2|q|][I(c_1, c_2, c_3, k) - k] \pm [1/2]\sqrt{-3[I(c_1, c_2, c_3, k) - k]^2 - 4p}. \tag{40}$$

4. Comments. If q be zero and p be negative, the hypotheses of the interpolation theorem do not then hold. However, actual substitution of $q = 0$ shows that (39) and (40) still hold if we take the precaution of dividing the common factor $p + k^2$ out of numerator and denominator in the first approximation, see (32), before applying the formula. This exception to the general solution can be avoided by taking for k a rational number larger than $\sqrt{|p|} + \sqrt[3]{|q|}$, preferably as nearly equal to the latter as can be easily discerned. Moreover, the extra steps required to attain the same degree of accuracy with the larger $|k|$ are not in general as tedious as the evaluation of $\sqrt{|p|} + \sqrt[3]{|q|}$. Inspection will show that such a choice of k would in no way invalidate the derivation of the above formulæ for the roots.

ILLUSTRATIVE EXAMPLE. Solve $x^3 - 3x + 1 = 0$. Using formula [2] (15),

$$c/(3 - [1]^2) = .5, \qquad c/[.5]^2 = 4/11 = .3636\ldots,$$
$$c/(3 - [4/11]^2) = .3487+, \qquad c/(3 - [.3487+]^2) = .3474+,$$

and the next four approximations are

.34730625, .347297184, .347296424

and

.34729636.

The last approximation is correct to eight places. Substitution in the left member of (a) gives a remainder of .000 000 012 +. Using formula [3] (19), five approximations give for a second

[1] In the trivial case $q = 0$, it is necessary to define $q/|q|$ as zero.

[2] This is of the type $I(c_1, c_2, c_3, k)$, for it can be written $iI(ic, 3, 0, i)$ where $i = \sqrt{-1}$. The function I is of interest *per se*. The study of this function, when p and q have the continuum for ranges, will be taken up in a later paper.

[3] Instead of using (19), we could have gotten the same formula by putting $p = -3$, $q = 1$ and $k = 2$ in (39). This value of k is less than $\sqrt{3} + \sqrt[3]{1}$ and gives a more rapid approach to

root

$$-\ 1.87938\ -.$$

Using (2), eleven [1] approximations give the third root,

$$1.531993\ +.$$

The sum of the three roots should be zero and it is .000 091 −.

POINTS ON THE CIRCUMCIRCLE.

By J. W. CLAWSON, Ursinus College.

There are several methods by which definite points on the circle circumscribing a triangle may be associated with a given straight line in the plane of the triangle; some of these methods are discussed in the first three sections of this note. Few of the facts in this part of the paper are new, but the arrangement, the point of view and most of the analytical results are original. If one of the remarkable lines of the triangle be taken as the given line, an associated point may be worthy of notice among the remarkable points of the triangle. Some of these points are listed in the fourth section; among them are two notable points and another point which has been mentioned; but there is considerable new material in this section. Sections 5 and 6 are entirely new, so far as I am aware.

1. It is well known that every point on the circumscribed circle of a triangle possesses a pedal (or Wallace or Simson) line [2] on which are the feet of the perpendiculars from that point to the sides of the triangle. It is also well known that if the lines, instead of being drawn perpendicular to the sides, are drawn so as to make any given angle with the sides the feet are again collinear.[3] All the lines so drawn envelop a parabola which has the given point at its focus.[4] Indeed if p, q, r are the trilinear normal coördinates of the point, and if a, b, c are the lengths of the sides of the triangle, the equation of the parabola is

$$a^4x^2/p^2 + b^4y^2/q^2 + c^4z^2/r^2 - 2b^2c^2yz/qr - 2c^2a^2zx/rp - 2a^2b^2xy/pq = 0,$$

and its directrix is

$$a\cdot\cos A\cdot qrx + b\cdot\cos B\cdot rpy + c\cdot\cos C\cdot pqz = 0,$$

where A, B, C are the angles of the triangle. It is easily verified that this directrix

the root than would the latter or the next larger integer, 3. In fact the evaluation of all the formulæ of this article can be expedited by giving increments to each approximation, these increments having been estimated roughly from the differences between preceding approximations.

[1] The best general procedure would be to use formula (19) or (20) according as c is positive or negative, then solve the reduced equation for the other roots.

[2] Leybourn's *Mathematical Repository*, old series, II (1798), p. 111. Mackay, *Proc. Edin. Math. Soc.*, IX (1890–1891), p. 83. This MONTHLY (*1916*, 61).

[3] Poncelet, *Propriétés Projectives*, vol. 2, p. 261, 2d edit. (1866).

[4] Steiner, *Gergonne's Ann.*, XIX (1828). (Mackay, *l.c.*)

is parallel to the pedal line of the point, whose equation is

$$(a\cdot\cos A - p\cdot\sin A)qrx + (b\cdot\cos B - q\cdot\sin B)rpy + (c\cdot\cos C - r\cdot\sin C)pqz = 0.$$[1]

Conversely, each line in the plane is associated with a single point on the circumcircle; it is a member of one of these families of lines.[2] Let ABC be the triangle, and let any line cross the sides at P, Q, R respectively. Let the equation of this line be $lx + my + nz = 0$. Then if lines are drawn from P, Q, R making the angle θ with BC, CA, AB, it can be shown[3] that these lines are concurrent for but one value of the angle, namely, that given by

$$\tan\theta = \frac{a^2mn(bn - cm) + b^2nl(cl - an) + c^2lm(am - bl)}{a^2mn(n\cdot\cos B + m\cdot\cos C) + b^2nl(l\cdot\cos C + n\cdot\cos A) + c^2lm(m\cdot\cos A + l\cdot\cos B) - lmn(a^2 + b^2 + c^2)}$$

Using this angle, the point of concurrency is

$$\frac{a}{l(bn - cm)},\quad \frac{b}{m(cl - an)},\quad \frac{c}{n(am - bl)}.$$

It is easily verified that this point lies on the circumcircle, whose equation is

$$ayz + bzx + cxy = 0.$$

The point, J, at which these lines concur can be located as follows. From A draw a line parallel to PQR cutting the circle at U. Then UP crosses the circle again at J. This may be proved analytically, or as follows. Angles AUJ, QPJ are equal; and angles AUJ, ACJ are equal; hence angles QPJ, QCJ are equal, and the points Q, P, C, J are concyclic. Hence angles JPC, JQC are equal, so JP, JQ make equal angles with BC, CA respectively. It is easily proved that if BV, CW are drawn parallel to PQR, cutting the circle at V, W, then VQ, WR cross the circle again at the same point J, and hence that the angle JRA is also equal to the angle JPC.[4]

2. Again, starting with a given line, a point on the circumcircle may be found whose pedal line is parallel to the given line.[5] The coördinates of this point are

$$\frac{1}{l - m\cdot\cos C - n\cdot\cos B},\quad \frac{1}{m - n\cdot\cos A - l\cdot\cos C},\quad \frac{1}{n - l\cdot\cos B - m\cdot\cos A}.$$

[1] Casey, *Analytical Geometry*, 2d edit. (1893), gives a different form on p. 136.

[2] I have not found this fact recorded in places where the pedal line theorems are collected.

[3] The analytical work is tricky and tedious here and also in obtaining the other results.

[4] This is new, but compare an example at the foot of p. 164 in Hutton's *Principles of Projective Geometry* (1913).

[5] The fact is stated by Alison, *Proc. Edin. Math. Soc.*, III (1884–1885), § 17, cor. 3. The construction which immediately follows is found in several places, for example in Richardson and Ramsay's *Modern Pure Geometry* (1893), p. 54.

This point may be located in three ways.

(*i*) Draw AU parallel to the given line cutting the circle at U; then draw UO perpendicular to BC cutting the circle at O; and O is the point required. For, if ON is drawn perpendicular to AB and if N is joined to L, the intersection of UO with BC, since OLB, ONB are right angles, O, N, L, B are concyclic and the angles NBO, NLO are equal. But since O, A, U, B are concyclic, the angles AUO, ABO are equal. Hence angles NLO, AUO are equal, and therefore LN is parallel to UA which was drawn parallel to the given line.

(*ii*) Draw AX perpendicular to the given line. Then draw AO so that the angles OAB, CAX are equal. This line crosses the circle at O, which is the point required.[1] For, draw AU parallel to the given line. Join UB and UO. Now angles OAB, OUB are equal. Hence angles CAX, OUB are equal. Also angles UAC, UBC are equal. Therefore the sum of angles CAX, UAC equals the sum of angles OUB, UBC. But AX is perpendicular to AU. Hence OU is perpendicular to BC, and therefore the point O is the same as that reached by the preceding construction.

(*iii*) Take any two points, S and T, on the given line. Draw lines from S and T respectively parallel and perpendicular to the three sides of the triangle and let corresponding lines intersect at A', B', C'; S and T are evidently extremities of a diameter of the circle which circumscribes the triangle $A'B'C'$, and it is easily seen that this triangle is similar to the triangle ABC. If lines are drawn from A, B, C parallel respectively to $B'C'$, $C'A'$, $A'B'$, these lines will meet at a point, O, on the circumcircle which is the same point as that found by the preceding methods.[2] For, if AO is drawn parallel to $B'C'$, angles BAO, $SC'B'$ are equal. Also, if AX is perpendicular to ST, and if $B'Z$ is also drawn perpendicular to ST, the angles CAX, $SB'Z$ are equal. Now $B'ST$ is the complement of $SB'Z$; so is $B'A'T$. Again, angles $SC'B'$, $SA'B'$ are equal. But $SA'B'$ is the complement of $B'A'T$. Hence the angles BAO, CAX are equal, and the point O is the same as that reached by the preceding construction.

Thus, by any one of the three methods given, it is easy to find a second point, O, on the circumcircle associated with the given line.

3. Again, if any point, S, whose coördinates are f, g, h be taken on the given line, and if AS, BS, CS meet the circle at U', V', W', then it is not hard to show that $U'P$, $V'Q$, $W'R$ are concurrent at a point,[3] O', on the circle, whose coordinates are a/fl, b/gm, c/hn. Thus to each point on the given line corresponds a point on the circumcircle. In fact, the J-point discussed in the first section is the O'-point corresponding to S at infinity.

4. Now if we take one of the well-known lines[4] connected with a triangle, points can be found on the circumcircle which are associated with these re-

[1] Lachlan, *Modern Pure Geometry* (1893), p. 68, ex. 4.

[2] This is a generalization of the well-known construction to find Steiner's Point by drawing parallels from the vertices to the sides of Brocard's First Triangle.

[3] A pure geometry proof of this theorem will be found in this MONTHLY (*1919*, 59).

[4] The coördinates and equations of the most important points and lines connected with the triangle are collected in Vigarié's "Inventoire," published in *Comptes Rendus*, 1887.

markable lines in the ways described above. Some of the points so obtained may be found to have enough remarkable properties to warrant their inclusion among the notable points of the triangle. Among them are the following:

(*i*) Point whose coördinates are

$$\frac{a}{b^2 - c^2}, \quad \frac{b}{c^2 - a^2}, \quad \frac{c}{a^2 - b^2}.$$

This is the J-point of Lemoine's line (the radical axis of the circumcircle and Brocard's circle); it is the O-point of Euler's line;[1] it is the O'-point of the orthocenter on Euler's line, of the symmedian point on Brocard's diameter, of the centroid on the line joining the symmedian point to the centroid, and of the inverse point of the incenter on the line joining that point to the incenter. This point is the intersection of the circumcircle with the ellipse (the isogonal transformation[2] of the orthic axis), which has the symmedian point for center and which also circumscribes the triangle; it is harmonically associated[3] with Brocard's diameter; and it is the focus of Kiepert's parabola.[4]

(*i'*) The point diametrically opposite to the point just considered. Its coördinates are

$$\frac{1}{2a^4 - b^4 - c^4 + 2b^2c^2 - a^2b^2 - a^2c^2},$$

and two similar expressions. It is the O-point of the orthic axis; and the intersection of the circumcircle with the isogonal transformation of Euler's line.

(*ii*) Steiner's Point, whose coördinates are

$$\frac{1}{a(b^2 - c^2)}, \quad \frac{1}{b(c^2 - a^2)}, \quad \frac{1}{c(a^2 - b^2)}.$$

This is the O-point of Brocard's diameter;[5] it is the O'-point of the symmedian point on the line joining that point to the centroid, of the incenter on the line joining that point to its inverse, of the orthocenter on the line joining that point to its inverse, and of the pole of the line joining Brocard's points on Brocard's diameter; it is also known as the intersection of the circumcircle with Steiner's ellipse (the isogonal transformation of Lemoine's line), having the centroid as center and which also circumscribes the triangle; it also has other properties.[6]

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[1] Gallatly, *Modern Geometry of the Triangle* (n.d.), p. 26.

[2] If lines SA, SB, SC are drawn joining a point, S, to the vertices, and if other lines are drawn from the vertices so that angles SAC, BAS'; SBA, CBS'; SCB, ACS' are respectively equal, these lines meet at a point S', which is isogonally conjugate to S; and if S moves along a given line, the locus of S' is a conic which is the isogonal transformation of the given line.

[3] *I.e.*, lines from the vertices through the given point cut the opposite sides in points which together with the points in which the given line cuts these sides form harmonic ranges.

[4] Casey, *l.c.*, p. 458.

[5] Gallatly, *l.c.*, p. 102.

[6] Casey, *l.c.*, p. 451.

(ii') The point diametrically opposite to Steiner's Point is known as Tarry's Point. Its coördinates are

$$\frac{1}{a(b^4 + c^4 - a^2b^2 - a^2c^2)},$$

and two similar expressions. This is the O-point of Brocard's diameter and also of Lemoine's line; it is the point where the circumcircle intersects Kiepert's hyperbola (the isogonal transformation of Brocard's diameter).

(iii) The point whose coördinates are

$$\frac{1}{b - c},\quad \frac{1}{c - a},\quad \frac{1}{a - b}.$$

This is the O-point of the line joining the circumcenter and the incenter; it is the O'-point of the symmedian point on the line joining that point to the incenter, of the centroid on the line joining that point to the incenter; it is also harmonically associated with the line joining the symmedian point to the incenter, and it is the intersection of the circumcircle with the ellipse which is the isogonal transformation of the antiorthic axis.

(iii') The point diametrically opposite to the point just considered. Its coördinates are

$$\frac{1}{1 - \cos B - \cos C},\quad \frac{1}{1 - \cos C - \cos A},\quad \frac{1}{1 - \cos A - \cos B}.$$

This is the O-point of the antiorthic axis of the triangle.

(iv) The point whose coördinates are

$$\frac{a}{b - c},\quad \frac{b}{c - a},\quad \frac{c}{a - b}.$$

This is the J-point of the antiorthic axis; it is the O'-point of the incenter on the line joining that point to the symmedian point, and of the centroid on the line joining that point to the incenter.

5. From a consideration of some of these special cases, the following general theorem was discovered:

THEOREM: *The O-point of any straight line is diametrically opposite to the point of intersection of the isogonal transformation of the given line with the circumcircle.*

The theorem is easily proved analytically.

6. Finally, the following theorems concerning a complete quadrilateral may be noted:

THEOREM: *The focal point*[1] *of a complete quadrilateral is the J-point of each side with respect to the triangle determined by the other three sides.*

THEOREM: *The intersection of the circle circumscribed to three sides of a complete quadrilateral with the circumcentric circle*[2] *of that quadrilateral is the O-point of*

[1] Clawson, *Annals of Math.*, XX (1919), p. 233.
[2] Clawson, *l.c.*, p. 236, § 5b.

the fourth side of the quadrilateral with respect to the triangle determined by the other three sides.

These theorems follow at once from theorems in the article referred to in the footnotes.

LOCI OF POINTS OF A MOVING BODY WHICH ARE EXCEPTIONAL POINTS OF THEIR TRAJECTORIES.

By E. L. REES, University of Kentucky.

In this paper we shall make a study of the instantaneous loci of points of a moving body which are exceptional points of their trajectories. The advantages in kinematics of vector methods which are used here have been noted by the writer in previous articles.[1]

1. Curve of Inflections. Let us find first the instantaneous locus of points of the body which are points of inflection of their trajectories. We shall see that this locus is in general a curve of the *third order* which we shall call the *curve of inflections.*

At a point of inflection the curvature is zero and the acceleration normal to the curve is zero. In other words the acceleration has the direction of the velocity at a point of inflection. Consequently such points are subject to the condition[2]

$$\dot{r} \times \ddot{r} = (\dot{p} + \dot{q}) \times (\ddot{p} + \ddot{q}) = 0, \tag{1}$$

in which p is the position vector of a point P fixed in the body, q the vectors from P to the points of the body satisfying the condition and r the position vectors of these latter points. It is understood of course that dots indicate time derivatives.

We shall consider first the general case in which $w \times \dot{w} \neq 0$, where w is the angular velocity vector. Taking P at the center of acceleration,[3] we have for our condition $(\dot{p} + \dot{q}) \times \ddot{q} = 0$. Multiplying this equation by $\dot{p}\cdot$, we get

$$[\dot{p}\dot{q}\ddot{q}] = [w\dot{w}q]\dot{p}\cdot q + w\cdot\dot{p}(w \times q)^2 = 0, \tag{2}$$

a homogeneous equation of the second degree in q. The curve of inflections therefore lies on a quadratic cone with the center of acceleration as vertex and with one element parallel to the central axis. The center of acceleration is of course a point of this curve, *i.e.*, the curve passes through the vertex of the cone.

If we divide the equation $(\dot{p} + \dot{q}) \times \ddot{q} = 0$ by q^2 and let $q = \infty$, we get $[w\dot{w}q_1]q_1 + (w \times q_1)^2 w = 0$, the subscript indicating unit vectors. Obviously

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[1] See this MONTHLY (*1923*, 290–296); also (*1924*, 131–135). For a synthetic treatment of the subject of this paper see Schönflies, *La Géométrie du Mouvement*, Paris, 1893, pp. 139–144.

[2] This condition is satisfied also by points whose velocities are zero and by those whose accelerations are zero if there are any such points; and there may be certain points other than points of inflection for which the velocity and acceleration have the same direction such as points of undulation. All such points are of course contained in the loci discussed in this article.

[3] See an article by Ziwet and Field, this MONTHLY (*1916*, 376–378).

this equation is satisfied when and only when q has the direction of w. Hence we may say that the curve of inflections contains one and only one point at infinity, namely, the infinitely distant point on the central axis.

The condition for a point of inflection may also be written $\ddot{q} = u(\dot{p} + \dot{q})$, or

$$\dot{w} \times q + w \times (w \times q) = u(\dot{p} + w \times q). \tag{3}$$

Each value of the parameter u determines in general a definite q, which locates a unique point of inflection. By letting u vary we get the locus of these points of inflection, *i.e.*, the curve of inflections. Recalling that $w \times \dot{w} \neq 0$, we may write the vector equation of this curve in the form

$$q = w \cdot qR + \dot{w} \cdot qS + w \times \dot{w} \cdot qT,$$

where R, S, T and $w, \dot{w}, w \times \dot{w}$ are reciprocal sets of vectors, and the coefficients are functions of the parameter u. These coefficients may be determined by multiplying equation (3) in turn by $w\cdot$, $\dot{w}\cdot$ and $w \times \dot{w}\cdot$. We find that these coefficients are of the form

$$u(au^2 + bu + c), \quad u(du^2 + eu + f), \quad gu,$$

in which a, b, c, d, e, f, g are scalar functions of $\dot{p}$, w and $\dot{w}$. Thus the locus is a third order curve.

We now discuss the special cases for which $w \times \dot{w} = 0$.

1. If $w \neq 0$ we may write $\dot{w} = kw$, and by taking P on the central axis we may also write $\dot{p} = lw$. Substituting these values of $\dot{w}$ and $\dot{p}$ in the equation $(\dot{p} + \dot{q}) \times (\ddot{p} + \ddot{q}) = 0$, we find

$$lw \times \ddot{p} + klw \times (w \times q) - lw^2(w \times q) + (w \times q) \times \ddot{p} + (w \times q)^2 w = 0. \tag{4}$$

Let us first assume that $w \times \ddot{p} \neq 0$ and $\dot{p} \neq 0$. Multiplying in turn by $w\cdot$ and $\ddot{p}\cdot$, we get respectively $w \times \ddot{p} \cdot Q = w^2$ and $\ddot{p} \cdot Q = l^{-1}\ddot{p} \cdot w^{-1} - k$, where $Q = (w \times q)^{-1}$. These two equations together with the additional relation $w \cdot Q = 0$ enable us to solve for Q in terms of the vectors reciprocal to w, $\ddot{p}$ and $w \times \ddot{p}$. We thus find

$$Q \equiv (w \times q)^{-1} = (l^{-1}\ddot{p} \cdot w^{-1} - k)(w \times \ddot{p})^{-1} \times w + w^2(w \times \ddot{p})^{-1}.$$

Let K represent the second member of this equation. Then $w \times q = K^{-1}$, and since $w \cdot K = 0$ this equation represents a line. In this case then the curve of inflections is a straight line. If $w \cdot \ddot{p} = 0$, this line is the axis of acceleration.

If $\dot{p} = 0$, then $l = 0$ and equation (4) reduces to $(w \times q) \times \ddot{p} + (w \times q)^2 w = 0$, which is obviously satisfied by $w \times q = 0$, *i.e.*, by points of the central axis.[1] This is the only solution unless $w \cdot \ddot{p} = 0$, in which case the equation $(w \times q)^{-1} \times \ddot{p} + w = 0$ defines $w \times q$ as the vector chords of a circle in a

[1] These points are points of zero velocity.

plane perpendicular to w, and the equation therefore represents a right circular cylinder. This case is illustrated by uniplanar motion.

Finally, if $w \times \ddot{p} = 0$, then again $w \times q = 0$ satisfies equation (4) and we get the central axis. Moreover $w \times q$ cannot be different from zero, otherwise the equation $w \times \ddot{p} \cdot Q = w^2$ would lead to a contradiction since $w \times \ddot{p} = 0$ and $w \neq 0$.

2. If $w = 0$ and $\dot{w} \neq 0$, the resulting equation $\dot{p} \times (\ddot{p} + \dot{w} \times q) = 0$ gives $\dot{w} \times q = \dot{p}^{-1} \times (\dot{p} \times \ddot{p}) + \mu\dot{p}$ if $\dot{p} \neq 0$. This represents a line parallel to $\dot{w}$ if μ be determined uniquely so as to satisfy the condition

$$[\dot{w}\,\dot{p}^{-1}\,\dot{p} \times \ddot{p}] + \mu\dot{w}\dot{p}\cdot = 0.$$

If $\dot{w} \cdot \ddot{p} = 0$, the line is the axis of acceleration.

If $\dot{w} \cdot \dot{p} = 0$ and $\dot{w} \cdot \ddot{p} \neq 0$, there would be no locus since $\mu = \infty$.

If $\dot{w} \cdot \dot{p} = 0$ and $\dot{w} \cdot \ddot{p} = 0$, the locus would be a plane perpendicular to $\dot{p}$ which would contain P if $\ddot{p} = 0$.

If $\dot{p} = 0$, the velocities of all points are zero and the equation is satisfied by every point.

3. Finally if $w = \dot{w} = 0$, our equation $\dot{r} \times \ddot{r} = 0$ reduces to $\dot{p} \times \ddot{p} = 0$, a relation independent of q. Hence there will be no locus unless the velocity and acceleration of some one point P are in the same direction (or one of them vanishes) in which case all points satisfy the condition.

We may now state the following

THEOREM: *The locus of points of the body which are points of inflection of their trajectories is in general a curve of the third order. If, however, the angular velocity and acceleration have the same direction, or if one of them vanishes, the locus is in general a straight line.*

2. Locus of Points of Contact of Stationary Osculating Planes. Let us now find the locus of points of the body at which the osculating planes of their trajectories are instantaneously stationary.[1]

Since the torsion $T\left(= \dfrac{[\dot{r}\ddot{r}\dddot{r}]}{(\dot{r} \times \ddot{r})^2}\right)$ of the trajectory $r = p + q$ is zero for such points, it follows[2] that

$$[\dot{r}\ddot{r}\dddot{r}] = [\dot{p} + \dot{q}\;\ddot{p} + \ddot{q}\;\dddot{p} + \dddot{q}] = 0.$$

[1] At such points the osculating planes have contact of order higher than the second. The order of contact is determined by the vanishing of the triple products of the derivatives of r, which is shown by the following Taylor's expansion:

$$\delta = [\Delta r\dot{r}\ddot{r}] = \{[\dot{r}\ddot{r}\dddot{r}](\Delta t)^3/3! + [\dot{r}\ddot{r}r^{(iv)}](\Delta t)^4/4! + \cdots\}\,|\dot{r} \times \ddot{r}|^{-1},$$

where δ is the distance from the osculating plane to a point of the curve near the point of contact. It is assumed in this expansion that $\dot{r} \times \ddot{r} \neq 0$. If this product is zero, a different pair of derivatives would have to be used.

[2] Conversely, if $[\dot{r}\ddot{r}\dddot{r}] = 0$ and the derivatives of r are not zero, the torsion is zero except at points of inflection. At such points the torsion is not zero unless also $[\dot{r}\ddot{r}\,r^{(iv)}] = 0$. See next footnote.

But $\dot{q}$, $\ddot{q}$ and $\dddot{q}$ are expressible linearly in terms of q. Hence the equation is generally of the third degree in q and represents a third order surface. Whence the

THEOREM: *The locus of points of the body at which the osculating planes of their trajectories are instantaneously stationary is in general a surface of the third order.*

If one point of the body is fixed and P is taken at this point, our equation becomes $[\dot{q}\ddot{q}\dddot{q}] = 0$, a homogeneous equation of the third degree in q. Hence if one point of the body is fixed, the locus of points for which the osculating planes of their trajectories are instantaneously stationary is a cubic cone. This is the cone of lines whose polar axes are stationary.

It is seen at once that points satisfying the condition for points of inflection, namely, $(\dot{p} + \dot{q}) \times (\ddot{p} + \ddot{q}) = 0$, satisfy also the condition $[\dot{p} + \dot{q}\ \ddot{p} + \ddot{q}\ \dddot{p} + \dddot{q}] = 0$. It follows that the curve of inflections lies on the surface represented by this equation.[1]

Assuming $w \times \dot{w} \neq 0$, the curve of inflections is thus a part of the intersection of this third order surface and a quadratic cone (Art. 1).

We note the following special forms of the third order surface:

If $w \times \dot{w} = 0$, $w \times \ddot{w} = 0$ but $w \neq 0$, all of the third degree terms vanish and the equation is of the second degree in $w \times q$. Hence the locus is a cylinder with elements parallel to the central axis and with a conic for directrix. If all of the derivatives of p are perpendicular to the central axis, the equation becomes an identity and all points satisfy it. This last case is illustrated by uniplanar motion.

If $w = 0$, the equation is $[\dot{p}\ \ddot{p} + \dot{w} \times q\ \dddot{p} + \ddot{w} \times q] = 0$, which represents a quadric surface in general; but if also $\dot{w} \times \ddot{w} = 0$, this locus is in general a plane.

If $w = \dot{w} = \ddot{w} = 0$, all points or no points satisfy the condition according as some one point P does or does not satisfy it.

3. Envelopes of Loci of Exceptional Points. If not only $T = 0$ but also $\dot{T} = 0$, we have in addition to the condition $[\dot{r}\ddot{r}\dddot{r}] = 0$ the further condition $[\dot{r}\ddot{r}r^{(iv)}] = 0$, and the locus is the curve of intersection of two third order surfaces. But the curve of inflections is on both of these surfaces. Consequently, if $w \times \dot{w} \neq 0$, the locus of points for which $T = \dot{T} = 0$ is a curve of the sixth order. It will be observed that the intersection of these two surfaces is the characteristic of the surface of points of contact of stationary osculating planes, which therefore consists of the curve of inflections and the sixth order curve; in other words, the locus of these curves as t varies is the envelope of this surface.

In general the osculating planes of points of the surface have contact of the third order with the trajectories of these points, but at points of the sixth order curve they have in general fourth order contact.

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[1] It is easy to show that the osculating plane at a point of inflection, which is usually determined by the vectors $\dot{r}$ and $\ddot{r}$, has contact of the third order, or higher. However, the osculating plane in general is not stationary at a point of inflection; for, in order that the osculating plane be stationary the condition $(\dot{r} \times \ddot{r}) \times (\dot{r} \times \ddot{r})_t = 0$, or $[\dot{r}\ddot{r}r^{(iv)}] = 0$ must be satisfied. If this condition is met, the osculating plane has contact of the fourth order, or higher, at a point of inflection.

If $T = \dot{T} = \ddot{T} = 0$, we have the additional condition

$$[\dot{r}\ddot{r}\,r^{(\text{iv})}] + [\dot{r}\dddot{r}r^{(\text{v})}] = \frac{d}{dt}[\dot{r}\dddot{r}r^{(\text{iv})}] = 0.$$

Among the intersections of the surface represented by this equation and the other two surfaces are those points at which the osculating planes have contact of the fifth order. The locus of these points as t varies is a part of the edge of regression of the envelope of the third order surfaces of points of third order contact and is the envelope of the sixth order curves of the points of fourth order contact. The locus of the other points of intersection of these surfaces is the envelope of the curves of inflections and it is easy to show that this curve is a part of the locus of points at which the osculating plane has contact of the fourth order.

We sum up these results in the following statements:

The envelope of the third order surface locus of points of the body for which the trajectories have, with their osculating planes, contact of the third order consists of two surfaces; one is the locus of the curves of inflections and contains the points for which the tangents to their trajectories have second order contact, *i.e.*, points for which the tangents are stationary; the other is the locus of points for which the trajectories have, with their osculating planes, contact of the fourth order.

The edge of regression of this envelope consists of two curves; one is the envelope of the curves of inflections and contains points for which the trajectories have stationary tangents and have contact of the fourth order with their osculating planes; the other is the envelope of the sixth order curves (each of which is the instantaneous locus of points whose trajectories have fourth order contact with their osculating planes) and contains points for which the trajectories have with their osculating planes contact of the fifth order.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The Department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. A GENERALIZATION OF A PROPERTY OF CONFOCAL CONICS.

BY NORMAN MILLER, Queen's University.

A common method of describing an ellipse—by means of a loop of string around two pegs—depends on the property that the sum of the focal distances of any point on an ellipse is constant. If, instead of being stretched around two

pegs, the string be stretched around an ellipse (supposed, for physical purposes, an elliptic cylinder), the locus traced will be a confocal ellipse. This theorem is due to Bishop Graves,[1] whose proof made use of the familiar property of two confocal conics that a tangent and normal at a point of the outer one bisect the angles made by the two tangents from this point to the inner conic. His proof, however, based on the so-called method of infinitesimals is unsatisfactory inasmuch as no enquiry is made regarding the orders of the infinitesimals.[1]

The theorem may be generalized, as is indeed suggested by Salmon. The following theorem includes that concerning the ellipse as a special case.

THEOREM: *If from an external point P two tangents be drawn to a curve which has at every point a definite curvature and if the excess of the lengths of the two tangents over the arc intercepted between their points of contact is constant, the locus of the point P is a curve having the property that the tangent at any point on it makes equal angles with two tangents from that point to the original curve.*

If the given curve is closed and is everywhere concave toward its interior, then the locus can be traced mechanically by winding a loop of string around the curve and holding a pencil in the loop so as to keep it taut.

Let $y = f(x)$ denote the given curve. The hypothesis requires that $f'(x)$ and $f''(x)$ exist at every point of the curve except possibly at a finite number of points where $f'(x)$ is infinite. Let (X, Y) be a position of the tracing point P and (x_1, y_1) and (x_2, y_2) the points of contact of tangents from (X, Y) to $y = f(x)$. The axes of coördinates may evidently be chosen so that $x_1 < X < x_2$ and that neither $f'(x_1)$ nor $f'(x_2)$ is infinite. Then we have for parametric equations of the locus of (X, Y)

$$Y - f(x_1) = f'(x_1)(X - x_1), \tag{1}$$

$$Y - f(x_2) = f'(x_2)(X - x_2), \tag{2}$$

$$\sqrt{(X - x_1)^2 + (Y - f(x_1))^2} + \sqrt{(X - x_2)^2 + (Y - f(x_2))^2} - \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}dx = k, \quad \text{a constant.} \tag{3}$$

By virtue of the first two the third equation may be written,

$$(X - x_1)\sqrt{1 + (f'(x_1))^2} - (X - x_2)\sqrt{1 + (f'(x_2))^2} - \int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2}dx = k. \tag{4}$$

Since $X < x_2$, the second term requires a negative sign in order to render the term positive. From these equations, involving the parameters x_1 and x_2, we may find dY/dX, the slope of the tangent at (X, Y). By differentiating (1), (2) and (4) in succession, and writing for brevity

$$\sqrt{1 + (f'(x_1))^2} = \alpha, \qquad \sqrt{1 + (f'(x_2))^2} = \beta,$$

[1] See Salmon, *Conic Sections*, London, 1900, p. 377; or Sommerville, *Analytical Conics*, p. 199.

we get

$$dY/dX - f'(x_1) = f''(x_1)(X - x_1)dx_1/dX. \tag{5}$$

$$dY/dX - f'(x_2) = f''(x_2)(X - x_2)dx_2/dX. \tag{6}$$

$$\begin{aligned}\left(1 - \frac{dx_1}{dX}\right)\alpha + (X - x_1)\frac{f'(x_1)f''(x_1)}{\alpha}\frac{dx_1}{dX} - \left(1 - \frac{dx_2}{dX}\right)\beta \\ - (X - x_2)\frac{f'(x_2)f''(x_2)}{\beta}\frac{dx_2}{dX} - \beta\frac{dx_2}{dX} + \alpha\frac{dx_1}{dX} = 0.\end{aligned} \tag{7}$$

When dx_1/dX and dx_2/dX are eliminated from (7) by means of (5) and (6), we get

$$\alpha + \frac{f'(x_1)}{\alpha}\left(\frac{dY}{dX} - f'(x_1)\right) - \beta - \frac{f'(x_2)}{\beta}\left(\frac{dY}{dX} - f'(x_2)\right) = 0,$$

or

$$\beta\left\{\alpha^2 + f'(x_1)\frac{dY}{dX} - (f'(x_1))^2\right\} - \alpha\left\{\beta^2 + f'(x_2)\frac{dY}{dX} - (f'(x_2))^2\right\} = 0.$$

Hence

$$\beta\left\{1 + f'(x_1)\frac{dY}{dX}\right\} - \alpha\left\{1 + f'(x_2)\frac{dY}{dX}\right\} = 0.$$

It follows that

$$\frac{dY}{dX} = \frac{\alpha - \beta}{\beta f'(x_1) - \alpha f'(x_2)}\,. \tag{8}$$

Now form the quotient

$$\begin{aligned}&\left\{\frac{dY}{dX} - f'(x_1)\right\}\Big/\left\{1 + \frac{dY}{dX}\cdot f'(x_1)\right\} \\ &\quad = \left\{\frac{\alpha - \beta}{\beta f'(x_1) - \alpha f'(x_2)} - f'(x_1)\right\}\Big/\left\{1 + \frac{\alpha - \beta}{\beta f'(x_1) - \alpha f'(x_2)}\cdot f'(x_1)\right\} \\ &\quad = \frac{\alpha - \beta - \beta(f'(x_1))^2 + \alpha f'(x_1)f'(x_2)}{\alpha(f'(x_1) - f'(x_2))} \\ &\quad = \frac{\alpha - \beta - \beta(\alpha^2 - 1) + \alpha f'(x_1)\cdot f'(x_2)}{\alpha(f'(x_1) - f'(x_2))} = \frac{1 - \alpha\beta + f'(x_1)f'(x_2)}{f'(x_1) - f(x_2)'}\,.\end{aligned}$$

Similarly

$$\begin{aligned}&\left\{\frac{dY}{dX} - f'(x_2)\right\}\Big/\left\{1 + \frac{dY}{dX}f'(x_2)\right\} = \frac{\alpha - \beta - \beta f'(x_1)f'(x_2) + \alpha(f'(x_2))^2}{\beta(f'(x_1) - f'(x_2))} \\ &\quad = \frac{\alpha - \beta - \beta f'(x_1)f'(x_2) + \alpha(\beta^2 - 1)}{\beta(f'(x_1) - f'(x_2))} = \frac{\alpha\beta - 1 - f'(x_1)f'(x_2)}{f'(x_1) - f'(x_2)}\,.\end{aligned}$$

Since dY/dX is the slope of the tangent to the curve traced by (X, Y) and $f'(x_1)$ and $f'(x_2)$ are the slopes of the tangents from (X, Y) to the original curve, these results establish the theorem.

II. A Problem in Speculation.

By J. P. Ballantine, Columbia University.

Let us consider a particular speculation. For every dollar that A puts up, if fortune favors him, he will receive q dollars, and if fortune is unkind to him, he will lose what he puts up. The probability of success is p.

The *expectation* is pq for each dollar risked, and must be greater than 1 in order to make the speculation attractive. If A were to try to make a maximum the sum of his cash on hand plus his expectation, by risking the proper number of dollars, he would find in the case of an attractive speculation that this sum can always be increased by increasing the stake.

The *moral expectation*, due to D. Bernoulli,[1] is

$$p \log (a - ax + aqx) + (1 - p) \log (a - ax),$$

where a and x are defined as below. It is possible to make the moral expectation a maximum by a proper choice of the stake. Moral expectation was defined as a working hypothesis, and with no especial regard to speculation. There is no *a priori* reason why the value of x which makes the moral expectation a maximum should be the best value for a speculator to use. In the present paper it is proved to be the best value to use in the long run.

In the mathematical treatment of the problem we cannot let enter such considerations as A's personal wants. In practice it may be, for instance, that what A has is enough for his wants, and a speculation would involve risking the possibility of owning something which he wants in return for the chance of having more money than he has wants for. We assume that A is a confirmed gambler, having in mind only to increase his fortune in the long run. This he does by entering into one speculation (for simplicity) at a time, and of the simple type described in the first paragraph. It further simplifies matters if all of the speculations are identical, *i.e.*, have the same values of p and q. This involves the assumption that the best thing to do in a single speculation is the best thing to do in one of a sequence of identical speculations. We assume that the best amount to put up in a given speculation (p, q) is proportional to a (the number of dollars A has) and otherwise depends only on the values of p and q. Another person B with twice as much money as A should put up twice as much on the same speculation.

If x denotes the proportional part of A's money which he puts up, so that ax is the amount he puts up, after one speculation his wealth is either

$$a - ax + aqx \qquad \text{or} \qquad a - ax$$

according as he wins or loses. After $h + k$ speculations, of which h are wins and k are losses, regardless of the order in which they occur, A's wealth is

$$a(1 - x + qx)^h(1 - x)^k.$$

[1] Jordan, "On Daniel Bernoulli's 'Moral Expectation' and on a New Conception of Expectation," Monthly (*1924*, 183–190).

The problem, then, is to choose x so that the above expression is a maximum. It is the same thing if we make

$$(a - ax + aqx)^{h/(h+k)}(a - ax)^{k/(h+k)}$$

a maximum; for multiplying an expression by a fixed power of a and extracting a positive root would have no effect on the value of x which produces a maximum. From the definition of probability, as h and k increase, the ratio $h/(h + k)$ approaches p. Hence if A is considering a large number of speculations, and since he has no way of knowing in advance the exact values of h and k separately—in fact, he knows p only by making a conservative guess—he may as well use the expression

$$(a - ax + aqx)^p(a - ax)^{1-p}.$$

This latter is rendered maximum in turn by making its logarithm

$$p \log (a - ax + aqx) + (1 - p) \log (a - ax) \tag{1}$$

maximum.

Expression (1) is the moral expectation of A. In general, the moral expectation of a speculator who has undertaken a speculation with several outcomes is defined by D. Bernoulli as the sum of the products of the probability of each outcome by the logarithm of the total amount of money he will have (not receive) in case of that outcome. The theorem which we have proved in the case of a speculation of only two outcomes, and which is easily extended to the case of several outcomes, is: *The best amount of money to venture in any speculation is that amount which makes the "moral expectation" a maximum.*

III. The Notion "Matrix of a Determinant."

By A. A. Bennett, University of Texas.

There can be no question that so far as historical development is concerned, the notion of a matrix as a hypercomplex number subject to combination with other matrices of a suitable type by addition or multiplication is an outgrowth and by-product of investigations in the theory of determinants. In this historical sense the matrix is subsequent to the determinant.

The use of such expressions as the rows of a determinant, its columns, the upper left-hand corner, the main diagonal, and others of similar sort, bear evidence to the feeling that where a determinant is given its matrix may be recognized. These expressions are certainly not thought of as mere forms of diction, mere figures of speech handed down from the past and devoid of immediate logical justification. To be sure some writers of an older school ignore the distinction between matrix and determinant but the use of these terms cannot be wholly attributed to a continuation of such a confusion. For example, Bôcher in his classical *Introduction to Higher Algebra* remarks with his usual clarity and in

accordance with orthodox precedents (page 21), "Although, as we have pointed out, square matrices and determinants are wholly different things, every determinant determines a square matrix, the *matrix of the determinant,* and conversely every square matrix determines a determinant, the *determinant of the matrix.*"

The question arises as to just what we mean when we say that a determinant has a matrix. It is certain that a determinant may be defined from a (square) matrix and so may be thought of as having this matrix as its origin provided that we know the matrix in question. But suppose, if possible, the determinant is encountered in some connection in which many distinct matrices are involved, and divorced from all identifiable association with any one particular matrix of the set. Is there in reality one and only one matrix which serves as "its matrix"?

The determinant in all ordinary uses is of course a number. If it be given as a numerical constant, there is little or no meaning to the term, element of the determinant. It is only as a function of certain variables, presumably of its elements, that the determinant could possibly define any matrix. If the matrix is assumed to be subject to some further special restriction, the problem is altered, as when the matrix is required to be symmetrical, skew-symmetrical, orthogonal, and so forth, and when the variables are not the elements, but the elements themselves are regarded as functions of one or more independent variables that figure throughout, as in the case of the so-called lambda-matrices, still further modifications of the discussion must be made.

Suppose then that the elements are given as independent variables. One still asks how are these elements to be identified? Do we have a natural number, n, and two subscripts, i and j, $i, j = 1, 2, \cdots, n$, and is each element, e_{ij}, recognized not only as an element but as the one which has these particular subscripts? If we do, there is no question that the determinant identifies the matrix, for then the determinant occupies the anomalous position of being all of a matrix and much besides. But such a convention is certainly contrary to a natural interpretation of the statement that certain familiar transformations alter the matrix but leave the determinant unaltered. Furthermore such a notion of a determinant is wholly inconsistent with the current definition of the determinant as a multilinear integral rational function of a certain prescribed sort, consisting of the algebraic sum of a number of signed products, since the character of a rational integral function in relation to its arguments is completely determined by internal properties and cannot depend upon assumed notation not consequent to the functional relationship, or upon extraneous order relations derived from some other representation. Nor is such a notion in accord with general mathematical tradition as illustrated in the case of polynomials. A polynomial identifies its roots and if it be agreed that the leading coefficient shall be unity, the roots in turn identify the polynomial. No one will ask that the separate roots be recognized individually as listed in the order in which they were selected in the formation of the polynomial considered. On the other hand, as one might insist, a polynomial identifies its coefficients and the set of coefficients identifies the

polynomial. Here, however, the several coefficients must be taken in an assigned order but they are further recognized not merely as coefficients but as coefficients of given powers of the independent variable. The question seems clearly to be one of the group of transformations leaving the expression invariant. While an arbitrary permutation of the roots of a polynomial leaves the expression invariant, and the roots are therefore identified only up to an arbitrary permutation among them, that is, they are recognized merely as an unordered set, yet on the other hand no non-identical permutation of the coefficients of the general polynomial of a given degree leaves this polynomial unaltered, and therefore its coefficients are individually recognizable.

If we are to apply such a criterion to the case under discussion, it would seem that we are compelled to say that the determinant identifies its matrix only as being one of a set of matrices all having the same determinant—which is almost verbal tautology, and if desired we could add—and convertible one into the other by a suitable group of transformations. It remains for one merely to settle upon the particular group of transformations to be used.

In any development of the theory of determinants one comes across unitary transformations of various sorts each of which is said to leave the determinant unaltered. The matrix after the unitary transformation does not usually have the same elements as it did previous to the transformation. Should the entire group of unitary transformations be considered, the problem is too general to have much interest. We have, merely, $D(x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}, \cdots, x_{n1}, x_{n2}, \cdots, x_{nn})$ containing n^2 unknowns, given as equal, say, to $D(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{n1}, a_{n2}, \cdots, a_{nn})$, and the problem is to find the values of the n^2 unknowns in terms of the values of the known a's. The D is a polynomial in its arguments and it is possible for this case to choose $n^2 - 1$ of the unknowns restricted only by certain natural inequalities and then to determine the final unknown to satisfy the given condition. In this plausible interpretation of the problem, a determinant falls far short of defining "its matrix."

Let us, then, greatly restrict the inquiry, so as if possible to give some appreciable content to the notion, matrix of a determinant. Suppose that a set of n^2 variables is given, say, $x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}, \cdots, x_{n1}, x_{n2}, \cdots, x_{nn}$, with explicit subscripts, and a matrix is formed of these in the manner suggested by the subscripts, and, finally, the determinant, D, of this matrix, is found. Suppose, in this determinant expressed as a polynomial, some one else should replace each x_{ij} by one of the variables selected from a set, $y_1, y_2, \cdots, y_{n^2}$, so that these latter constitute merely a definite but unknown permutation of the x's. The problem might be to show how far it is possible to identify the original variables. In any one term of the expanded determinant only elements from different rows and different columns appear. By inspecting in turn each of the terms, it would be readily possible to subject the y's to a double classification, which we might call into lines of the first type and lines of the second type. The names of the two types are assigned wholly arbitrarily, but otherwise the classification is determined by the expanded form of the determinant. The

lines of one of these types will be columns and the lines of the other type will be rows, although which will be which is inherently ambiguous. In the determinantal expansion, there are terms of two sorts, positive and negative. As a consequence, half of the terms, namely the $n!/2$ negative terms, cannot serve as principal term (product of the elements down the main diagonal). Any one of the $n!/2$ positive terms may however serve. More generally we may observe that any even permutation of rows, of columns, or of both, or a complete interchange of all rows for columns or vice-versa, leaves elements as elements and does not alter the value of the determinant, as indeed is known to every one interested in this theory.

One may say that in the restricted sense of the term a determinant classifies its elements doubly, and thus serves to identify its own determinant minors, their complements, cofactors, and so forth. Half of the possible $n!$ choices of a set of n elements, one from each set of each of the two types, are eligible for principal term and the other half are not. Any further methods for identification of the matrix are consequences of these facts only. For example, the determinant $uy - xv$ with elements, u, v, x, y, is of any one of the four forms,

$$\begin{vmatrix} u & v \\ x & y \end{vmatrix}, \quad \begin{vmatrix} u & x \\ v & y \end{vmatrix}, \quad \begin{vmatrix} y & v \\ x & u \end{vmatrix}, \quad \begin{vmatrix} y & x \\ v & u \end{vmatrix}.$$

Would it not therefore seem more desirable when speaking of specific rows or columns to call these as *of the matrix* and not as *of the determinant,* and never to use the term *matrix of a determinant* except in the obvious case in which a particular matrix is already known and exhibited and is the patent and unambiguous source from which the given determinant under discussion was obtained?

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Relativity and Modern Physics.[1] By G. D. BIRKHOFF, with the coöperation of R. E. LANGER. Harvard University Press, 1923. xi + 283 pages. Price $4.00.

Relativity is a subject which is due to the physicist. It was developed to meet the needs that have arisen from the modern refinements of physical theory and experiment. Most of the writers on this subject have presented it as a branch of physics, requiring, indeed, a large amount of mathematical technique, but supposed to depend chiefly upon its physical aspects for its significance. It is possible, however, to take the mathematical part and make of it as logical a development as that of non-Euclidean geometry, and a study of relativity

[1] Reviewed by L. P. Eisenhart in *Science,* December 28, 1923, pages 539–541, and by O. Veblen in *Bulletin of the American Mathematical Society,* vol. 30, pp. 365–367.

as a branch of pure mathematics, dispensing with extraneous circumstances and assuming ideal conditions, would be of great interest, and would, indeed, add much to an understanding of its applications.

On the other hand, the physicist's point of view is what the mathematician should cultivate, and the physical books on relativity, written for advanced students, tend to bring the two classes together, just as in their earlier courses they are brought together by the type of mathematical text-book that has come into use in recent years.

Professor Birkhoff stands among the foremost of our mathematicians, and when his book on this subject was announced some mathematicians may have hoped for a purely mathematical treatise. This is not what he has written. As the title indicates, it is a book on "Relativity and Modern Physics." It would appear from the preface that it is intended equally for the mathematician and the physicist. The "postulational method," to which attention is particularly called, is a mathematical method, but the phraseology of the book seems to be that of the physicist. The constant use of physical terms, and the attempts to justify the assumptions made, as in accord with physical experiment, and to show the physical significance of the results,—these things would seem to appeal especially to the physicist. But it is possible to select some of the mathematics and study it in a purely mathematical way.

The book is a work of great originality, with new results and especially with new methods and points of view. Indeed, as far as we have observed, results which are not new are always obtained by new methods. Even to one who has devoted considerable time to the study of relativity the contents of these pages will come as new matter, leading to a study that will eventually give new insight into the entire subject. The book is not, however, easy to understand, even in the mathematical parts. The author hopes that it "will prove serviceable as a text for an undergraduate course," but it seems to be more difficult than he thinks it is. We have attempted to use it with a class of graduate students, and now after further study we should like to try it again, not, however, for its help in the explanation of difficulties, but for its suggestiveness. We have ourselves mastered only portions, if any of it, and this paper will not be a comprehensive review, but only an attempt to explain some of these portions.

A "Review of Classical Physics" takes up the first chapter and it is in the second that the new theories are introduced. In this chapter and for a considerable portion of the book the discussion is confined to a space of one dimension or line. A particle[1] A is supposed to have a system of time called its *local time*, measured by an "ideal clock" that belongs to it and has certain physical properties. By this clock it determines directly the time of any event at which it is present.[2] The local time of one particle cannot be compared with the local

[1] The individual on this line is sometimes called an observer and sometimes a particle according to the point of view. In this paper we shall use the term particle in all cases.

[2] In a mathematical treatment the notion of clocks is not necessary. A particle's local time, instead of being determined by the physical nature of such a mechanism, may be assumed to be perfectly arbitrary, subject only to the usual order relations of time. Certain particular ways

time of another except when they happen to meet, nor does a particle's local time tell it directly about the time of an event at which it is not actually present. By means of light signals, however, it can determine what it may call this time, assuming that a flash of light is always reflected at the moment midway between the time of sending and the time of receiving the flash, and this time can be compared with any local time at the event.

The same light signal serves also to determine what the particle may call the position of the event, assuming that the distance is proportional to the interval of time taken.

These determinations are made as follows: A sends a flash at time t_1 by its local time to a second particle B and receives the reflected flash at time t_2. A then considers that the time at B when the flash reaches B is $t = (t_2 + t_1)/2$ and that the position of B at that moment is denoted by the abscissa [1] $x = (t_2 - t_1)/2$. The reflection of the flash at B is an event. t is called the *apparent time* and the numerical value of x the *apparent distance* of the event.[2] It would be well always to put in the words "according to A," or "with respect to A," because the apparent time of an event with respect to one particle may be very different from its apparent time with respect to another particle.

Any event may be thought of as determined by its time and position, t and x, with respect to a given particle A, or by the quantities t_1 and t_2. Either pair of quantities may be changed to the other at any time, and many formulæ are simpler when expressed in terms of t_1 and t_2.

Events can be plotted on a chart, t and x being taken as coördinates or t_1 and t_2. In fact, with proper units the same graph will serve for both kinds of coördinates, the axes of the two systems bisecting each other's angles. A particle, passing through a series of events, is represented by the locus of the points that represent these events, which is called the particle's *world-line*. The *velocity* of the particle according to A, dx/dt, is represented by the slope of its world-line, and in the t_1t_2 chart this is always positive. When the variables belong to the system that is based on A's local time the chart is called A's chart.

A second particle B has a local time of its own and a system of variables, t_1', t_2', t' and x', based on its local time in the same way that t_1, t_2, t and x are based on A's local time. B's time for an event (the apparent time according to B) will not generally be the same as A's, but the two systems are connected by transformation equations which may be expressed as equations connecting t'

of determining the local time of different particles may then be selected for special study because they are interesting, as well as others that we think may have been imposed on the world by the nature of "space-time."

[1] A constant factor c, depending on the unit of length, is put equal to 1.

[2] Einstein in his first paper (1905) starts with these two equations, which he writes

$$t_B - t_A = t_A' - t_B, \qquad 2AB/(t_A' - t_A) = c,$$

c being obtained thus as the velocity of light. *The Principle of Relativity.* Memoirs by Lorentz, Einstein, Minkowski and Weyl, translated by W. Perrett and G. B. Jeffery, London, 1923, page 194. The idea of determining distances and setting up a system of coördinates by means of light signals, assuming c to be given, is developed very clearly in a paper by E. V. Huntington, "A new approach to the theory of relativity," *Festschrift, Heinrich Weber*, Leipzig, 1912, pages 146–169.

and x' with t and x (as, for example, the Lorentz equations), or t_1' and t_2' with t_1 and t_2. Here the advantage of the latter letters is strikingly shown, for only t_1 and t_1' occur in one equation and only t_2 and t_2' in the other equation. The equations are linear and those expressing t_1 as a function of t_1' and t_2 as a function of t_2' are reduced to symmetrical form by the use of a quantity λ, which is introduced casually as a "factor of proportionality" (page 31), but turns out to be of great importance and is called the *characteristic function*. A more concrete meaning for λ comes from its relation to B's local time pointed out at the bottom of page 31. If we make the variables refer to events of B and let v denote the velocity of B with respect to A ($v = dx/dt$), then we shall have $dt_1dt_2 = (1 - v^2)dt^2$ (see bottom of page 40), and as $dt'^2 = \lambda^2 dt_1 dt_2$, $\lambda\sqrt{(1 - v^2)}$ will be equal to the ratio of the differential of B's local time to the differential of A's time at B, or what A would call the rate of B's local time. This fact is referred to again on page 39, but not otherwise mentioned in connection with the discussion of λ, which is always called the "characteristic function."

It will appear in a moment that λ is equal to 1 in the space-time of relativity, or at least of the restricted theory. With the variables referring to events of B the differential of B's time as it appears to A takes the form $\sqrt{(dt^2 - dx^2)}$, and as B is any particle this is a general formula for the differential of local time in the restricted theory when the space considered is one-dimensional. $dt^2 - dx^2$ is the "fundamental quadratic differential form" in this case, and so our author calls it ds^2 instead of dt'^2; and then he carries the idea into the general theory and throughout the book describes s as local time. This is certainly easier to understand than such terms as "interval" or "separation" or "linear element" (for ds), used with the idea of representing something analogous to distance in ordinary space.

The author emphasizes the graphical method of representing all these quantities. Thus the equations connecting the two systems of variables based on A's and B's local times are said to express the distortion that makes one of two charts fit the other. For ourselves we do not find the consideration of these charts very helpful. It is difficult to become familiar with the Minkowski method of combining time with space as another dimension of just the same kind as the dimensions of space, and the difficulties of this theory may as well be postponed till the difficulties of relativity are mastered. Moreover, there are too many charts. Thus, B being any particle, it is better to think of λ without thinking of B's chart, using A's chart alone. Every particle will then have a definite world-line, a definite velocity v with respect to A, represented by the slope of this world-line, and a definite time-rate represented by $\lambda\sqrt{(1 - v^2)}$.

It is possible to go farther and drop A's chart too, regarding all events and the world-lines of all particles as represented on the chart of a single arbitrary particle otherwise unmentioned. Indeed, this is what the author seems to do in his study of the function λ in chapter III. This chart is referred to at the bottom of page 33 and in line 7 on page 36. There are transformation equations connecting A with this arbitrary particle, and a characteristic function, and A

has with respect to it a velocity and a time-rate, but for the most part it is not necessary to think of these. We may then consider any number of particles A, and for each of them the velocity and time-rate of a particle B with respect to it, without thinking of any chart except that of the given arbitrary particle to which everything is referred.

He speaks of λ as the characteristic function of A when it is really a function [1] of B; that is, its value depends on the velocity and position of B with respect to A and the rate of B's local time. When, however, we consider different particles A and set up for each a system of reckoning time and position, there will be for each of them a function λ of the variable particle B. It is, therefore, convenient to speak of λ as A's characteristic function, thinking perhaps of its form rather than its value as what is determined by A.

In the third chapter he studies the function λ and bases upon it a classification of different kinds of *space-time*, space-time being a complete system of reckoning local times. Supposing that the external world possesses a definite kind of space-time, he would try to find what it is by making certain sets of assumptions and determining from the results which is most probably correct.

In the first place he assumes that space-time is *isometric:* given any two events O and O' and a particle A at O with characteristic function λ, there is at O' at least one particle A' whose characteristic function is also λ, the same as for A. It does not seem necessary to say that O and O' are the epochs of A and A' (events of local time zero), nor, indeed, to mention O at all.

He makes another assumption (stated on page 35) that might have been included in his definition of isometric space-time: Given A and its characteristic function λ, there is also at O' at least one particle whose characteristic function would be λ if the positive and negative directions along their line were reversed, this reversal causing an interchange of the subscripts 1 and 2 and a change in the signs [2] of x and v.

Two kinds of space-time are of special interest, *æolotropic* space-time and *isotropic* space-time.

In æolotropic space-time there is at the event O' only one particle A' whose characteristic function is a given λ and only one whose characteristic function would be λ if the direction of positive and negative were reversed.

If the space-time is æolotropic, let us take an event O and let M be any

[1] We may call attention to a similar form of expression in connection with coördinates. On A's chart the coördinates of a point P are spoken of as the coördinates of A, and the coördinates on B's chart as the coördinates of B. Thus on page 49 equations (15) are described as expressing the relation between the coördinates of A and the coördinates of B (see also page 156). At the top of page 61 he does, indeed, speak of the coördinates of P, but further along on the same page he refers again to the coördinates of the observer, here called O.

[2] For a given particle B the value of A's characteristic function λ is the same whichever direction is positive, but when we consider λ as a function of the velocity and position of B a change in the signs of these two quantities will usually make it a different function. In the cases considered it turns out that λ is a function of the velocity only of B and not of its position. The assumption means that a particle A' can be taken in place of A with a characteristic function whose value for B is the same function of minus B's velocity with respect to A' that the value of λ is of its velocity with respect to A.

particle at O. As the assumptions apply to M and the event O there will be a particle M' at O (and only one) that will have the same characteristic function when the directions of positive and negative for one are the opposite of their directions for the other. M and M' are called homologous functions, each homologous to the other.

The chart on which the world-lines of these two particles are supposed to be represented is taken with O for origin. On it these are homologous world-lines and on it the t_1 and t_2 axes are also said to be homologous world-lines. Therefore, if the two given world-lines are rotated about O, they will rotate in opposite directions, and in some unique position they will coincide and be a self-homologous world-line.

An interchange of the two directions will involve an interchange of t_1 and t_2, but the question might be asked what are the characteristic functions of t_1 and t_2, or how do t_1 and t_2 satisfy the definition of being homologous. The following explanation may enable us to obtain the self-homologous particle and its world-line without using the t_1 and t_2 axes: M and M' are homologous particles at O. A particle at O different from M or M' will have a different characteristic function, and its homologous particle will also be different from M or M'. We may speak of varying[1] M and M'. Particles at the same event are distinguished only by their velocities, velocities with respect to the system of reference, represented by the directions of their world-lines on the chart of this system. If the velocity of M varies until it becomes the same as the original velocity of M', then the velocity of M' will become the same as the original velocity of M. If M varies continuously and we assume that M' varies continuously when M varies continuously, there will be some point in this process where M and M' have the same velocity and are the same particle, the particle which is called a self-homologous particle, its characteristic function being the same whichever direction is taken as positive.

It is shown that all self-homologous particles (the self-homologous particles at different events) are relatively at rest, and then, because they form a class that is "uniquely singled out," they may be regarded as absolutely at rest and other particles as in absolute motion (page 47). Thus even in the æolotropic space-time of classical physics the only ground for saying that a particular kind of time and space are absolute is the uniqueness of the set of particles that have this time and space!

In isotropic space-time there is absolutely no distinction between any two particles. Each has the same characteristic function, and from the equations it appears that each has numerically the same velocity with respect to the other. By applying the equations to the two particles themselves we find that the characteristic function must be equal to 1 (page 48), and so the local time of each will appear to the other to have a rate which is equal to $\sqrt{(1 - v^2)}$, where v is their

[1] This "varying" does not take time. We may turn our attention first to one pair of particles and then to another, but the particles at O are all there at the same time, O itself belonging only to a single instant of time.

relative velocity. Isotropic space-time is the space-time of relativity, or at least of the restricted relativity.

One or two matters of detail may be mentioned in connection with this part of the subject.

There seems to be some ambiguity in the phrase "future for a particle" used at the bottom of page 25. Apparently it refers here to an event at which the particle *could* be present if it should move just fast enough to get to the place of the event at the right time. In this sense any one event represented by a point in the first quadrant of A's chart is in the "possible future" for any one particle at A's epoch. But there are at least two other ways of interpreting the expression. This subject is referred to again on page 30.

The paragraph at the bottom of page 28 could have been placed at the end of section 12 on page 26 and used in the proof that follows. But we do not need section 12 at all for a geometrical proof of the relation between the two charts. We need only assume that straight lines correspond to straight lines, parallel lines to parallel lines, and, in particular, horizontal lines to horizontal lines and vertical lines to vertical lines. For then equal horizontal segments will correspond to equal horizontal segments, and all segments of this kind on one chart must be in a given ratio to the corresponding segments on the other chart, any two on one chart being proportional to the corresponding two on the other chart. This is proved, first,. when they are commensurable, and then by the usual methods when they are incommensurable. It follows that t_1 is a linear function of t_1', and in the same way we prove that t_2 is a linear function of t_2'.

The explanation of the theory of tensors is clearer than the treatment of other topics, and this part of the book is the easiest to understand (at least for the mathematician). It comprises most of chapters VI and VIII. The symbolism for summation is fully explained and the author makes his formulæ much easier to read by agreeing to use Greek letters for summation indices and Roman letters for other indices. He assumes at first that the transformations are linear, so that the coefficients of the tensor transformation equations are constants, and the ordinary derivatives of tensors are themselves tensors (see bottom of page 81). This does not help very much, but, on the other hand, the results can be generalized without difficulty (page 108), and much of his later development of tensor analysis is based on the use of variables whose transformations are linear. He also assumes for the most part that the number of variables is two, but points out that the discussion, almost without change, applies to any number of variables (for an exception see pages 108–111).

Chapter VIII [1] takes up the general theory of tensors. The theory is based

[1] It is interesting to compare this part of Birkhoff's work with the treatment of the same subject by Eddington in his *Mathematical Theory of Relativity* reviewed by Philip Franklin in this MONTHLY (*1924*, 444). The two books were published at about the same time and cover about the same ground so far as the topics mentioned in this review are concerned, but scarcely any subject is discussed in the same way or any theorem given the same proof. In general, Eddington's methods seem to be more in the spirit of the tensor calculus, while Birkhoff follows more closely the methods of the older mathematical analysis. References to Eddington in this review will be to the *Mathematical Theory of Relativity.*

on a *fundamental quadratic differential form* $ds^2 = g_{\alpha\beta}dx_\alpha dx_\beta$. The three-index symbols of the first and second kinds are denoted by $\Gamma_{i,\,jk}$ and Γ^i_{jk}, $\Gamma_{i,\,jk} = (\partial g_{ij}/\partial x_k + \partial g_{ik}/\partial x_j - \partial g_{jk}/\partial x_i)/2$, and in connection with the fundamental form there is a system of *geodesics* whose differential equations are

$$\frac{d^2x_i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0. \tag{71}$$

Two sets of variables are connected by equations of the form

$$x_i = c_{i\alpha}\bar{x}_\alpha + c_{i\beta\gamma}\bar{x}_\beta\bar{x}_\gamma + \cdots. \tag{64}$$

To begin with he lets $c_{ij} = \delta^i_j$ where δ^i_j is used[1] to denote 1 when $i = j$ and 0 when $i \neq j$. The equations are then

$$x_i = \bar{x}_i + c_{i\alpha\beta}\bar{x}_\alpha\bar{x}_\beta + \cdots, \tag{72}$$

and the two systems of coördinates may be said to be *associated at O*.

If also $c_{ijk} = -\frac{1}{2}\Gamma^i_{jk}|_0$ the system of coördinates $\bar{x}_i$ will be one for which the first derivatives of the coefficients of the fundamental form, and therefore all the three-index symbols of both kinds, vanish at O (page 117). Our author describes such a system as *geodesic at O*, and for a system geodesic at O he uses the letter τ in place of x and h_{ij} in place of g_{ij}.

Finally, there is introduced a special system of coördinates that he calls *Riemannian coördinates*. For this system he uses the letter ξ in place of x and γ_{ij} for g_{ij}, and he defines it by writing $\xi_i = \partial x_i/ds|_0 s$, where s is the parameter in the equations of the geodesic from O through the point x_i. The coördinates x_i and the coördinates $\bar{x}_i$ connected by equations (72) have the same Riemannian system, and the term "associated" may well be applied to any two such systems of coördinates. Two Riemannian systems at O are connected by a linear transformation. Thus if x_i and $\bar{x}_i$ are connected by (64) the associated Riemannian systems will be connected by the equations $\xi_i = c_{i\alpha}\bar{\xi}_\alpha$.

The "admissibility" of these coördinates is then shown by deriving the transformation of the ξ's, which is

$$x_i = \xi_i - \tfrac{1}{2}\Gamma^i_{\alpha\beta}|_0\xi_\alpha\xi_\beta - \tfrac{1}{6}(\partial\Gamma^i_{\alpha\beta}/\partial x_\gamma - 2\Gamma^i_{\alpha\mu}\Gamma^\mu_{\beta\gamma})_0\xi_\alpha\xi_\beta\xi_\gamma + \cdots. \tag{79}$$

This is obtained from the solution of (71) in power series in s, in which occur the expressions given above in the definition of ξ_i.

The derivatives of a tensor satisfy the tensor transformation equations when the variables are connected by a linear transformation. Taking a tensor expressed in terms of any variables x_i, he transforms it by passing from the x's to the associated Riemannian system at a point O, differentiates it, and then transforms the derivative by changing back to the original variables. In this way is obtained an expression which satisfies the tensor equations at O, and so

[1] This is Einstein's notation. Eddington writes g^i_j for the same thing, associating it with g_{ij} according to the rules of the tensor calculus.

anywhere, O being any point. This expression is the *covariant derivative* of the given tensor. Thus one use of the Riemannian system is to give him the covariant derivative.

Another use is to give him a certain "fundamental tensor." Since a Riemannian system is a particular kind of geodesic system, the first derivatives of the γ's are zero at O, but the second derivatives are not zero, and when the coördinates are transformed back to general coördinates these second derivatives become the elements of important tensors.

In particular, certain differences of these second derivatives become the elements of what is called by our author the *Riemannian tensor* and written R_{ijkl}, this expression being the transform of

$$\left(\frac{\partial^2\gamma_{il}}{\partial\xi_j\partial\xi_k} - \frac{\partial^2\gamma_{jl}}{\partial\xi_i\partial\xi_k}\right)_0 .$$

The tensor $R^l_{ijk} = g^{\alpha l}R_{ijk\alpha}$, contracted with respect to i and l, is the tensor that Einstein puts equal to zero for the condition of a gravitation field in empty space. It is called R_{jk} and may be written[1] $g^{\alpha\beta}R_{\alpha jk\beta}$.

The difficulties in the transformation that gives us the formula for R_{ijkl} are partly removed by the introduction of an intermediate system of coördinates, an associated system geodesic at O. This system is defined by the equations

$$\tau_i = x_i + \tfrac{1}{2}\Gamma^i_{\alpha\beta}\big|_0 x_\alpha x_\beta, \tag{94}$$

the Γ's being formed from the fundamental form for the x's. These coördinates have the same Riemannian system as the x's and are connected with it by the equations

$$\tau_i = \xi_i - \tfrac{1}{6}\frac{\partial\Gamma^i_{\alpha\beta}}{\partial\tau_\gamma}\Big|_0 \xi_\alpha\xi_\beta\xi_\gamma + \cdots,$$

the Γ's here formed from the τ's, and equal to zero at O. The equations defining the τ's show that their third derivatives with respect to the x's are zero, and the last equations show that the second derivatives of the τ's with respect to the ξ's are zero at O.

Among all the Riemannian systems at a point O there are some that are *normal;* that is, some for which $\gamma_{ij}|_0 = \delta^i_j$, so that at O the fundamental form reduces to $d\xi_\alpha{}^2$ (page 124). But some of the statements that Birkhoff makes about normal coördinates seem to be true of any Riemannian system, and we have almost come to the conclusion that he sometimes uses the term normal when he means simply Riemannian. In particular, the equations

$$\Gamma^i_{\alpha\beta}x_\alpha x_\beta = 0, \tag{78}$$

[1] R_{ijkl} is what Eddington would write B_{kjil}, i and j forming one pair and k and l the other pair in the pairing of the indices. This difference in notation, though a little confusing, comes naturally from the difference in the method of obtaining these quantities. For R^l_{ijk} Eddington writes B^l_{kji} following Einstein. Einstein does not mention the former tensor, but calls the latter the Riemann-Christoffel tensor. The contracted tensor R_{jk} is written by Einstein (and Eddington) G_{jk}. Eddington, pages 72 and 81, Einstein, *Principle of Relativity*, pages 141 and 142.

which are the conditions that given coördinates be Riemannian, are referred to on page 210 as the conditions that the coördinates be normal. In some of his proofs he uses normal coördinates when Riemannian coördinates, or even coördinates geodesic at O, would have all the advantages of the normal coördinates, these advantages coming usually from the fact that the first derivatives of the g's are zero at the point. This is true, in particular, of the proof on page 136. The discussion on page 126 and the proof on page 210 require that the coördinates be Riemannian, since they make use of equations (78), but they do not need to be normal. The formula

$$\left[\frac{\partial \Gamma^i_{jk}}{\partial \xi_l} + \frac{\partial \Gamma^i_{kl}}{\partial \xi_j} + \frac{\partial \Gamma^i_{lj}}{\partial \xi_k}\right]_0 = 0 \tag{85}$$

is perhaps the most important formula for Riemannian coördinates, since it expresses the condition imposed on the second derivatives of the g's. The corresponding formula, written with the three-index symbols of the first kind, is also true, and so on page 126 it is sufficient that the coördinates be Riemannian.

We may add that the discussion beginning on page 128 would have to be modified only slightly if carried through without the assumption that $h_{ij}|_0 = \delta^i_j$.

Several writers have made use of coördinates for which the first derivatives of the g's all vanish at a given point, and Eddington has introduced a system with a still closer approximation to Birkhoff's Riemannian system; namely, a system for which the second derivatives of the g's satisfy equations (85). These are introduced by him, however, after the theory of covariant differentiation and the Einstein equations have been developed. Eddington calls them *canonical coördinates*, and in the series which express the values of the original coördinates by means of these the terms of the first, second and third degrees have the same coefficients as in the series which express their values in terms of Riemannian coördinates (equations 79 given above).[1]

Professor Birkhoff has been able to give simple proofs for some important theorems.

At the beginning of chapter XIII he proves that the equations $R_{ijkl} = 0$ are the sufficient as well as necessary conditions that a system of coördinates with a given fundamental form can be transformed so that the g's shall be constants. The proof is indirect and does not show how to get the transformation that will do this as does the proof given by Eddington (pages 76–77), but it is simpler, being based on Euler's theorem concerning homogeneous polynomials.

At the end of the chapter on tensor analysis he proves for a Riemannian system at O that we have at this point the four equations

$$\frac{\partial R^\alpha_i}{\partial \xi_\alpha} - \frac{1}{2}\frac{\partial R^\alpha_\alpha}{\partial \xi_i} = 0. \tag{104}$$

[1] Eddington also (pages 77–79) uses a particular geodesic system as intermediate between the original coördinates and his canonical coördinates, but in writing the transformation resulting from the combination of the two transformations (equations 36.8) he has put into one group of terms three-index symbols based on the original coördinates and into another group of terms those based on the intermediate coördinates.

The corresponding equations will then be true for any coördinates at any point if the symbol "∂" is understood to denote covariant derivative. These are "the four identical relations of the Einstein theory," which have attracted considerable attention. Three or four proofs of these equations have been published. The most direct and elegant is by A. E. Harward, published in *The Philosophical Magazine*, volume 44, pages 380–382. It depends, however, on certain complicated formulæ which Birkhoff does not use. The other proofs, like Birkhoff's, are verifications and are made possible by the use of coördinates geodesic at O.

It is stated by Eddington (page 81) and others, as well as by Birkhoff, that because of these four identical relations only six of the ten Einstein equations are independent. It is a little puzzling to see just how these identities affect the independence of the Einstein equations. The identities are linear and homogeneous in the R's and their first derivatives, and if the R's were all zero they would be satisfied. But if all but four of the R's are zero and these equations serve then to make the remaining four zero, it must be because the remaining four, but not their derivatives, occur in them and occur in such a way that the equations can be solved with respect to them.

Professor Birkhoff discusses this question on pages 219–220, but his discussion is very concise. At first he seems to have in mind the case of normal coördinates, for which $R_i^j = R_j^i$ at O. He writes the identities with the variables ξ and supposes "that we satisfy some six of the equations $R_i^j = 0$, omitting, for instance, the four equations $R_i^i = 0$."

But at this point the author changes from ξ to x, that is, as we understand it, from normal coördinates to general coördinates. He says, "By means of the identity just written we find four differential equations in R_i^i,

$$\frac{\partial R_i^i}{\partial x_i} + \varphi_i = 0,$$

where φ_i is linear in R_j^j with $j = 1, 2, 3, 4$." In the general case the expressions in (104) are covariant derivatives and so there are terms in the R's themselves as well as in their derivatives. Also in the general case there are sixteen of the functions R_i^j that are distinct and not ten. But the sixteen are connected by six linear homogeneous equations corresponding to the equations $R_{ij} = R_{ji}$, and in some cases at least it may be supposed possible to put six of the functions R_i^j for which $i \neq j$ equal to zero, and express the other six in terms of the four R_i^i, and in this way to reduce the four identities to equations in these four R_i^i alone, linear and homogeneous in them and their first derivatives.

Suppose that in the equations so reduced derivatives of all four of the R_i^i occur and that we can select one particular derivative of each function and solve the equations for the four derivatives chosen. If, for example, these four derivatives are $\partial R_i^i/\partial x_i$, then the equations can be written as above, but φ_i will usually contain terms in the other derivatives of the R's as well as terms in the R's themselves.

This seems to be the meaning, but the wording in the book would indicate that the derivatives $\partial R_i^i/\partial x_i$ are necessarily the ones to be chosen, or rather that they are the only derivatives that occur.

The determination of the four R's from the equations

$$\frac{\partial R_i^i}{\partial x_i} + \varphi_i = 0$$

will involve four arbitrary functions, and if these functions are taken as equal to zero the R_i^i will be zero. This may be worked out in detail after the manner of certain existence proofs (see C. Jordan, *Cours d'Analyse,* 2d edition, volume 3, chapter 3). A function is determined by its value and the values of all of its derivatives at a single point, say at the point O. If R_i^i and all of its derivatives that do not involve differentiation with respect to x_i are made to vanish at O, this being done for each of the four R_i^i, it will follow by virtue of the above equations and the equations that can be obtained from them by successive differentiation that all of the remaining derivatives of R_i^i will vanish at O, and, therefore, that R_i^i must vanish throughout space-time.[1]

He expresses the result by saying that "six of the differential equations $R_i^j = 0$, joined with four appropriate boundary conditions, are equivalent to the entire set of ten equations."

Professor Birkhoff now passes to the functions R_{ij}. This he can do, for the functions of one set are all zero whenever those of the other set are all zero. He concludes that "the general solution of the equations $R_{ij} = 0$ will involve at least four arbitrary functions."[2] This does not refer to the arbitrary functions introduced in the integration of the four identities, they being taken equal to zero. The equations $R_{ij} = 0$ involve ten g's, and if there are only six of these equations four of the g's could be put equal to arbitrary functions and still leave as many unknowns as there are equations. These arbitrary functions seem to be the ones referred to. Whether with the "boundary conditions" it would still be possible to make these four functions entirely arbitrary might be a question. But the functions more or less arbitrary that are introduced in the integration of the six equations $R_{ij} = 0$ will perhaps take care of the "boundary conditions."[3]

[1] The author does not require that any derivatives of R_i^i shall vanish at O as seems to be necessary, but only that the R_i^i themselves do this.

[2] These are supposed to provide for an arbitrary transformation of coördinates.

[3] In the existence proofs as usually given the equations are solved for the highest derivatives with respect to one particular variable, and it would seem as if the present problem could be treated very simply in that way, especially as there is a considerable choice of equations and unknown functions. Moreover the functions R_i^j may be used or the functions R_{ij}, and of the latter there are only ten.

To give only an outline of the method, let it be supposed, for example, that the six equations $R_{ij} = 0$, $i \neq j$, contain the six derivatives $\partial^2 g_{ij}/\partial x_1^2$, $i \neq j$, and can be solved for them, so that we have

$$\frac{\partial^2 g_{ij}}{\partial x_1^2} = \varphi_{ij},$$

With some breaks in the continuity it will be possible to include in this paper a mention of the three famous tests that have been proposed for the application of the Einstein theories to our universe, and to bring this study to a fitting climax.

Professor Birkhoff determines first (chapter XV) the form of a quadratic differential that will be unaltered by any space rotation about the origin, so that, so far as the quadratic is concerned, the space is symmetrical with respect to this point, supposed to be a unique point of attraction. Such a quadratic, expressed in polar coördinates, is shown to be of the form

$$\alpha dt^2 + 2\beta dt dr + \gamma dr^2 + \delta(d\theta^2 + \sin^2 \theta d\varphi^2),$$

α, β, γ and δ functions of t and r at most. This form is then supposed to be reduced to

$$\lambda(dt^2 - dr^2) + \mu(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{153}$$

(page 245; see also page 109) and the coefficients λ and μ are made to satisfy the Einstein equations, $R_{ij} = 0$, for this form. At first it is assumed that λ and μ are functions of r alone and do not contain t, and under these conditions the Schwarzschild form is derived from the Einstein equations. Later (pages 253–256) it is shown that the above form can always be reduced by a transformation to the Schwarzschild form, so that, except for purposes of proof, it was not necessary to assume that λ and μ do not contain t.

This theorem is very important. It may be well to add to Birkhoff's proof one or two details.

He points out that f', g' and λ must not be zero at O, and says that this requirement is easily met because λ is not zero identically and there is only a single homogeneous relation between $f'(0)$ and $g'(0)$. Now he might have added

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where φ_{ij} contains the g's, their first derivatives, and their second derivatives except those for which the equations are solved.

With the six R_{ij} put equal to zero the four identities will contain only the four R_{ii} and their first derivatives. Suppose that they actually contain the four derivatives $\partial R_{ii}/\partial x_1$ and can be solved for them. Then if these four R_{ii} are made zero when $x_1 = 0$, whatever the values of the other x's, they will be zero identically and all of the Einstein equations will be satisfied.

Let the four functions g_{ii} be arbitrary functions of all of the x's. They can be regarded as given, or the equations as containing only the six unknown functions g_{ij}, $i \neq j$, and their derivatives. Put $\bar{g}_{ij} = g_{ij}|_{x_1=0}$ and $\bar{g}_{ij}' = \partial g_{ij}/\partial x_1|_{x_1=0}$. These are functions of x_2, x_3 and x_4, and when they are given the values of the six g_{ij} will be completely determined.

The four equations $R_{ii}|_{x_1=0} = 0$ involve the functions $\bar{g}_{ij}$ and $\bar{g}_{ij}'$, the first and second derivatives of the former and the first derivatives of the latter. They are differential equations for these twelve functions with x_2, x_3 and x_4 as the independent variables. All that is necessary, then, is to satisfy these equations, and this can be done without placing any restriction upon the g_{ii} as arbitrary functions of the four x's.

To carry through the solution thus outlined it is necessary first to determine the possibility of solving the equations in the manner indicated. Professor Birkhoff refers to this, pointing out that the equations must be compatible (see also Jordan, *l.c.*, page 303). If the solution can be completed we have a determination of the g's that involves four arbitrary functions of all the x's, functions necessary to make possible a most general transformation of coördinates, and then there will usually be arbitrary functions of three of the variables. It is to these that he seems to refer when he says, "the degree of indeterminateness would appear to be that of arbitrary functions of only three variables."

the reason, also necessary, that[1] μ_2 is not identically equal to μ_1 nor to $-\mu_1$; if it were we should have $\mu_{11} = \pm \mu_{12} = \mu_{22}$, and the equation $R_{33} = 0$ would be impossible.

In identifying the transformed quadratic with the Schwarzschild form he has three constants, k, $\bar{r}_0$ and m, that enable him to give preassigned initial values to $\bar{\lambda}$, $\bar{\lambda}_2$ and $\bar{\mu}_2$. But the initial value of $\bar{\mu}$ is also preassigned, and therefore there is a relation which these four values must satisfy if they are to be obtained by means of the three constants. This relation will be found to be

$$[\bar{\lambda}\bar{\mu}_2^2 + 2\bar{\mu}\bar{\lambda}_2\bar{\mu}_2 + 4\bar{\lambda}^2\bar{\mu}]_0 = 0.$$

With $\bar{\mu}_1 = 0$ at O it makes $\bar{\mu}_{11} = 0$ at O. Thus it might have been obtained, not as necessary to make a formal identification with the Schwarzschild form, but as necessary to make $\bar{\mu}_{11} = 0$ at O, and if at O $\bar{\lambda}$, $\bar{\mu}$, $\bar{\lambda}_2$ and $\bar{\mu}_2$ have assigned values satisfying this relation, then $\bar{\lambda}$ and $\bar{\mu}$ may be determined uniquely as functions of $\bar{r}$ alone, satisfying the equations $\bar{R}_{ij} = 0$, about as explained on page 255.

The relation given above may be expressed as a relation connecting the values of λ, μ, f' and g' at O, and when λ and μ are given it becomes an equation for $f'(0)$ and $g'(0)$ to satisfy in addition to the equation at the bottom of page 253.

In the solution of the two equations for $f'(0)$ and $g'(0)$ there will appear what seems to be a possible case of inconsistency; namely, when

$$\mu_1^2 - \mu_2^2 - 4\lambda\mu = 0.$$

λ being equal to $\frac{1}{2}(\mu_{11} - \mu_{22})$, this is a differential equation in μ whose solution may be expressed by saying that μ is equal to minus the square of the difference of two functions, one of $t - r$ and the other of $t + r$, while λ is equal to four times the product of their first derivatives. If these two functions are put equal to $\frac{1}{2}(\bar{t} - \bar{r})$ and $\frac{1}{2}(\bar{t} + \bar{r})$ the quadratic will reduce to the "isotropic" form (given at the bottom of page 245), a particular case of the Schwarzschild form. The reason that the transformation does not fail in this case is that the absolute term in the second of the two equations for $f'(0)$ and $g'(0)$ contains $\bar{\lambda}_2$ as a factor and this is zero in the present case, $\bar{\lambda}$ being equal to 1. Thus the two equations are not inconsistent, but are dependent, and the transformation desired is always possible.[2]

[1] In this paper subscripts will be given to λ and μ to denote derivatives with respect to t and r. The author uses the same letters, t, r, λ and μ, after the transformation that he uses before it, which is liable to cause some confusion. We will try to make our explanations clearer by writing $\bar{t}$, $\bar{r}$, $\bar{\lambda}$ and $\bar{\mu}$ for the new letters, and $\bar{R}_{ij} = 0$ for the new Einstein equations. Subscripts to $\bar{\lambda}$ and $\bar{\mu}$ will then denote derivatives with respect to $\bar{t}$ and $\bar{r}$.

[2] The above discussion will suggest that it may be possible to derive the Schwarzschild form directly by integrating the equations $R_{ij} = 0$ without assuming that λ and μ do not contain t, and thus dispense with the proof given on pages 253–256.

Take the equations

$$R_{11} + R_{22} = R_{33} = R_{12} = 0.$$

In the last chapter the author explains the three tests referred to, taking the plane $\varphi = \pi/2$ for the plane of operations.

The mathematical problem of the first test is to determine the motion of a particle moving so that $\int ds$ shall be an extremum. The resulting equations

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These are

$$\mu_{11} + \mu_{22} = \frac{1}{2\mu}(\mu_1^2 + \mu_2^2) + \frac{1}{\lambda}(\lambda_1\mu_1 + \lambda_2\mu_2),$$

$$\mu_{11} - \mu_{22} = 2\lambda,$$

$$\mu_{12} = \frac{1}{2\mu}\mu_1\mu_2 + \frac{1}{2\lambda}(\lambda_1\mu_2 + \lambda_2\mu_1).$$

Adding twice the third to the first and dividing by $\mu_1 + \mu_2$ we have

$$\frac{(\mu_1 + \mu_2)_1 + (\mu_1 + \mu_2)_2}{\mu_1 + \mu_2} = \frac{\mu_1 + \mu_2}{2\mu} + \frac{\lambda_1 + \lambda_2}{\lambda}.$$

Similarly, subtracting twice the third from the first and dividing by $\mu_1 - \mu_2$ we have

$$\frac{(\mu_1 - \mu_2)_1 - (\mu_1 - \mu_2)_2}{\mu_1 - \mu_2} = \frac{\mu_1 - \mu_2}{2\mu} + \frac{\lambda_1 - \lambda_2}{\lambda}.$$

These equations may be integrated as partial differential equations of the first order. For example, with u written for $\log (\mu_1 + \mu_2)/\lambda\sqrt{(-\mu)}$ the first reduces to $u_1 + u_2 = 0$, and its solution is of the form u = function of $(t - r)$. The result of these integrations may therefore be expressed in the form

$$-(\mu_1 + \mu_2)f' = (\mu_1 - \mu_2)g' = \lambda\sqrt{(-\mu)},$$

f' and g' arbitrary functions of $t - r$ and $t + r$, respectively, which it is convenient to write as derivatives.

If λ and μ are given, f' and g' will be determined by these equations.

For the partial differential equation

$$\mu_1\cdot(g' + f') - \mu_2\cdot(g' - f') = 0,$$

one auxiliary equation is

$$\frac{dt}{g' + f'} + \frac{dr}{g' - f'} = 0;$$

or

$$g'(dt + dr) - f'(dt - dr) = 0;$$

whence

$$g - f = \text{const.},$$

and μ is a function of the single quantity $g - f$.

Therefore

$$\mu_1 = \mu'\cdot(g' - f') \qquad \text{and} \qquad \mu_2 = \mu'\cdot(g' + f'),$$

so that both equations above reduce to

$$\lambda\sqrt{(-\mu)} = -2\mu'f'g'.$$

The derivatives of μ_1 and μ_2 with respect to t and r, respectively, will be

$$\mu_{11} = \mu'\cdot(g'' - f'') + \mu''(g' - f')^2,$$

and

$$\mu_{22} = \mu'\cdot(g'' - f'') + \mu''(g' + f')^2;$$

whence

$$2\lambda = \mu_{11} - \mu_{22} = -4\mu''f'g'.$$

Now the elimination of λ from the preceding equation leads to

$$\mu''\sqrt{(-\mu)} = \mu',$$

are the equations of the path in which it is supposed to move when free except for the attraction at the origin. Applied to the planet Mercury they give us the orbit and the motion of the planet in it, the result, as we know, in remarkable agreement with observation.

For the path of a ray of light ds is said to be equal to zero. The equations (157) and (158) were obtained as first integrals of the extremum problem. Multiplying the latter by Uf he points out that they are then true in the present case if the constant f is taken equal to zero. But how the deductions that follow (158) are valid now is not clear, the first step of these deductions being the elimination of the factors whose vanishing makes the equations true. Granted, however, that equation (159) is true with $f = 0$, the completion of the integrations is much like that of the preceding theory, and from the result is obtained the deflection determined by the Einstein theory.

The displacement of the spectral lines is discussed on the last two pages of the book. If we suppose that a vibrating atom of a specified element is the "clock" by which local time is determined, and if the differential of local time at any particle, as it appears to an observer whose system of coördinates is denoted by t, r, θ and φ, is represented by ds, then for the sun and earth

$$f_s : f_e = ds_s : ds_e,$$

where f_s denotes the number of vibrations received in a given interval of time from the atom on the sun and f_e the number received from the atom on the earth. But the frequency of vibrations received determines the position of the spectral lines, and therefore a difference in the rates of local time in the two places will cause a difference in the position of the spectral lines.

The Einstein theories have caused much discussion, some believing that the above tests have established the correctness of these theories, a few perhaps

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and on integration

$$\tfrac{1}{2}\mu' = -\sqrt{(-\mu)} + 2m,$$

$2m$ the constant of integration.

Put $\mu = -\bar{r}^2$ and therefore $\lambda = 4\bar{r}'f'g'$. The μ' equation becomes

$$\bar{r}' = 1 - 2m/\bar{r}.$$

$\bar{r}'$ means $d\bar{r}/d(g - f)$, so that $d(g - f) = d\bar{r}/\bar{r}'$; *i.e.*,

$$g'(dt + dr) - f'(dt - dr) = d\bar{r}/(1 - 2m/\bar{r}).$$

Put also

$$g'(dt + dr) + f'(dt - dr) = d\bar{t}.$$

The product of the sum and difference of these expressions will be

$$4f'g'(dt^2 - dr^2) = d\bar{t}^2 - [d\bar{r}/(1 - 2m/\bar{r})]^2.$$

But

$$\lambda = 4f'g'(1 - 2m/\bar{r}).$$

so that, finally, we have

$$\lambda(dt^2 - dr^2) = (1 - 2m/\bar{r})d\bar{t}^2 - d\bar{r}^2/(1 - 2m/\bar{r}).$$

This shows that (153) can always be reduced to the Schwarzschild form by the transformation

$$2g = \bar{t} + \int d\bar{r}/(1 - 2m/\bar{r}),$$
$$2f = \bar{t} - \int d\bar{r}/(1 - 2m/\bar{r}).$$

doubtful or actively opposed. This discussion does not touch the mathematician; we might almost say that to him it is meaningless. The mathematician finds that the mathematics is true. Some of it, the theory of tensors and tensor transformations, was in his hands long before the beginning of Einstein's work. These theories belong to analysis, but have sprung into prominence on account of their application to relativity. But there are other things in relativity, the Lorentz transformation, the FitzGerald contraction, the determination of the time and position of an event by means of light signals, and things considered in that branch of mathematics that we call kinematics, that may be treated mathematically without any reference to the external world, or to any physical causes producing them. This mathematics is of value to the mathematician apart from any physical application, just as are all of the various branches of mathematics in their relations to one another.

Personally we have found relativity a most baffling and elusive study. Time and again has it shown an open path to rapid progress, and time and again has the path suddenly closed with some stage of this progress not quite completed. Obstacles would remain perhaps for weeks or months; and then they would be removed and the way opened, only soon to be closed again by other obstacles.

In the explanations attempted here we have by no means covered all of the mathematics in Professor Birkhoff's book—there are still obstacles to a full completion of that task—and we have not touched the physics. But the explanations in this paper will serve, we hope, to show that the book is stimulating and suggestive. Whatever the cost, it is well worth study. It will contribute much to the advancement of mathematical and physical science, and a long time will pass before the store of ideas that it contains is exhausted.

H. P. MANNING.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 47, no. 1, January, 1925: "L'expression la plus générale de la 'distance' sur un droit" by M. Fréchet, 1–10; "Second paper on tensor analysis" by G. Y. Rainich, 11–24; "Generalization of certain theorems of Bohl" by F. H. Murray, 25–44; "Expansion problems in connection with the hypergeometric differential equation" by B. P. Reinsch, 45–70.

ANNALS OF MATHEMATICS, second series, volume 25, no. 4, June, 1924: "Differential equations from the group standpoint" by L. E. Dickson, 287–378.

JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, volume 4, no. 1, January, 1925: "Note on a paper of O. Perron" by N. Wiener, 21–32; "The torsion problem of curved beams" by P. Heymans and W. J. Heymans, 33–61; "A Taylor's expansion of a determinant" by L. H. Rice, 62–63.

MONIST, volume 34, no. 4, October, 1924: "Number: an introduction to the theory of analytic functions" by G. Mittag-Leffler, 481–510; "Foundations of mathematics" by S. Klyce, 615–637.

PHILOSOPHICAL MAGAZINE, volume 49, no. 289, January, 1925: "The field of an electron at rest and in uniform motion" by H. Bateman, 1–18; "Note on the quantitative formulation of Bohr's correspondence principle" by R. C. Tolman, 130–136.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 12, December, 1924: "A general type of singular point" by E. Hille, 488–493; "Topological invariants of manifolds" by J. W. Alexander, 493–494.

SCHOOL SCIENCE AND MATHEMATICS, volume 25, no. 1, January, 1925: "Mathematics in the junior and senior high schools" by M. Gugle, 29–35; "A little-understood principle in multiplication" by H. I. Jones and B. P. Jones, 36–43; "Some formulæ for checking correlation tables" by J. N. Mallory, 44–48.

Science, volume 61, no. 1566, January 2, 1925 and no. 1567, January 9, 1925: "The foundation of the theory of algebraic numbers" by H. Hancock, 5–10 and 30–35.

Transactions of the American Mathematical Society, volume 26, no. 4, October, 1924: "Algebras which do not possess a finite basis" by J. H. M. Wedderburn, 395–426; "Determination of all the prime power groups containing only one invariant sub-group of every index which exceeds the prime number" by H. A. Bender, 427–439; "Invariants of the linear group, modulo $\pi = p_1^{\lambda_1}p_2^{\lambda_2} \cdots p_n^{\lambda_n}$" by C. Gouwens, 435–440; "A uniqueness theorem for Legendre and Hermite polynomials" by K. P. Williams, 441–445; "A new type of class number relations" by E. T. Bell, 446–450; "A new method in the equivalence of pairs of bilinear forms" by R. G. D. Richardson, 451–478; "Relative extrema of pairs of quadratic and hermitian forms" by R. G. D. Richardson, 479–493.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

The Mathematics Club of Brown University, Providence, R. I. [*1924*, 145.]

The following is the program of the Mathematics Club of Brown University for the year 1924–1925:

November 14, 1924. "Linkages" by Professor Raymond Kurtz Morley.

December 12, 1924. "The Josephus problem" by Edson Clark Lockwood '25. "Caroline Herschel" by Lena Mae Dailey '26. "The four fours and related problems" by Leslie Thomas Fagan '26.

January 16, 1925. "Fibonacci series and related geometric paradox" by George Toyoharu Tsukuno '27. "Bhaskara's Lilavati" by Rose Mary Fogarty '25.

February 27, 1925. "A prismatoid formula" by William Elliot Cruise '26. "Abel" by Elizabeth Pearce Armstrong '27. "Lightning calculators" by Eugene Anderson Conant '27.

April 14, 1925. "Mathematics as an element in the history of thought." Open to all members of the University, by Professor Alfred North Whitehead, of Harvard University.

May 1, 1925. "Rational triangles and quadrilaterals" by John Joseph Bauer '25. "The Greek anthology" by Mary Virginia Kenny '26.

May (Date and placed to be announced later). Picnic.

(Report by Mr. G. Sauté.)

Mathematics Round Table, Illinois Wesleyan University, Bloomington, Ill.

The following is a list of the subjects which were discussed in the Mathematics Round Table of the Illinois Wesleyan University:

"Curve tracing," "Fourth dimension," "Graphical machinery," "Magic squares," "The history of mathematics," "The history of Pi," "Linkages," "The life and work of Newton," "Negative and complex numbers," "Time and its measurement," "The theory of relativity," "Algebraic notations," "Mathematical prodigies," "Calculating machines," "René Descartes."

(Report by Mr. H. P. Bicknell.)

The Mathematics Club of University of North Carolina, Chapel Hill, N. C. [*1922*, 26.]

The officers of the club for the session 1923–1924 were:

Vinton A. Hoyle, president; C. H. Benson, vice-president; Professor E. T. Browne, secretary and treasurer.

The club was organized primarily for the graduate students in mathematics, for seniors, and the better students among the juniors. The attendance at all meetings was uniformly good.

The program was as follows:

October 23, 1923. "Nine proofs of the Pythagorean theorem" by V. A. Hoyle.
November 27, 1923. "Curiosities in numbers" by W. V. Parker.
January 29, 1924. "The history and development of logarithms" by C. H. Benson.
February 26, 1924. "Algebraic and geometric fallacies" by S. B. Smiethey.
April 7, 1924. "The parallel postulate" by G. S. Bruton.
May 27, 1924. "The trisection of the angle" by H. A. Breard. "Magic squares" by R. G. Florance.

(Report by Professor Browne.)

MATHEMATICS CLUB OF BUCKNELL UNIVERSITY, Lewisburg, Pennsylvania. [*1922*, 177.]

The program for the meetings during the remainder of 1921–1922 were as follows:
March 27. "Algebraic fallacies" by Paul Schmidt '22. "Value of mathematics in high school" by Catherine Stahl '22.
May 8. "Division of the circle" by Professor G. L. Lowry.
May 13. Annual hike of the Club.

Election of officers: president, Kathryn Kinble '23; vice-president, Harold Schaefer '24; secretary-treasurer, Frieda Ebner '23.

1922–1923:
October 2, 1922. "Energy of the sun" by Professor William C. Bartol.
November 6. "New type of examination" by Frieda Ebner '23. "Practical value of high school geometry" by Harold Schaefer '24.
December 5. "Reasons why students fail in mathematics" by Mary E. Bailey '23. "Calculating prodigies" by Rachel Steckel '24.
January 8, 1923. "The status of mathematics in schools to-day" by Mrs. Anna C. Clark, Instructor in Mathematics.
February 5. "Relation of mathematics to commerce" by Chas. R. Birch '23. "Super-power survey" by Professor Rhodes '03, Professor of Electrical Engineering.
March 26. A mathematics bee with puzzles, questions, problems, symbols, definitions and magic properties and numbers, conducted by Professor H. S. Everett '12.
May 7. Reading: Leacock's "Education made attractive." Election of officers for the year 1923–1924: president, Harold Schaefer '24; vice-president, Hilda DeWitt '24; secretary-treasurer, Margaret Ackerman '25; faculty adviser, Professor H. S. Everett.

1923–1924:
November 19, 1923. Social meeting, mathematical games and puzzles.
January 14, 1924. "Teaching mathematics" by Professor Whyte, Professor of Oral English.
February 11. "Mathematics—A great inheritance" by Professor H. S. Everett.
May 19. "Roots of unity" by Professor J. S. Gold. Election of officers: president, Myron F. Decker; vice-president, Eugene D. Carstater; secretary-treasurer, Hulda J. Baxter.
May 24. Hike.

(Report by Professor Everett.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3125. Proposed by W. J. SIDIS, New York City.

If n is a prime number and r is a prime number not of the form $kn + 1$, then in the scale of radix r, a perfect nth power can be found ending in any given digit. Also, provided the given last digit is not 0, a perfect nth power can be found ending in any given set of digits.

3126. Proposed by V. M. SPUNAR, Chicago, Illinois.

Interpret the equations:

$$\begin{aligned} \sqrt{\alpha \sin A} + \sqrt{\beta \sin B} + \sqrt{\gamma \sin C} &= 0, \\ \sqrt{\alpha \cos A} + \sqrt{\beta \cos B} + \sqrt{\gamma \cos C} &= 0, \\ \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma} &= 0. \end{aligned}$$

3127. Proposed by HARRY LANGMAN, New York City.

Assuming $x^3 + y^3 = z^3$ impossible, show that neither of the expressions $4q^3 - 27p^6$, $12q^3 - 3p^6$, can be squares.

3128. Proposed by J. ROSENBAUM, Milford, Connecticut.

Given the mid-points of the sides of a polygon, to construct the polygon.

3129. Proposed by C. N. SCHMALL, New York City.

Three hyperbolas are described each touching one side of a given triangle and having the remaining sides as asymptotes. Show that the product of their three latera recta is equal to the cube of the diameter of the inscribed circle of the triangle.

3130. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that if d, D are the double polar distances, or diameters, of the inscribed and circumscribed circles of a spherical triangle, and d', D' the corresponding diameters for its polar triangle,

$$d' + D = 180° = d + D'.$$

3131. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment intercepted by two fixed tangents to a given conic on a variable tangent to the same conic.

3132. Proposed by J. W. CLAWSON, Ursinus College.

The trilinear coördinates of the circumcenter of a triangle are: $\cos A$, $\cos B$, $\cos C$; that is, the distances of this point from the sides of the triangle are proportional to $\cos A$, $\cos B$, and $\cos C$ respectively. The points ($\sin A$, $\sin B$, $\sin C$), ($\sec A$, $\sec B$, $\sec C$), and ($\csc A$, $\csc B$, $\csc C$) are also well-known points (symmedian, orthocenter and centroid) which can be located by simple geometrical constructions.

Can the points ($\tan A$, $\tan B$, $\tan C$) and ($\cot A$, $\cot B$, $\cot C$) be found by simple geometrical constructions? Have the points been named?

SOLUTIONS.

3081 [1924, 255]. Proposed by HARRY LANGMAN, New York City.

Show that

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 1 & \binom{4}{1} & \binom{6}{2} & \cdots & \binom{2m-2}{m-2} & \binom{2m}{m-1} \\ 0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \binom{2m}{1} \end{vmatrix} = 2,$$

where the parentheses represent binomial coefficients.

SOLUTION BY J. F. REILLY, University of Iowa.

From each element of the second row subtract half the corresponding element of the first row (the elements of the first row are even integers), and the given determinant becomes

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2k}{m-k} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 0 & 1 & \binom{5}{1} & \cdots & \binom{2m-2k-1}{m-k-2} & \cdots & \binom{2m-3}{m-3} & \binom{2m-1}{m-2} \\ 0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2k}{m-k-2} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \binom{2m}{1} \end{vmatrix}.$$

In this determinant subtract each element of the second row from the corresponding element of the third row; in the resulting determinant subtract each element of the third row from the corresponding element of the fourth row; again in the resulting determinant subtract each element of the fourth row from the corresponding element of the fifth row, and so on, effecting the subtractions by using the theorem

$$\binom{n}{j} - \binom{n-1}{j} = \binom{n-1}{j-1}.$$

There will result finally the determinant

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2k}{m-k} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 0 & 1 & \binom{5}{1} & \cdots & \binom{2m-2k-1}{m-k-2} & \cdots & \binom{2m-3}{m-3} & \binom{2m-1}{m-2} \\ 0 & 0 & 1 & \cdots & \binom{2m-2k-1}{m-k-3} & \cdots & \binom{2m-3}{m-4} & \binom{2m-1}{m-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \binom{2m-1}{1} \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

This determinant is equal to 2, since the cofactor of $\binom{2}{1}$, its upper left hand element, is unity.

Also solved by E. P. BUGDANOFF, THEODORE BENNETT, and P. J. DA CUNHA.

3083 [1924, 305]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that if three distinct normals to the parabola, $y^2 = 4px$, are concurrent, the sum of their slopes is zero, and that if the sum of three numbers is zero, they are the slopes of concurrent normals to the parabola, $y^2 = 4px$. If two of the three concurrent normals are perpendicular and the third bisects the angle between them, show that they meet at $(3p, 0)$, $(8p, \sqrt{5}p)$ or $(8p, -\sqrt{5}p)$.

SOLUTION BY H. W. BAILEY, Champaign, Illinois.

Let the normals be N_1, N_2, N_3, respectively, at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) on the parabola $y^2 = 4px$; their slopes are then $m_1 = -y_1/2p$, $m_2 = -y_2/2p$, $m_3 = -y_3/2p$. The condition that the sum of their slopes be zero is then $y_1 + y_2 + y_3 = 0$. Now the equations of the normals are $y_ix + 2py = (2p + x_i)y_i$, $i = 1, 2, 3$, respectively, for N_1, N_2, N_3. The necessary

and sufficient condition that these three normals be concurrent may be written after reduction as

$$\begin{vmatrix} y_1 & 1 & y_1^3 \\ y & 1 & y_2^3 \\ y_3 & 1 & y_3^3 \end{vmatrix} = 0;$$

but if this be expanded and factored the result is $(y_1 - y_2)(y_2 - y_3)(y_1 - y_3)(y_1 + y_2 + y_3) = 0$. Hence the necessary and sufficient condition that three distinct normals be concurrent is that the sum of their slopes be zero.

Suppose that N_1 and N_2 are perpendicular and that N_3 bisects the angle between them. Then $m_2 = -1/m_1$, and $m_3 = (m_1 - 1)/(m_1 + 1)$ and imposing the condition that the sum of the slopes be zero we get the equation $m_1^3 + 2m_1^2 - 2m_1 - 1 = 0$. The roots of this equation are 1, $(-3 - \sqrt{5})/2$, $(-3 + \sqrt{5})/2$. Solving for m_2 and m_3, we get the sets of slopes $(1, -1, 0)$, $([-3 - \sqrt{5}]/2, [3 - \sqrt{5}]/2, \sqrt{5})$, $([-3 + \sqrt{5}]/2, [3 + \sqrt{5}]/2, -\sqrt{5})$. From these values and the original definitions of the slopes, we can compute the $x_1, y_1, x_2, y_2, x_3, y_3$ numerically; for example, for the first set of slopes they are, in order, $p, -2p, p, 2p, 0, 0$. The points of intersection are then found by solving two simultaneous equations for each set of roots, for example, the equations:

$$\begin{cases} x - y = 3p, \\ x + y = 3p, \end{cases} \qquad \begin{cases} x(3 + \sqrt{5}) + 2y = (24 + 10\sqrt{5})p, \\ x\sqrt{5} \qquad - \ y = 7\sqrt{5}p, \end{cases} \qquad \begin{cases} x(3 + \sqrt{5}) - 2y = (24 + 10\sqrt{5})p, \\ x\sqrt{5} \qquad + \ y = 7\sqrt{5}p. \end{cases}$$

Solving, we find the points of intersection to be $(3p, 0)$, $(8p, \sqrt{5}p)$, $(8p, -\sqrt{5}p)$ respectively.

NOTE BY THE EDITORS: A proof of the first statement follows readily from the fact that the cubic in m,

$$pm^3 + m(2p - x') + y' = 0,$$

lacks the m^2 term.

The condition that two of the normals be perpendicular and the third bisect one of the angles between them would be satisfied also if $m_3 = (1 + m_1)/(1 - m_1)$; but, since this can be reduced to the condition already considered by interchanging m_1 and m_2, it need not be discussed further.

Also solved by W. L. AYERS, THEODORE BENNETT, E. P. BUGDANOFF, MAURICE BAUDIN, R. W. BRINK, E. H. CLARKE, P. J. DA CUNHA, S. E. FIELD, C. N. MILLS, A. PELLETIER, and J. B. REYNOLDS.

3084 [1924, 305]. Proposed by H. S. UHLER, Yale University.

Given that a and b are constants, evaluate the integral

$$\int \{[x^3 \arcsin (b/x)]/\sqrt{a^2 - x^2}\} dx.$$

I. SOLUTION BY FREDRICK WOOD, Lake Forest College.

By integration by parts this integral becomes

$$\left[\frac{(a^2 - x^2)^{3/2}}{3} - a^2\sqrt{a^2 - x^2}\right] \arcsin\frac{b}{x} + b\int\left[\frac{(a^2 - x^2)^{3/2}}{3} - a^2\sqrt{a^2 - x^2}\right]\frac{dx}{x^2\sqrt{1 - \frac{b^2}{x^2}}}.$$

Then, by using the substitution

$$a^2 - x^2 = (a^2 - b^2)\sin^2\theta,$$

the second term can be put in the form

$$\frac{b(a^2 - b^2)}{3}\int\left[\sin^2\theta - \frac{2a^2}{a^2 - b^2} + \frac{2a^4}{(a^2 - b^2)(a^2\cos^2\theta + b^2\sin^2\theta)}\right]d\theta.$$

After integrating, and changing variables, the final result is

$$\frac{1}{6}\left\{-2(x^2 + 2a^2)\sqrt{a^2 - x^2}\arcsin\frac{b}{x} - b(3a^2 + b^2)\arcsin\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - b^2}} - b\sqrt{(a^2 - x^2)(x^2 - b^2)} + 4a^3\arctan\frac{b\sqrt{a^2 - x^2}}{a\sqrt{x^2 - b^2}}\right\} + C.$$

II. SOLUTION BY P. J. DA CUNHA, University of Lisbon.

An integration by parts applied to the proposed integral gives

$$I = -x^2\sqrt{a^2-x^2}\arcsin\frac{b}{x} + 2\int x\sqrt{a^2-x^2}\arcsin\frac{b}{x}\,dx - b\int\frac{x\sqrt{a^2-x^2}}{\sqrt{x^2-b^2}}\,dx.$$

Again applying the same method to the first integral of this result, we find, after combining several terms,

$$I = -\frac{(2a^2+x^2)\sqrt{a^2-x^2}}{3}\arcsin\frac{b}{x} - \frac{b}{3}\int\frac{(2a^2+x^2)\sqrt{a^2-x^2}}{x\sqrt{x^2-b^2}}\,dx.$$

Setting $t = \sqrt{a^2-x^2}/\sqrt{x^2-b^2}$, the integral on the left, omitting the factor $-b/3$, becomes

$$-(a^2-b^2)\int\frac{t^2[3a^2+(2a^2+b^2)t^2]}{(1+t^2)^2(a^2+b^2t^2)}\,dt.$$

The integrand is then decomposed into

$$-\frac{1}{(1+t^2)^2} - \frac{a^2+b^2}{(a^2-b^2)(1+t^2)} + \frac{2a^4}{(a^2-b^2)(a^2+b^2t^2)}.$$

The integration of these three fractions gives the final result:

$$I = -\frac{(2a^2+x^2)\sqrt{a^2-x^2}}{3}\arcsin\frac{b}{x} - \frac{b\sqrt{(a^2-x^2)(x^2-b^2)}}{6}$$
$$-\frac{b(3a^2+b^2)}{6}\arctan\sqrt{\frac{a^2-x^2}{x^2-b^2}} + \frac{2a^3}{3}\arctan\frac{b}{a}\sqrt{\frac{a^2-x^2}{x^2-b^2}} + \text{constant}.$$

Also solved by H. W. BAILEY, THEODORE BENNETT, C. K. ROBBINS and the PROPOSER.

3088 [1924, 306]. Proposed by C. J. COE, University of Michigan.

If a sphere of radius $2r$ is cut by a circular cylinder of radius r so that the center of the sphere lies on the surface of the cylinder, show that the length of the curve of intersection of the two surfaces is twice that of the curve of intersection of this cylinder with a plane cutting its elements at an angle of 45°.

SOLUTION BY MAURICE BAUDIN, Miami University.

With reference to rectangular axes, the origin being the center of the sphere, an element of the cylinder being the Z-axis, and u the angle which the radius r of the cylinder makes with the positive X-axis, we have as the equations of the curve of intersection

$$x = r(1+\cos u), \qquad y = r\sin u, \qquad \text{and} \qquad z = \pm 2r\sin(u/2).$$

If s_1 be the length of this curve, we have

$$\tfrac{1}{4}s_1 = \sqrt{2}r\int_0^{\pi}\sqrt{1-\tfrac{1}{2}\sin^2\frac{u}{2}}\,du,$$

or, setting $u/2 = \varphi$,

$$\tfrac{1}{8}s_1 = \sqrt{2}r\int_0^{\pi/2}\sqrt{1-\tfrac{1}{2}\sin^2\varphi}\,d\varphi.$$

The equation of the given ellipse may be written

$$x = r(1+\cos\varphi), \qquad y = z = r\sin\varphi,$$

and its length s_2 is given by

$$\tfrac{1}{4}s_2 = \sqrt{2}r\int_0^{\pi/2}\sqrt{1-\tfrac{1}{2}\sin^2\varphi}\,d\varphi.$$

Hence, $s_1 = 2s_2$.

Also solved by J. A. BULLARD, THEODORE BENNETT, and J. B. REYNOLDS.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Mr. W. B. LORING, of the University of Maine, has been appointed assistant professor of mathematics at Colby College, to fill the vacancy caused by the illness of Professor B. E. Carter.

Professor E. A. SHAW, for twenty-five years professor of mathematics at Norwich University, has retired from active teaching.

Professor R. G. D. RICHARDSON, of Brown University, has been granted leave of absence for the second semester of the academic year 1924–1925.

Professor L. P. EISENHART, of Princeton University, has been elected dean of the faculty.

Dr. R. E. GLEASON, of Princeton University, has been appointed professor of mathematics at Temple University, Philadelphia.

Mr. D. H. MACPHERSON, of Brooklyn Polytechnic Institute, died December 28, 1924, at the age of twenty-two.

Professor A. G. HALL, registrar of the University of Michigan, died January 11, 1925.

Dr. W. E. STANTON, formerly professor of mathematics at Iowa State College, died recently at the age of seventy-four.

The first Carus Monograph has been ordered by sixty per cent. of all members who have thus far paid their dues. Each member is entitled to order one copy at the cost price either now or at any future time. All orders should be sent to Secretary W. D. Cairns, Oberlin, Ohio. Non-members should send their orders to the Open Court Publishing Company, Chicago, Illinois.

The following reports of Summer Sessions to be held in 1925 have been received:

University of Chicago, first term, June 20 to July 29; second term, July 30 to September 4. In addition to the usual courses in Trigonometry, College algebra, Plane and solid analytical geometry and Calculus given by regular members of the staff and by Professors H. E. BUCHANAN and H. L. SMITH, the following advanced courses are announced: By Professor G. A. BLISS: Calculus of variations; Thesis work in analysis. By Professor H. E. SLAUGHT: Differential equations; Definite integrals. By Professor E. T. BELL: Algebraic numbers; Elliptic and theta functions. By Professor F. R. MOULTON: Periodic orbits; Analytic mechanics. By Professor G. C. EVANS: Mathematical theory of economics; Integral equations and Potential theory. By Professor A. C. LUNN: Relativity and the quantum phenomena; Theory of functions of a complex variable. By Professor E. P. LANE: Analytical projective geometry; Metric differential geometry.

Columbia University, July 6 to August 14. In addition to courses in Trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, and

a series of courses for teachers of secondary mathematics, the following advanced courses are offered: By Professor H. F. BLICHFELD: Elementary exposition of selected topics in modern mathematics; Theory of continuous groups. By Professor W. B. FITE: Introduction to higher algebra. By Professor G. A. PFEIFFER: Theory of functions of a real variable. By Dr. K. W. LAMSON: Differential equations.

Cornell University, July 6 to August 14. By Professor W. A. HURWITZ: Analysis. By Professor VIRGIL SNYDER: Projective geometry. The following reading and research courses are also offered: By Professor C. F. CRAIG: Functions of a complex variable. By Professor SNYDER: Algebraic geometry. By Professor F. R. SHARPE: Hydrodynamics and elasticity. By Professors D. C. GILLESPIE and HURWITZ: Analysis. By Professors W. B. CARVER and F. W. OWENS: Projective geometry.

University of Illinois, June 22 to August 15. In addition to the usual courses in College algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered: By Professor J. B. SHAW: Philosophy of mathematics. By Professor R. D. CARMICHAEL: Partial differential equations. By Professor E. B. LYTLE: Teachers course; History of mathematics. By Professor Arnold EMCH: Projective geometry. By Mr. L. L. STEIMLEY: Advanced calculus. By Mr. B. R. REINSCH: Theory of equations.

University of Iowa, first term, June 15 to July 24. In addition to courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following courses are offered: By Dr. D. H. MENZEL: Astronomy. By Dr. ROSCOE WOODS: Projective geometry. By Professor W. H. WILSON: Differential equations; Subject matter and teaching of mathematics. By Professor J. F. REILLY: Advanced algebra; Interpolation and graduation. Second term, July 27 to August 28. By Professor W. H. WILSON: Theory of equations. By Professor E. W. CHITTENDEN: Differential equations; Advanced calculus.

Johns Hopkins University, June 30 to August 7. In addition to courses in College algebra, College geometry, and Introductory calculus, the following graduate course is offered: By Professor F. D. MURNAGHAN: Functions of a complex variable with applications to differential equation theory.

University of Michigan, June 22 to August 14. In addition to courses in Algebra, Plane and solid geometry, Trigonometry, Analytic geometry, Calculus, Mathematical statistics, and the Mathematical theory of interest and insurance, the following advanced courses are offered: By Professor W. B. FORD: Advanced calculus; Determinants and the theory of equations. By Professor L. C. KARPINSKI: Teaching of geometry; History of mathematics. By Professor J. W. BRADSHAW: The figures of solid geometry. By Professor T. H. HILDEBRANDT: Theory of functions of a complex variable. By Professor T. R. RUNNING: Empirical formulas. By Professor PETER FIELD: Mechanics. By Professor H. C. CARVER: Theory of probability; Finite differences. By Professor NORMAN ANNING: Solid analytic geometry. By Mr. S. E. FIELD: Differential equations.

University of Minnesota, first term, June 22 to August 1; second term, August 2 to September 5. The department of mathematics will offer an intensive course entitled: Selected topics in advanced mathematics. The topics for 1925 are: First term: By Professor DUNHAM JACKSON: Fourier series and other special series. By Professor A. L. UNDERHILL: Differential equations. By Dr. ELIZABETH CARLSON: Advanced geometry. Second term: By Professor W. L. HART (topic to be announced later).

University of Oklahoma, June 10 to August 4. By Professor S. W. REAVES: Integral calculus; Modern geometry. By Professor J. O. HASSLER: Differential calculus; Theory of functions of a complex variable; Teachers course in mathematics. By Associate Professor E. D. MEACHAM: Analytic geometry; Projective geometry.

University of Pennsylvania, July 6 to August 15. In addition to the usual courses in Solid geometry, Trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered: By Professor G. G. CHAMBERS: Elementary statistics. By Professor H. H. MITCHELL: Algebras and their arithmetics. By Professor J. R. KLINE: Integral equations.

University of Texas. By Professor E. L. DODD: Theory of irrational numbers; Probability and least squares. By Professor R. L. MOORE: Theory of sets; Introduction to the foundations of geometry. By Associate Professor H. J. ETTLINGER: Complex numbers and vector analysis; Infinite processes; Theory of the Riemann integral. By Associate Professor A. A. Bennett: Fundamentals in elementary mathematics; Ruler and compass constructions.

University of Wisconsin, June 29 to August 7. By Professor L. W. DOWLING: Projective geometry; Selected topics in higher geometry. By Professor H. W. MARCH: Definite integrals. By Professor W. W. HART: The teaching of secondary mathematics. By Professor M. H. INGRAHAM: Finite number fields; The elementary properties of number systems. By Professor W. WEAVER: Theory of probability; Vector analysis. By Dr. F. E. ALLEN: Analytic projective geometry. By Mr. E. B. MILLER: Differential equations; Theory of equations.

Mathematical Research will be directed by Professors DOWLING, MARCH, WEAVER, and INGRAHAM.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 8–9, 1925;

Tenth Annual Meeting, Kansas City, Mo., December, 1925.

The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Peoria, May 8–9, 1925

INDIANA, Purdue Univ., April

IOWA, State Teachers College, Cedar Falls, May 1–2

KANSAS, Topeka, February 7

KENTUCKY, Univ. of Kentucky, April or May

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 20–21

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Baltimore, December 22, 1924

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OHIO, Ohio State Univ., Columbus, April 3

ROCKY MOUNTAIN, Laramie, April

SOUTHEASTERN, Birmingham, Ala., Spring

SOUTHERN CALIFORNIA, February 28

TEXAS, Dallas, November 27–28, 1925

THE AMERICAN

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ELEVENTH REGULAR MEETING OF THE KANSAS SECTION.

The eleventh regular meeting of the Kansas Section was held at Topeka, February 7, 1925, in connection with the annual meeting of the Kansas Association of Mathematics Teachers. Two sessions were held of which the first was a joint session with the Kansas Association. Professor J. J. Wheeler, Chairman of the Kansas Section, presided part of the time during the joint (forenoon) session and all of the time during the afternoon session.

The attendance was forty-six including the following twenty-eight members of the Association:

C. H. Ashton, Wealthy Babcock, T. Bell, Florence Black, Lucy Dougherty, P. L. Evans, W. H. Garrett, W. A. Harshbarger, Emma Hyde, W. H. Hill, W. C. Janes, C. F. Lewis, Viola Lindberg, O. B. Loewen, W. A. Luby, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza Mossman, C. A. Reagan, B. L. Remick, J. A. G. Shirk, G. W. Smith, Edith Steininger, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, W. H. Garrett; Vice-Chairman, W. T. Stratton; Secretary-Treasurer, U. G. Mitchell.

The following papers by members of the Association were presented:

(1) "Report of the Toronto meeting, International Mathematical Congress" by Professor J. A. G. Shirk.

(2) "The evolution of our number system" by Professor C. F. Lewis.

(3) "Books mathematics teachers should read" by Professor O. B. Loewen.

(4) "Invariants of determinants with linear elements" by Miss Edith Steininger.

(5) "Fermat and Mersenne numbers" by Professor U. G. Mitchell.

(6) "Reduction of the formulæ for standard deviation and coefficient of correlation" by Professor A. E. White.

(7) "Future policies of the Section," general discussion led by Professor J. J. Wheeler.

The first three of the above papers were read at the joint session.

Abstracts of the papers are given below, the numbers corresponding to the numbers in the list of titles above.

(1) Professor Shirk spoke of the varied program of the Mathematical Congress meeting held at Toronto, the number of papers and most important results presented. He emphasized, in particular, the evidence given of new fields of employment for mathematicians.

(2) Professor Lewis's paper traced the historical development of the real number system, with special attention to the development of various notations.

(3) Professor Loewen discussed certain types of books which should be of especial interest to teachers of mathematics, mentioning a number of valuable

publications and noting that in certain allied fields no books of very great value to teachers of mathematics have yet appeared.

(4) Using as the basis of her paper an nth order determinant whose elements are $a_{ij}x_1 + b_{ij}x_2$ $(i, j = 1, 2, 3, \cdots, n)$, Miss Steininger developed theorems with regard to the invariants and covariants of such a determinantal form. The theorems are in every case analogous to the theorems for an ordinary binary form, but the proofs are considerably simpler.

(5) Professor Mitchell reviewed the history of Fermat and Mersenne numbers and presented tables of the known results as to their factorization. He spoke of the various methods of attack previously used and suggested that since if a Fermat number, $F_n = 2^{2^n} + 1$, is composite there must exist an integer k satisfying the identity $F_n = (2^{2^{n-1}} - k)(2^{2^{n-1}} + k) + k^2 + 1$, which, if determined, will yield a factor of F_n, it might be possible to secure further results by making use of this relation. Similarly, if a Mersenne number, $M_n = 2^p - 1$ (p an odd prime), is composite, there must exist an integer k satisfying the identity $F_p = (2)(2^n - k)(2^n + k) + 2k^2 - 1$, where $p = 2n + 1$. In working with this second identity he had discovered that $M_{571} = 2^{571} - 1$ has the factor 5711, a result which is believed to be new.

(6) By definition $\sigma = \sqrt{\frac{\Sigma fd^2}{N}}$. Let x_i be the measures and f_i the frequencies, $(i = 1, 2, 3, \cdots, n)$. Then $d_i = x_i - M_x$, $f_i d_i^2 = f_i x_i^2 - 2M_x f_i x_i + f_i M_x^2$ and $\Sigma fd^2 = \Sigma fx^2 - 2M_x \Sigma fx + M_x^2 \Sigma f$. But $M_x = \frac{\Sigma fx}{\Sigma f}$ or $\frac{\Sigma fx}{N}$ where $N = \Sigma f$. Therefore, $\sigma x = \frac{1}{N}\sqrt{N(\Sigma fx^2) - (\Sigma fx)^2}$, the value of σx in terms of the original measurements.

Again, by definition, $r = \frac{\Sigma fdxdy}{N\sigma x \sigma y}$. Let x_i and y_i $(i = 1, 2, 3, \cdots, n)$ be the original measurements with frequencies f_i. Then

$$f_i dx_i dy_i = f_i(x_i - M_x)(y_i - M_y) = f_i x_i y_i - f_i x_i M_y - f_i y_i M_x + f_i M_x M_y.$$
$$\Sigma fdxdy = \Sigma fxy - M_y \Sigma fx - M_x \Sigma fy + M_x M_y \Sigma f.$$

Eliminating M_x and M_y,

$$\Sigma fdxdy = \frac{N\Sigma fxy - (\Sigma fx)(\Sigma fy)}{N}.$$

Therefore,

$$r = \frac{N\Sigma fxy - (\Sigma fx)(\Sigma fy)}{\sqrt{N(\Sigma fx^2) - (\Sigma fx)^2}\sqrt{N(\Sigma fy^2) - (\Sigma fy)^2}},$$

the value of r in terms of the original measurements.

(7) The general discussion of future policies of the section was concerned chiefly with questions of time and place of meeting and suggestions for future

programs. As a result of the discussion it was voted that, beginning with the meeting of January, 1927, the section adopt the policy of pooling the railway fares of all members attending, and the determination of the time and place of the next meeting was delegated to the officers. It is expected that the January, 1926, meeting will be held at Kansas City in connection with the meeting of the American Association for the Advancement of Science.

U. G. MITCHELL, *Secretary-Treasurer.*

IS THE UNIVERSE FINITE?[1]

By ARCHIBALD HENDERSON, University of North Carolina.

1. On one occasion, after finishing a highly theoretical piece of research, the English mathematician H. J. S. Smith is reported to have made the delightful remark:[2] "Thank God, there is something which can never by any possibility have a practical application!" It was perhaps with some such feeling as this that Bolyai and Lobatchewsky developed the non-Euclidean geometry associated with their names, although the latter did dabble a little in parallaxes; and perhaps too, little thought of practical considerations animated the minds of Klein and Newcomb in the analysis of elliptical and spherical space. Little did Ricci and Levi-Civita, I daresay, as they were evolving it, dream of the extraordinary rôle in a new mechanics their theory of the absolute differential calculus was so soon destined to play.

Prophets and seers of the Einsteinian physics and cosmogony, to be sure, were Gauss, Riemann, Helmholtz and Clifford. Despite his dread of the "outcry of the Bœotians," Gauss triangulated in Hanover, using the peaks Inselberg, Brocken and Hoher Hagen, in the effort to discover a deviation of the sum of the angles of a large triangle from two right angles, and thus to decide between Euclidean and non-Euclidean geometry, in regard to validity in space. A herald of today was Clifford who clearly anticipated and foreshadowed the conceptions of Fitzgerald, Lorentz, and Einstein, notably in such a passage as this, from his *Common Sense of the Exact Sciences*:

> "Because a solid figure appears to us to retain the same shape when it is moved about in that portion of space with which we are acquainted, it does not follow that the figure really does retain its shape. The change of shape may be either imperceptible for those distances through which we are able to move the figure, or if they do take place we may attribute them to physical causes,—to heat, light, or magnetism (he should have added 'electricity')—which may possibly be mere names for variations in the curvature of our space."

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[1] This paper, on behalf of the Mathematical Association of America, was delivered on January 1, 1925, before the American Mathematical Society, the Mathematical Association of America, and Sections A, B, and D of the American Association for the Advancement of Science, at Washington, D. C.

[2] Smith's words, probably spoken in regard to some method in the theory of numbers, are thus quoted by D. E. Smith, *History of Mathematics*, I, 467: "It is the peculiar beauty of this method, gentlemen, and one which endears it to the really scientific mind, that under no circumstances can it be of the smallest possible utility."

In his *The Origin and Meaning of the Geometrical Axioms* and cognate papers, Helmholtz gives lucid exposition of fundamental aspects of geometry and space conceptions; and I cannot forbear quoting his words:

"We have no other mark of rigidity of bodies or figures but congruence, whenever they are applied to one another at any time or place, and after any revolution. We cannot however decide by pure geometry and without mechanical considerations whether the coinciding bodies may not both have varied in the same sense."

The foundation stone of relativity is, from the larger aspect, Riemann's study of the differential quadratic form with its creative notion of the "interval," and his extension, to space of any number of dimensions, of Gauss' idea of the curvature of a surface. In his famous memoir: *Concerning the hypotheses which lie at the base of geometry*, delivered at Göttingen in 1854, Riemann uses these memorable words:

"That space is an unbounded three-fold manifoldness is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought body are constructed, and which by these applications is forever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience; but its infinite extent by no means follows from this; on the other hand, if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite, provided this curvature has ever so small a positive value."

2. Certain physicists and astronomers during the past thirty years have raised the query as to whether the Newtonian mechanics, while holding with remarkable accuracy for our planetary system, also holds exactly for bodies at immeasurably great distances apart. In his paper, "Concerning Newton's law of gravitation" (*Astronomische Nachrichten*, 137, 1895), Seeliger affirms that we must make one or the other assumption: (1) If the common mass of the universe is immeasurably large, then the Newtonian law is not valid as a mathematically strict expression for the controlling powers; or (2) If the Newtonian law is absolutely exact, then the common mass of the universe must be finite. Moreover, either an absolutely empty space or one filled with infinitely tenuous matter would not be in conformity with Newtonian mechanics. For instance, if the universe were infinite and there were an attenuated swarm of fixed stars of approximately the same kind and density, however far we might penetrate the interstellar spaces, then matter would have a finite mean density; and in accordance with Newton's law of gravitation and a theorem due to Gauss, a body at the surface of a very large spherical portion of the universe would be attracted by a force proportional to the product of the radius of the sphere and the mean density of matter. As the radius of the sphere increases without limit, the intensity of the gravitational field at the boundary of the universe would be infinite. This is manifestly impossible, as it would give rise to velocities of a magnitude unobserved by astronomers.

On the other hand, if the mean density of matter were infinitesimally small, the cosmos must present the picture of an island of finite extent surrounded on all sides by infinite empty space. Such a view is repugnant to our minds, since the light of the stars and isolated stars themselves would drift away into the

infinite space devoid of matter, and this ephemeral cosmos would gradually melt away and disappear. Astronomical observation and physical research on the whole do not support the view that the energy of the cosmos is continually being dissipated.[1]

Seeliger advanced two hypotheses to meet the dilemma presented by Newtonian mechanics: either the possibility of matter of negative density in order to produce a null mean density; or to substitute in the numerator of Newton's law of attraction the quantity $e^{-\lambda r}$, where e is the base of natural logarithms, which would require λ to have the value 0.00000038, in order to account for the discrepancy in the advance per century in the perihelion of Mercury, taken as $40''$. Professor Hall suggested a modification of Newton's law by which, for great distances, the force of attraction between two masses diminishes more rapidly than would result from the inverse square law, the increment to the exponent 2 being chosen as 0.00000016, in order to explain Mercury's movement. Unfortunately none of these *ad hoc* hypotheses, which have neither empirical nor theoretical foundation, will serve; for while setting right the outstanding anomaly in the node of Venus, as de Sitter has pointed out,[2] they at the same time introduce greater discrepancies in other elements.

3. Is the universe infinite? If a voyager of the skies travel deep into the inter-stellar spaces, past the great blue helium stars of Orion, past Betelgeuse and Antares, beyond the white variable Cepheids, the gaseous red and yellow giant-stars, the faintest of the super-nebulæ, "lying like silver snails in the garden of the stars" but whirling in fiery spirals in the dim void of remoter space—will he ever reach any limit to the universe? Astronomers are not yet agreed that the amount of matter in the material universe is finite. It is significant that the density of matter falls off quite rapidly the deeper we penetrate into the stellar universe. For example, Hale says that there is probably an actual thinning out of the stars towards the boundary of the stellar universe. However, the problem with which we are here concerned is a very different one, *viz.*, has space a curvature? In advancing this question, we are flying in the face of poets, philosophers, scientists, theologians from time immemorial. In his "The dæmon of the world," Shelley says:

Below lay stretched the boundless universe!
  There, far as the remotest line
That limits swift imagination's flight,
Unending orbs mingled in mazy motion,
  Immutably fulfilling
  Eternal Nature's law.
  Above, below, around,
  The circling systems formed
  A wilderness of harmony—
  Each with undeviating aim
In eloquent silence through the depths of space
  Pursued its wondrous way.

[1] Professor F. R. Moulton regards Seeliger's reasoning as imperfect; and in his *Astronomy* long ago gave an elaborate argument to show that the notion of an infinitely extended universe is not incompatible with a finite mean density.

[2] "On Einstein's theory of gravitation and its astronomical consequences," *M. N. R. A. S.* vol. LXXVI, no. 9, supplementary number, 1916.

On metaphysical grounds alone Kant affirmed that space is infinite and sown with similar stars in all parts. Descartes, confronted with the question, "What lies beyond?" always maintained that a finite universe was impossible. In his *Our Place among Infinities*, the astronomer, Proctor, says: "The teachings of science bring us into the presence of the unquestionable infinities of time and of space, and the presumable infinities of matter and of operation—hence therefore into the presence of infinity of energy." An awful image of an infinite void is procured us by the Book of Job: "He stretcheth out the north over the empty space, and hangeth the earth upon nothing."

In a paper entitled "Cosmological observations concerning general relativity," published in the *Report of the Berlin Academy of Sciences*, February 8, 1917, Einstein advanced the view that the universe is finite, but unbounded. Various and cogent reasons led him to this conclusion. He was in search of a theory of the universe in conformity with the principle of general relativity, by which, in contradistinction to Newtonian mechanics, no preference is given to any reference system, and the laws of nature remain unchanged irrespective of the frame of reference to which they may be assigned. Ignoring the local concentrations of matter, represented by bodies and systems of bodies, Einstein assumed that the matter of the universe is distributed with uniform density—expecting thereby to arrive at some approximate conception of the metrical character of space as a whole. In order to differentiate between the coördinates of space and the coördinate of time, he made the reasonable assumption that the stellar system is approximately at rest, since the motion of matter is very small as compared with the velocity of light. Einstein was now confronted with two alternatives: either to assume that the universe is infinite and Euclidean at infinity; or else to adopt the view of Mach that inertia depends upon a mutual action of matter. Although the former is consonant with our conventional view that Galilean behavior tends to set in as we recede from a massive body, it was rejected by Einstein on the ground that it involves the far-reaching limitation, lacking in a physical basis, namely that B_{iklm} shall vanish at infinity, twenty independent conditions, while only ten curvature components G_{ik} enter into the laws of the gravitational field. It is not difficult, however, to show that, in the important case of the radially symmetrical field, these remaining ten conditions, on being complied with, lead to a solution indistinguishable from the familiar Schwarzschild form—which when r becomes infinite gives the Galilean values for the metrical tensor g_{ik}.

In his choice, Einstein was controlled by his desire to retain Mach's principle of the "relativity of inertia"—according to which the inertia of a body is entirely due to all the remaining matter in the universe. Brief consideration suffices to show that the only set of boundary values at infinity for the g_{ik} which would be invariant for all transformations is that all g_{ik} be zero. This postulate has been termed by de Sitter the "mathematical postulate of relativity of inertia." In Einstein's original theory of 1915, the g_{ik} are determined by the covariant field equations inside matter

$$G_{ik} = - \kappa T_{ik} + \tfrac{1}{2}\kappa g_{ik} T,$$

or

$$G_{ik} - \tfrac{1}{2}g_{ik}G = -\kappa T_{ik},$$
$$G = \kappa T,$$

where T_{ik} is the energy tensor of matter, T is the invariant of the energy tensor, and[1] $g^{ik}T_{ik} = T$.

In the modification of the Newtonian potential used by Neumann, which amounts to replacing the Laplace-Poisson equation by

$$\nabla^2\Omega - \lambda\Omega = -4\pi K\rho,$$

it had been shown by de Sitter that thereby was secured a distribution of matter of a non-vanishing though very small mean density, maintaining its equilibrium without an extra pressure. Guided by this alteration, Einstein modified his original field equations to the form

$$G_{ik} - \lambda g_{ik} = -\kappa(T_{ik} - \tfrac{1}{2}g_{ik}T),$$

which may be written

$$G_{ik} - \tfrac{1}{2}g_{ik}(G - 2\lambda) = -\kappa T_{ik},$$

since

$$G - 4\lambda = \kappa T.$$

On the original assumption that all matter is at rest, with an absence of all pressures, then the tensor T_{ik} becomes

$$T_{44} = g_{44}\rho_0 \text{ and all other } T_{ik} = 0,$$

where ρ_0 is the average density of world matter. Hence the final field equations are:

$$G_{ij} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{ij} = 0,$$
$$G_{44} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{44} = 0.$$

These can be satisfied by the following:

$$(A) \qquad ds^2 = -dr^2 - R^2\sin^2\frac{r}{R}(d\psi^2 + \sin^2\psi d\theta^2) + c^2dt^2,$$

if $\kappa\rho_0 = 2\lambda$, $\lambda = 1/R^2$. (Einstein.)

$$(B) \qquad ds^2 = -dr^2 - R^2\sin^2\frac{r}{R}(d\psi^2 + \sin^2\psi d\theta^2) + \cos^2\frac{r}{R}c^2dt^2,$$

if $\rho_0 = 0$, $\lambda = 3/R^2$. (de Sitter.)

$$(C) \qquad ds^2 = -dr^2 - r^2(d\psi^2 + \sin^2\psi d\theta^2) + c^2dt^2,$$

if $\rho_0 = 0$, $\lambda = 0$. (Newton.)

[1] Here $\kappa = 8\pi K/c^2$ where $K = (6\cdot6)10^{-8}$ and $c = (3)10^{10}$ in C.G.S. units.

The transformation $r = R\chi$ immediately gives for the *three-dimensional line-element* in both (A) and (B)

$$d\sigma^2 = R^2\{d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi d\theta^2)\}$$

—easily recognizable as representing *Riemann spherical space,* with constant curvature, $1/R^2$. For very great distances, it is easy to show that (B) but not (A) satisfies the "mathematical postulate of relativity." In this respect, de Sitter's space-time world is preferable to Einstein's. Einstein escaped the problem of boundary conditions at infinity by considering the world to be cylindrical—curved in the three space dimensions and straight in the time dimension. This required him to postulate the existence of vast quantities of matter, called by de Sitter "world-matter," far in excess of the amount now known to exist. It is easy to show that for Einstein's space-time world

$$2/R^2 = \kappa\rho_0 = 8\pi K\rho_0/c^2,$$

whence

$$R = \sqrt{2/\kappa\rho_0}, \qquad M = \rho_0\cdot 2\pi^2R^3 = 4\pi^2R/\kappa = \sqrt{32}\,\pi^2/\sqrt{\rho_0\kappa^3},$$

where

$$\kappa = (1\cdot 87)10^{-27}$$

and

$$2/\kappa = (1\cdot 08)10^{27}.$$

It is obvious that if $\rho_0 = 0$, $R = \infty$—*i.e.*, an infinite world.

4. I shall not regale you here with a description of the fantastic properties of the Einstein and de Sitter space-time universes, each finite but unbounded in their space extensions. Of the two types of elliptical space, Einstein deals with the antipodal, de Sitter with the polar. In the case of the former

$$M = 7\cdot 10^{41}/\sqrt{\rho_0}, \qquad V = 7\cdot 10^{41}/\sqrt[3]{\rho_0}$$

of the latter

$$M = \rho_0\cdot\pi^2R^3, \quad \text{etc.}$$

The formula for the Einstein world, $2/R^2 = \kappa\rho_0$, connecting R and ρ_0 may be described as "arresting" and "captivating," since, in the present state of astronomical knowledge, neither the density nor the world-radius can be arrived at with any degree of accuracy. It is self-protecting, since as we penetrate farther into space picking up very remote celestial objects, we reduce enormously the average density and thereby increase correspondingly the predicted radius. Were the spiral nebulæ, for example, found to be at distances greater than a computed world-radius, a new value of the world-radius greater than the former value would arise from the new value of the mean density. All that can be done, from this mode of approach, is to set up reasonable assumptions, and see if the conclusions arrived at are consistent. For example, the latest estimates of

the diameter of our galaxy range from 30,000 (Curtis) to 300,000 light-years (Shapley)—the yardstick here being the distance that light, with a velocity of 186,000 miles per second, travels in a year. Hence the radius of our spherical universe, if there are no extra-galactic objects, must be at least 150,000 light-years. The average mean density in the universe may be taken as equal to that of the Milky Way. Using as a yardstick the distance from the earth to the sun, namely 93 million miles, the radius of the Einstein universe is one million times ten millions times the distance from the earth to the sun. It would take a ray of light one billion years to go around the Einstein universe. The weight of this universe, in grams, would be 10^{54}—the weight of a trillion suns.[1]

In his paper, "On Einstein's theory of gravitation,"[2] de Sitter has made certain interesting computations for the radius of the finite universe of the polar type, so designed as not to contradict the known data of astronomical observations; and these are given in astronomical units. On the assumption that some of the spiral nebulæ or globular clusters are galactic systems comparable with our own in size, he finds from the formula for apparent angular diameter for elliptical space, at the greatest distance $\frac{1}{2}\pi R$, that

$$R \geqq 10^{12}.$$

If we choose for the mean density of the universe the star-density at the center of the galactic system, *viz.*, $\rho_0 = 10^{-17}$, we find

$$R = (9)10^{11}.$$

It is very probable that in the part of space which immediately surrounds our galactic system there are many similar systems whose mutual distances are large compared with their dimensions. If we assume as the average shortest distance between neighboring systems 10^{10}; and further suppose that the whole universe is thus filled with galactic systems, each of mass $\frac{1}{3}10^{10}$, then $\rho_0 = \frac{1}{3}10^{-20}$ and

$$R \leqq (5)10^{13}.$$

From the consideration of certain selective attenuation effects on starlight as due to molecular scattering, King finds that the density of interstellar residual gas amounts to 6,300 suns per cubic parsec. Using Shapley's determination, which is one fiftieth of this, we find

$$\rho_0 = (1\cdot 5)10^{-14},$$

giving

$$R = (2)10^{10}.$$

In de Sitter's universe, $g_{44} = \cos^2 \chi$; and hence the frequency of light-vibration, derived according to the familiar method, diminishes with increasing

[1] In this connection consult the writer's paper, "The size of the universe," *Science*, September 7, 1923.

[2] *M. N. R. A. S.*, vol. 78, 1917, 8 (no. 1, 1917).

distance from the origin of coördinates. Consequently on the assumption of the permanence of atoms, we may expect the spectrum lines of very distant stars or nebulæ to be displaced systematically toward the red. Assigning one third of this systematic displacement as an Einstein-effect due to the star's own gravitational field, the rest—corresponding to a receding radial velocity of 3 km./sec. for the helium or B stars—may be set down as an apparent displacement due to the diminution of g_{44}. Since the average distance of the helium or B stars has been computed to be about $(3)10^7$, we find

$$R = (0 \cdot 67)10^{10}.$$

Applying like reasoning to the Lesser Magellanic Cloud, which shows a spectrum shift corresponding to a radial velocity of $+ 150$ km./sec., with $r > (6)10^9$ (Hertzsprung), de Sitter finds

$$R > (2)10^{11}.$$

It is worthy of note that Weyl's new theory (1918), welding together electricity and gravitation, suggests a new mode of computing the curvature radius of space-time. The theory indicates that the ratio of the gravitational to the electrical radius of an electron should be of the same order as the ratio of the electrical radius of an electron to the radius of curvature of the world. Now the gravitational mass or "radius" of an electron is $(7)10^{-56}$ cms., the electrical radius $(2)10^{-13}$ cms. Hence it follows that the radius of space would have to be of the order $(6)10^{29}$ cms. or $(2)10^{11}$ parsecs. Remembering that 1 parsec equals $(2 \cdot 06)10^5$ astronomical units, this gives a value of R appreciably larger than those found by de Sitter.

5. In a series of letters and papers,[1] Silberstein has followed up the suggestion of de Sitter, and utilized results known with reasonable accuracy concerning certain remote celestial objects, in the effort to arrive at a closer value for R. For the elliptical space of de Sitter he has derived a formula for the Doppler shift, $d\lambda/\lambda$, of a star in inertial relative motion to the observer

$$d\lambda/\lambda = \left(1 - \frac{v_0^2}{c^2}\right)^{-1} \left\{ 1 \pm \left[1 - \cos^2 \frac{r}{R} \Big/ \left(1 - \frac{v_0^2}{c^2}\right)\right]^{1/2} \right\} - 1,$$

which for small or moderate values of r/R reduces to $d\lambda/\lambda = \pm\, r/R$, where

v_0 = individual velocity of star at some past or future epoch,
c = velocity of light,
r = distance of star from observer,
R = curvature radius of space-time,

since the expression $(1 - (v_0^2/c^2))^{-1/2}$ as suggested by astronomical observations, according to Lundmark, will be ≤ 1.0000444 and for most of the stars ≤ 1.000001. There is excellent reason for regarding the spectral displacements as of the

[1] *Nature*, January 1, March 8, April 26, June 7, September 6, 1924; *M. N. R. A. S.*, March 1924; *Philosophical Magazine*, May and October 1924.

Doppler type; but it is as yet an open question whether or not the spectral shifts are due to real motion and show no influence of the world-curvature. Contrary to the formulas used by Weyl and Eddington, Silberstein's formula has a $\pm$ sign, which accounts as well for receding as for approaching velocities. By using eight globular clusters and the Greater and Lesser Magellanic Clouds, Silberstein finds as a mean value:

$$R = (5 \cdot 9)10^{12}$$

or about *six million million* astronomical units.

Employing a statistical formula, an equation for R in terms of distances and Döppler effects alone,

$$\overline{D}_1^2 - \overline{D}_2^2 = \frac{1}{R^2}(\bar{r}_1^2 - \bar{r}_2^2),$$

when $D = d\lambda/\lambda$, obtained by choosing a mean distance for different groups, Silberstein finds for eleven globular clusters and two clouds:

$$R = (8 \cdot 8)10^{12}.$$

Again, giving up what he regards as the "artificial limitation" to radial motions, Silberstein has derived a spectrum-shift formula for any inertial motion, which when applied to the same thirteen celestial bodies, as before, gives:

$$R = (7 \cdot 2)10^{12}.$$

6. In the latest contribution to this subject,[1] the work of Silberstein is sharply criticized by Lundmark. Globular clusters are not regarded as nearly far enough away to be appreciably affected by the slowing down of atomic vibrations; and even if used, Silberstein is taken to task for not using all sixteen known velocities—instead of only those which give a rather constant value of the R. Silberstein rejected the globular clusters having a small radial velocity, *i.e.*, markedly under 100 km./sec. Using all the available data (eighteen in all, including two unpublished, supplied by Dr. Slipher), Lundmark found the values of R to range from $(2 \cdot 3)10^{12}$ to $(96 \cdot 5)10^{12}$ (Shapley's Parallaxes) and from $(7 \cdot 4)10^{12}$ to $(62 \cdot 5)10^{12}$ (Lundmark's Parallaxes)—a dispersion in R so large as to warrant the prediction that, for the present at least, the curvature of space-time cannot be determined with any accuracy by using the displacements in the spectra of globular clusters.

Lundmark goes on to examine other clusters of rather distant objects: Cepheids, Novæ, the O stars, the eclipsing variables, certain classes of red stars, and the spiral nebulæ. Nothing favorable to Silberstein's value of between 6 and 7 times 10^{12} astronomical units for R arises until we come to the O stars, which, on using the mean distance 1,560 parsecs, give the value

$$R = (4)10^{12},$$

[1] *M. N. R. A. S.*, October, 1924.

but unfortunately the values computed from individual velocities show a very large dispersion. The eclipsing variables (31 velocities) give the value (using mean distance):

$$R = (2\cdot 7)10^{12}.$$

The R stars (29) give values for R ranging from $(0.6)10^{12}$ to $(6\cdot 7)10^{12}$. The N stars (192) give, using mean radial velocity 18.8 km./sec.,

$$R = (2\cdot 3)10^{12}.$$

The real interest centers in the results given by the spiral nebulæ, which from the figures of Curtis (66 objects), Lundmark (82), Van Maanen (7 objects) and the same (Smart's values) give the following table:

R	*Material*
$(1\cdot 0)10^{12}$	Curtis
$(0\cdot 9)10^{12}$	Lundmark
$(6\cdot 6)10^{12}$	Van Maanen
$(5\cdot 2)10^{12}$	Van Maanen (Smart's values).

The last two results are very striking, as tending to confirm Silberstein. But Lundmark warns against their too ready acceptance, because the distances are probably too small; and further, because the parallaxes were derived by using the Döppler effects as indicating true motions, whereas the above procedure involved the supposition that the spectral shifts represent radial motions only.

The five values of R found by de Sitter ranging from

$$(0\cdot 67)10^{10} \text{ to } (5)10^{13},$$

which showed an agreement not to be expected *à priori;* the values of Silberstein ranging from

$$(5\cdot 9)10^{12} \text{ to } (8\cdot 8)10^{12}$$

and the cited values of Lundmark, *viz.*,

$$(2\cdot 7)10^{12},\ (4)10^{12}, \text{ and } (1\cdot 0)10^{12} \text{ to } (6\cdot 6)10^{12},$$

indicate a tendency to accumulate in the neighborhood of

$$(6)10^{12} \text{ to } (8)10^{12}.$$

Lundmark is in agreement with Silberstein that the latter's formula, in which the v_0^2 for one group of objects is assumed to be the same for another group placed at another distance, is practicable for application to moving clusters and some of the open clusters.

Recent investigations, reported by H. D. Curtis,[1] indicate with something like certainty that the spiral nebulæ are isolated stellar systems, at least a hundred

[1] Meeting of the A. A. A. S., in a paper read at the Bureau of Standards, December, 1924.

million light-years away.[1] Shapley was the first to point out and insist upon the significance of the systematic recessional motions of the spiral nebulæ. Whether above or below the plane, they recede from the Milky Way at high speed, all except the brighter and nearer ones. The displacement of the spectral lines to the red was thus interpreted as a Doppler effect. If, as now appears probable, the spirals are isolated stellar systems, this recession must be explained, it appears, either as a wholesale error or else as a relativistic effect. The former view is untenable; the latter lends strong support to the notion of a finite universe.[2] Much additional data will be required and many further researches made before it will be possible categorically to decide between the infinite, limitless, Euclidean universe of Newton and the finite, unbounded, non-Euclidean universe of Einstein or of de Sitter.

THE EULER DIFFERENTIAL EQUATION OF INFINITE ORDER.

By H. T. DAVIS, Indiana University.

1. The Problem. Originating in the important memoirs of C. Bourlet and S. Pincherle published in 1897,[3] the method of solving functional equations by first reducing them to differential equations of infinite order seems to promise a fruitful field for mathematical investigation. Although several articles on the subject have recently appeared in foreign journals, the theory does not seem to be generally known nor widely used. It is with the idea of indicating the application of the methods of this new calculus to the problems of analysis that the author in the present paper has developed the theory of an elementary type of differential equation of infinite order. In the last section of the paper an application, thought to be new, is made of the so-called "reduction principle" (principe des réduites) of the theory of linear equations in an infinite number of unknowns. The use of this principle in the present problem is of especial interest since it affords in many cases a method for computing the zeros of a function defined by a factorial series.

In this paper we propose to discuss the following Euler differential equation of infinite order:

$$F(x) = a_0\phi(x) + a_1 x\phi'(x) + \frac{a_2}{2!}x^2\phi''(x) + \frac{a_3}{3!}x^3\phi'''(x) + \cdots, \tag{1}$$

where the a_n are bounded as $n \to \infty$ and $F(x)$ is analytic about $x = 0$.

We may observe that we are led to an equation of this type in solving the

[1] This is one hundred times the figure suggested by Eddington in his *Space, Time and Gravitation* (1921).

[2] Letter of Professor Harlow Shapley to the writer, November 16, 1924.

[3] C. Bourlet: *Annales de l'École Normal Supérieure*, 3d series, vol. 14 (1897), pp. 133–190; S. Pincherle: *Mathematische Annalen*, vol. 49 (1897), pp. 325–381.

following rather general type of functional equation:

$$\int_0^x \frac{\phi(t)}{(x-t)^\nu}dt = x^{1-\nu}\sum_{i=0}^{m} c_i\phi(\mu_i x) + f(x), \qquad 0 \leqq \nu < 1, \tag{2}$$

where $f(x)$ is a known function of the form $x^{1-\nu}g(x)$, $g(x)$ being analytic about $x = 0$, and where c_i and μ_i are any set of constants.[1]

It will be seen that this equation generalizes the functional equation,

$$\int_0^x \phi(t)dt = \frac{x}{6}[\phi(0) + 4\phi(\tfrac{1}{2}x) + \phi(x)], \tag{3}$$

which has attracted considerable attention in the MONTHLY.[2] It also includes as special cases Abel's well-known integral equation,

$$\int_0^x \frac{\phi(t)}{(x-t)^\nu}dt = f(x),$$

and Weddle's formula for which a solution has recently been sought.[3]

We now assume that $\phi(t)$ and $\phi(\mu_i x)$ can be developed into the following Taylor's series:

$$\phi(t) = \phi(x) + (t-x)\phi'(x) + \frac{(t-x)^2}{2!}\phi''(x) + \cdots,$$

$$\phi(\mu_i x) = \phi(x) + (\mu_i - 1)x\phi'(x) + \frac{(\mu_i - 1)^2}{2!}x^2\phi''(x) + \cdots.$$

When these values have been substituted in equation (2), we obtain a differential equation of Euler type where $F(x) = x^{\nu-1}f(x)$ and

$$a_n = (-1)^n\left[\frac{1}{n-\nu+1} - \sum_{i=0}^{m} c_i(1-\mu_i)^n\right].$$

In equation (3) these values of a_n are easily seen to reduce to

$$(-1)^n\left[\frac{1}{n+1} - \frac{1}{6} - \frac{1}{3\cdot 2^{n-1}}\right]$$

and for Abel's equation we have

$$a_n = \frac{(-1)^n}{n+1-\nu}.$$

[1] This equation is related to a functional equation studied by P-J. Browne: *Annales de Toulouse*, vol. 4 (3), pp. 63–198. See also C. Popovici: *Comptes Rendus*, vol. 158, pp. 1866–1869.

[2] MONTHLY: Question **34** (*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19).

[3] MONTHLY: Problem **3076** (*1924*, 254).

2. The Non-Homogeneous Equation. Because of our assumption as to the analyticity of $F(x)$ we may differentiate[1] equation (1) n times and thus obtain the following system of linear equations in the infinitely many unknowns ϕ, ϕ', ϕ'', $\cdots$:

$$
\begin{aligned}
F(x) &= a_0\phi + a_1 x\phi' + \frac{a_2}{2!}x^2\phi'' + \cdots + \frac{a_n}{n!}x^n\phi^{(n)} + \cdots,\\
F'(x) &= (a_0+a_1)\phi' + (a_1+a_2)x\phi'' + \cdots + \frac{a_{n-1}+a_n}{(n-1)!}x^{n-1}\phi^{(n)} + \cdots,\\
F''(x) &= (a_0+2a_1+a_2)\phi'' + \cdots + \frac{a_{n-2}+2a_{n-1}+a_n}{(n-2)!}x^{n-2}\phi^{(n)} + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
F^{(n)}(x) &= (a_0+na_1+\cdots+a_n)\phi^{(n)} + \cdots.
\end{aligned}
\tag{4}
$$

Suppressing terms which contain derivatives of order higher than n, we are left with a system of $n+1$ linear equations in the $n+1$ unknowns ϕ, ϕ', $\cdots$, $\phi^{(n)}$, which we can solve formally for ϕ. If we designate by D_n the determinant of the system and by D_{0n}, D_{1n}, $\cdots$, D_{nn} the cofactors of the elements of the first column, then this formal solution, which we may denote by ϕ_n, will be

$$\phi_n = \frac{F(x)D_{0n} + F'(x)D_{1n} + \cdots + F^{(n)}(x)D_{nn}}{D_n}. \tag{5}$$

When the limit of the right-hand member exists for $n \to \infty$, we shall call $\lim_{n=\infty} \phi_n = \phi(x)$ the solution of the differential equation of infinite order if, for this function, the right-hand member of the equation converges uniformly to $F(x)$.

Denoting by $D(n)$ the product

$$D(n) = a_0(a_0+a_1)(a_0+2a_1+a_2)\cdots\left(a_0+na_1+\frac{n(n-1)}{2!}a_2+\cdots+a_n\right),$$

and assuming that $D(n) \neq 0$, the formal solution (5) is easily seen to reduce to

$$
\begin{aligned}
\phi_n(x) = \frac{1}{D(0)}F(x) - \frac{\Delta_1}{D(1)}xF'(x) + \frac{\Delta_2}{D(2)}x^2F''(x) - \cdots&\\
+ (-1)^n\frac{\Delta_n}{D(n)}x^nF^{(n)}(x),&
\end{aligned}
\tag{6}
$$

where Δ_1, Δ_2, $\cdots$, Δ_n are the leading principal minors of the first, second, $\cdots$, nth orders respectively of the determinant formed from the constant coefficients of the cofactor D_{nn}. It is obvious that this solution does not exist if any of the factors of $D(n)$ are equal to zero; the discussion of this case will be postponed to §4.

[1] For a treatment of the general problem by this method see E. Hilb: *Mathematische Annalen*, vol. 82 (1920–21), pp. 1–39.

Formula (6) will be recognized as the analogue of the operator

$$\left[\frac{1}{a_0 + a_1 D_x + a_2 D_x^2 + \cdots + a_n D_x^n + \cdots}\right] F(x),$$

which has long been known to give the formal solution of the non-homogeneous differential equation with constant coefficients.[1]

It is clear that series (6) will converge provided

$$\lim_{n=\infty} \sqrt[n]{\frac{\Delta_n}{D(n)} F^{(n)}(x)} = 0,$$

but it is desirable to find a criterion which applies separately to $F^{(n)}(x)$ and $\Delta_n/D(n)$.

If we make use of Hadamard's well-known theorem on the maximum value of a determinant, which states that the absolute value of a determinant does not exceed the square root of the product of the sums of the squares of the absolute values of the elements of each row,[2] we arrive at the inequality,

$$\left|\frac{\Delta_n}{D(n)}\right| \leqq \frac{\sqrt{\sigma_0 \sigma_1 \cdots \sigma_{n-1}}}{|a_0 + na_1 + \cdots + a_n|},$$

where for brevity we have set

$$\sigma_r = 1 + \sum_{m=1}^{n-r} \left(\frac{a_m + ra_{m+1} + \frac{r(r-1)}{2!} a_{m+2} + \cdots}{a_0 + ra_1 + \frac{r(r-1)}{2!} a_2 + \cdots} \right)^2 \left(\frac{1}{m!}\right)^2.$$

Then if constants A and K can be found such that we have

$$\left(\frac{a_n + ra_{n+1} + \cdots}{a_0 + ra_1 + \cdots}\right)^2 \leqq n!A \qquad \text{and} \qquad |a_0 + na_1 + \cdots + a_n| > K \geqq 1, \quad (7)$$

for all values of r and n, we shall have

$$\lim_{n=\infty} \sqrt[n]{\left|\frac{\Delta_n}{D(n)}\right|} < (1 + Ae)^{1/2} = C.$$

We thus arrive at the result that, *if the coefficients of the differential equation satisfy conditions* (7), *and* $D(n) \neq 0$ *for all values of* n, *series* (6) *will converge provided* $\lim_{n=\infty} \sqrt[n]{F^{(n)}(x)} = 0$.

[1] For a complete discussion of this operator see F. Schürer: *Leipziger Berichte*, vol. 70 (1918), pp. 185–246.

[2] J. Hadamard: Résolution d'une question relatif aux determinants. *Bull. des. Sc. Math.*, vol. 17 (1893), pp. 240–246.

We must next show that when series (6) is substituted in the differential equation, the right-hand member converges uniformly to $F(x)$. In order to do this, let us denote by S_n the series

$$S_n = a_0\phi(x) + a_1 x\phi'(x) + \frac{a_2}{2!}x^2\phi''(x) + \cdots + \frac{a_n}{n!}x^n\phi^{(n)}(x),$$

and for simplicity write

$$\phi(x) = A_0 F(x) + A_1 x F'(x) + \frac{A_2}{2!}x^2 F''(x) + \cdots + \frac{A_n}{n!}x^n F^{(n)}(x) + \cdots,$$

where $A_n = (-1)^n n!(\Delta_n/D(n))$.

We wish to show that the difference $A(x) = F(x) - S_n(x)$ approaches zero uniformly as $n \to \infty$.

Since we have

$$a_0\frac{A_r}{r!} + a_1\frac{A_{r-1}+A_r}{(r-1)!} + \frac{a_2}{2!}\frac{(A_{r-2}+2A_{r-1}+A_r)}{(r-2)!} + \cdots + \frac{a_r}{r!}(A_0 + rA_1 + \cdots + A_r) \equiv 0$$

for all values of r, it follows that

$$A(x) = \left\{a_0 A_{n+1} + a_1(n+1)(A_n + A_{n+1}) + \frac{a_2}{2!}(n+1)n(A_{n-1} + 2A_n + A_{n+1}) + \cdots + \frac{a_n}{n!}(n+1)!\,(A_1 + nA_2 + \cdots + A_{n+1})\right\}\frac{x^{n+1}}{(n+1)!}F^{(n+1)}(x) + \cdots.$$

Then since the a_n are bounded, $|a_n| < a$, and since

$$|A_{n+k}| = (n+k)!\left|\frac{\Delta_{n+k}}{D(n+k)}\right| < (n+k)!\,C^{n+k}, \quad n > n',$$

it will be seen that

$$|A(x)| < aC^{n+1}\left[1 + (n+1)2 + \frac{(n+1)n}{2!}2^2 + \cdots + \frac{(n+1)!}{n!}2^n\right] |x|^{n+1}|F^{(n+1)}(x)| + \cdots < aC^{n+1}3^{n+1}|x|^{n+1}|F^{(n+1)}| + aC^{n+2}3^{n+2}|x|^{n+2}|F^{(n+2)}| + \cdots.$$

But by hyopthesis on $F(x)$ we can always find an n' to match an arbitrarily small ϵ, so that $|F^{(n)}(x)| < \epsilon^n$, for $n > n'$ and $|x| < k$.

Hence it follows that

$$|A(x)| < a(C3k\epsilon)^{n+1}[1 + 3k\epsilon + (3k\epsilon)^2 + \cdots] = a\frac{(C3k\epsilon)^{n+1}}{1 - 3k\epsilon}.$$

Consequently if ϵ is chosen so that $\epsilon < 1/3Ck$, it is clear that $A(x)$ will approach zero uniformly as $n \to \infty$.

Another form may be given to series (6) by replacing $F^{(m)}(x)$ by the equivalent Cauchy integral

$$F^{(m)}(x) = \frac{m!}{2\pi i}\int_C \frac{F(t)}{(t-x)^{m+1}}\,dt,$$

so that we have

$$\lim_{n=\infty} \phi_n(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t-x}\left[\frac{1}{D(0)} - \frac{\Delta_1}{D(1)}\frac{x}{t-x} + \frac{2!\,\Delta_2}{D(2)}\frac{x^2}{(t-x)^2} - \cdots\right]dt.$$

It has been proved by Bourlet that the right-hand member of this equation will exist for all functions $F(t)$, analytic in a region of radius ρ about x, provided the expression in brackets converges for all values of t satisfying the condition

$$|t - x| = \rho.$$

3. The Method of Operators. We shall next apply to the problem the method of operators as developed in Bourlet's memoir already referred to. We can write equation (3) in the symbolic form

$$F(x) = \left(a_0 + a_1 x\frac{d}{dx} + \frac{a_2}{2!}x^2\frac{d^2}{dx^2} + \cdots\right)\phi(x). \tag{8}$$

The function

$$\Theta(xz) = a_0 + a_1 xz + \frac{a_2}{2!}x^2z^2 + \cdots \tag{9}$$

has been called the *generatrix function* and is of fundamental importance in the calculus of operators.[1]

It is easily established by a theorem due to Hadamard [2] that (9) is an entire function in xz of genus 1 or 0.

Therefore $\Theta(xz)$ may be written

$$\Theta(xz) = e^{cxz}\prod_{i=0}^{\infty}\left(1 - \frac{xz}{\beta_i}\right), \tag{10}$$

where the β_i are the zeros of the entire function $\Theta(y)$.

Product operators of this kind have been little studied. In a paper published in 1917 [3] J. F. Ritt has developed the properties of an operator of genus zero of the form

$$A = \prod_{i=0}^{\infty}\left(1 - \frac{D}{\alpha_i}\right)$$

where it is assumed that $\sum_{i=1}^{\infty}\frac{1}{|\alpha_i|}$ converges.

---

[1] We employ here the name used by T. Lalesco: *Journal de Mathématique,* ser. 6, vol. 4 (1908), p. 193. Bourlet calls this function the "fonction opérative."

[2] J. Hadamard: *Journal de Mathématique,* vol. 9 (4) (1893), p. 172. See also Bourlet: *loc. cit.*, p. 162.

[3] J. F. Ritt: *Trans. of the Amer. Math. Soc.*, vol. 18 (1917), pp. 27–49.

In case the number of zeros is finite, however, the analysis becomes much easier, and we can then write (10) in the form

$$\Theta(xz) = e^{cxz}P(xz),$$

where $P(xz)$ is a polynomial.

Bourlet has proved that the generatrix of the operator $A\left(x, \frac{d}{dx}\right)\left[B\left(x, \frac{d}{dx}\right)\right]$, where A and B are linear operators and the symbol denotes the operation of A on B, can be obtained from the following formula in which $A(x, z)$ and $B(x, z)$ are the generatrix functions of the two operators. Thus we have

$$\text{Generatrix}\left\{A\left(x, \frac{d}{dx}\right)\left[B\left(x, \frac{d}{dx}\right)\right]\right\} = B(x, z)\cdot A(x, z)$$
$$+ \frac{\partial B}{\partial x}\frac{\partial A}{\partial z} + \frac{1}{2!}\frac{\partial^2 B}{\partial x^2}\frac{\partial^2 A}{\partial z^2} + \cdots + \frac{1}{n!}\frac{\partial^n B}{\partial x^n}\frac{\partial^n A}{\partial z^n} + \cdots.$$

Applying this formula to the functions $A(x, z) = e^{cxz}$ and $B(x, z) = B(xz)$, we obtain

$$\text{Generatrix}\left\{A\left(x, \frac{d}{dx}\right)\left[B\left(x, \frac{d}{dx}\right)\right]\right\} = e^{cxz}B[(1 + c)xz].$$

Since this is the generatrix $\Theta(xz)$, we must have

$$P(xz) = B[(1 + c)xz]$$

and we see that the original operator can be resolved into the product of two operators. Hence we may write equation (8) in the form

$$e^{cx(d/dx)}\left\{P\left(\frac{x\dfrac{d}{dx}}{1 + c}\right)\phi(x)\right\} = F(x).$$

If we operate on both sides of this equation by the inverse of $e^{cx(d/dx)}$, it follows from a simple application of formula (6) that we shall obtain

$$P\left(\frac{x\dfrac{d}{dx}}{1 + c}\right)\phi(x) = F(x) - \frac{c}{1 + c}xF'(x) + \frac{c^2}{(1 + c)^2}\frac{x^2}{2!}F''(x) - \cdots,$$
$$= F\left(\frac{x}{1 + c}\right).$$

The problem of solving equation (8) is thus reduced to that of solving an Euler equation of finite order.

Let us consider the following example:

$$\tfrac{1}{4}x^2 = \phi(x) + \tfrac{1}{2}x\phi'(x) + \left(\frac{1}{2^2 2!} - 1\right)x^2\phi'' + \left(\frac{1}{2^3 3!} - \tfrac{1}{2}\right)x^3\phi''' + \cdots. \quad (11)$$

The generatrix function is easily seen to be

$$\Theta(xz) = e^{(1/2)xz}(1 - x^2z^2).$$

The equation is then equivalent to

$$e^{(1/2)x(d/dx)}\left(\phi - \frac{4}{9}x^2\frac{d^2\phi}{dx^2}\right) = \tfrac{1}{4}x^2,$$

which reduces to

$$\tfrac{4}{9}x^2\phi'' - \phi = -\tfrac{1}{9}x^2.$$

The solution of this equation is $\phi = c_1x^{\alpha_1} + c_2x^{\alpha_2} + x^2$, where $\alpha_1 = \frac{1}{2}(1 + \sqrt{10})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{10})$. By direct substitution, this solution will be found to satisfy the original equation as well.

4. The Homogeneous Equation. In the last section we saw how the method of operators could be used to reduce the problem to an equation of finite order, provided the generatrix had a finite number of zeros. In general this will not be the case so we must approach the homogeneous problem in another way.

By analogy with the method used in solving the Euler equation of finite order, we seek solutions of the form $\phi = x^r$ and are thus led to the auxiliary function

$$f(r) = a_0 + a_1r + \frac{a_2}{2!}r(r-1) + \frac{a_3}{3!}r(r-1)(r-2) + \cdots, \tag{12}$$

which we observe is a factorial series.[1] The region of convergence of this series is, in general, a half plane, limited on the left by a straight line perpendicular to the axis of reals and cutting it in a point λ, called the abscissa of convergence. In particular, the value of λ may be $-\infty$. Within the domain of convergence, then, for every simple zero r_i of this function there will exist one solution of the differential equation of the form

$$\phi(x) = x^{r_i},$$

and for every multiple root r_i of multiplicity μ_i there will exist μ_i solutions. These solutions will be

$$x^{r_i},\ x^{r_i}\log x,\quad \cdots,\quad x^{r_i}(\log x)^{\mu_i-1}.$$

It is thus seen that the solution of the homogeneous equation depends upon the convergence of series (12) and upon the determination of the zeros of the function defined by this series. A simple sufficiency condition for the convergence of (12) is given by the following lemma:

LEMMA. *The abscissa of convergence of* (12) *is never greater than the smallest*

[1] For a study of the properties of this factorial series, see S. Pincherle: Sur les fonctions déterminantes, *Annales de l'École Normale Supérieure*, vol. 22 (3), (1905), pp. 57–68.

value of p for which the series

$$\sum_{n=1}^{\infty} \frac{|a_{n+1}|}{n^{p+1}}, \tag{13}$$

converges.

To prove this let

$$P_n(r) = \left| \frac{a_{n+1}}{(n+1)!} r(r-1) \cdots (r-n) \right| = \left| \frac{a_{n+1} \prod_{k=1}^{n} r \left(1 - \frac{r}{k}\right) e^{r/k}}{(n+1) \prod_{k=1}^{n} e^{r/k}} \right|.$$

Since $\prod_{k=1}^{n} r\left(1 - \frac{r}{k}\right) e^{r/k}$ is a convergent product, there will exist a number N such that for all values of $n > N$ the absolute value of the product will never exceed a value K.

Now replace r by $p + qi$. Then we shall have

$$P_n(r) \leqq \left| \frac{a_{n+1}K}{(n+1) \prod_{k=1}^{n} e^{p/k} e^{(q/k)i}} \right| \leqq \left| \frac{a_{n+1}K}{(n+1) \prod_{k=1}^{n} e^{p/k}} \right|, \quad n > N.$$

But since $\lim (1 + \frac{1}{2} + \cdots + (1/n) - \log n) = C$, where C is Euler's constant, there will exist a number N' such that for $n > N'$ we shall have

$$1 + \tfrac{1}{2} + \cdots + \frac{1}{n} > C + \log n - \epsilon,$$

where ϵ is an arbitrarily chosen positive number independent of n.

Then, for n greater than N and N', we have

$$P_n(r) \leqq \frac{K}{e^{p(C-\epsilon)}} \frac{|a_{n+1}|}{(n+1)n^p} < \frac{K}{e^{p(C-\epsilon)}} \frac{|a_{n+1}|}{n^{p+1}}.$$

If $\bar{p}$ is the smallest value of p for which the series (13) converges, then it will clearly converge for all greater values of p. Hence the abscissa of convergence is equal to or less than $\bar{p}$ and series (12) is uniformly convergent in the plane to the right of $\bar{p}$.

Convergence theorems for series (13) are well known since it is a Dirichlet's series which has been extensively studied.[1]

We now consider an interesting special case, already referred to in section 2, where one or more of the factors of $D(n)$ are zero. It is easily seen that if the $(m-1)$st factor of this product is zero, then the factorial series (12) will vanish for $r = m$.

[1] G. H. Hardy and M. Riesz: *The General Theory of Dirichlet's Series.* Cambridge, 1915.

Thus in (12) let $r = m$ and we shall have

$$f(m) = \left[a_0 + ma_1 + \frac{m(m-1)}{2!} a_2 + \cdots + a_m \right] + \frac{a_{m+1}}{(m+1)!} m(m-1) \cdots (m-m) + \cdots,$$

which clearly reduces to zero.

In order to obtain the solution of the non-homogeneous differential equation corresponding to this case, we may let $\phi = x^m u(x)$ and thus obtain a new equation of the form

$$x^{-m-1} F(x) = A_1 u'(x) + A_2 x u''(x) + \cdots. \tag{14}$$

The left-hand member of this equation will be analytic about $x = 0$ provided $F(x)$ has a zero of order $m + 1$ at the origin and we can then solve for $u'(x)$ by the method given in section 2. By repetition of this process all of the zero factors of $D(n)$ can be removed.

An interesting example is furnished by the equation

$$c = u(x) - xu'(x) + \frac{x^2}{2!} u''(x) - \frac{x^3}{3!} u'''(x) + \cdots,$$

where c is a constant, which has for its solution every analytic function satisfying the condition $u(0) = c$.

5. Solution by the "Reduction Principle." In general, however, it will be difficult to compute directly the zeros of the auxiliary function. If they are finite in number, they must necessarily correspond in number to the zeros of the generatrix function. In many cases where these roots are real, and even in some cases where they are complex, the so called "reduction principle" (principe de réduites) of the theory of linear equations in an infinite number of variables can be used to make this computation. A discussion of the theoretical basis of this principle is beyond the scope of this paper, but results derived by its use can always be tested in the original equation. For further information the reader is referred to F. Riesz: *Les Systèmes d'équations linéaires a une infinité d'inconnues*, Paris (1913), Chapter I.

This method consists in solving the system, composed of the homogeneous equation and its derived equations, for $\phi'(x)$ in terms of $\phi(x)$ by a succession of approximations, using first two equations in the unknowns $\phi'(x)$ and $\phi''(x)$, then three equations in the unknowns $\phi'(x)$, $\phi''(x)$, and $\phi'''(x)$, etc. If these approximations approach a limit as n is indefinitely increased, then the resulting equation of first order may be solved for ϕ.

Thus referring to system (4), in which $F(x)$ is to be replaced by zero, we have

for these successive approximations the equations:

$$x\phi_0' = -\frac{a_0}{\Delta_1}\phi_0, \qquad \phi_0 = cx^{-a_0/\Delta_1},$$

$$x\phi_1' = -\frac{a_0\delta_1}{\Delta_2}\phi_1, \qquad \phi_1 = cx^{-a_0(\delta_1/\Delta_2)},$$

$$\cdots\cdots\cdots\cdots$$

$$x\phi_n' = -\frac{a_0\delta_n}{\Delta_{n+1}}\phi_n, \qquad \phi_n = cx^{-a_0(\delta_n/\Delta_{n+1})},$$

where the δ_i are the leading principal minors of the negative co-factor of a_1 in the determinant of the constant coefficients of (4) and Δ_i has the same meaning as in section 2.

In case $\lim_{n=\infty}\dfrac{\delta_n}{\Delta_{n+1}}$ does not exist, a derived equation of second order can perhaps be found by removing both ϕ and ϕ' to the right-hand side of the equations in system (4) and solving for $\phi''(x)$. It is obvious that this reduction can not, in general, be carried on indefinitely since the order of an equation equivalent to system (4) can not exceed the number of zeros of the generatrix function.

As an example consider the homogeneous equation obtained by replacing the left-hand member of equation (11) by zero. By the reduction principle we obtain successively

$$\phi_0 = cx^{-2}, \qquad \phi_1 = cx^{1.6}, \qquad \phi_2 = cx^{2.23};$$

the last function is seen to be an approximation to the correct solution $\phi = x^{2.081}$.

In order to find the second solution we may reduce equation (11) to one of second order by solving system (4) for $\phi''(x)$. The first three approximations are easily found to be the following:

$$x^2\phi_0''(x) - \tfrac{2}{3}x\phi_0'(x) - \tfrac{4}{3}\phi_0 = 0,$$

$$x^2\phi_1''(x) - .323x\phi_1' - 1.883\phi_1 = 0,$$

$$x^2\phi_2''(x) - .112x\phi_2' - 2.312\phi_2 = 0.$$

The solutions of the last equation are found to be $\phi_2 = x^{2.175}$ and $\phi_2 = x^{-1.063}$, which are seen to approximate the correct solutions $\phi = x^{2.081}$ and $\phi = x^{-1.081}$.

THE GENERALIZED KRONECKER SYMBOL AND ITS APPLICATION TO THE THEORY OF DETERMINANTS.[1]

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1. Definition of the Generalized Kronecker Symbol. There is a symbol of frequent occurrence in algebra, and its applications to geometry, known as Kronecker's symbol. It is defined by the following properties. Let r and s be

[1] Read before the Md.-Va.-D. C. Section of the Association, May 10, 1924.

two labels to which we may assign independently any one of the n integral values $1, 2, 3 \cdots n$; then Kronecker's symbol[1] $\begin{bmatrix} r \\ s \end{bmatrix}$ has n^2 values as the n numerical values are assigned independently to the labels r and s. (The label r is here used as a superscript, and s as a subscript, but this not essential; it is, however, convenient in certain applications.) These n values are, by definition of the symbol, as follows:

$$\begin{bmatrix} r \\ s \end{bmatrix} = 0; \text{ if } r \neq s, \qquad \begin{bmatrix} r \\ s \end{bmatrix} = 1; \text{ if } r = s.$$

We shall now introduce a generalization of this symbol which we shall call the generalized Kronecker symbol and shall denote by $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}$; the integer m being any one of the n integers $1, 2, \cdots n$ and the labels $(r_1, r_2, \cdots r_m)$, $(s_1, s_2, \cdots s_m)$ being capable of taking, independently, any one of the n integral values. The symbol has then n^{2m} values and these are by definition as follows:

(*a*) $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix} = 0$ if the set $(r_1, r_2, \cdots r_m)$ of m out of the n numbers $(1, 2, 3, \cdots n)$ is not the same as the set $(s_1, s_2, \cdots s_m)$ of m out of the same n numbers. Thus if $m = 2$, $\begin{bmatrix} 1 \; 2 \\ 2 \; 3 \end{bmatrix} = 0$.

(*b*) $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}$ is alternating in both the superscripts and the subscripts. In other words a change in position of any two of the m superscripts (or subscripts) changes the sign but not the numerical value taken by the symbol. Thus $\begin{bmatrix} 1 \; 2 \\ s_1 \, s_2 \end{bmatrix} = -\begin{bmatrix} 2 \; 1 \\ s_1 \, s_2 \end{bmatrix}$ and $\begin{bmatrix} r_1 \, r_2 \\ 1 \; 2 \end{bmatrix} = -\begin{bmatrix} r_1 \, r_2 \\ 2 \; 1 \end{bmatrix}$. As an immediate consequence we have the result that $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix} = 0$ if the numerical values taken by any two of the superscripts (or subscripts) are the same. *E.g.*, $\begin{bmatrix} 1 \; 1 \\ s_1 \, s_2 \end{bmatrix} = 0$ and $\begin{bmatrix} r_1 \, r_2 \\ 1 \; 1 \end{bmatrix} = 0$.

(*c*) When $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$ are merely arrangements of the same set of m distinct numbers out of the set $(1, 2, \cdots n)$, the symbol $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}$ is to be assigned the value 1 if it requires an even number, and the value -1 if it requires an odd number, of inversions to get from the arrangement $(r_1, r_2, \cdots r_m)$ to the arrangement $(s_1, s_2, \cdots s_m)$. The three defining properties (*a*), (*b*), (*c*) of the symbol $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}$ may be resumed briefly as

[1] The usual notation is δ_s^r but the notation adopted here has, for the generalized symbol at any rate, distinct advantages from the typographical point of view.

follows: $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}$ is to be assigned the value zero unless $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$ are arrangements of the same set of m distinct numbers out of the set $(1, 2, \cdots n)$. In this event it is to be assigned the value ± 1 according as these arrangements are of the same class or not.

2. Properties of the Generalized Kronecker Symbol. It will be convenient in stating these properties to adopt the convention that a *Greek* letter in an expression plays the rôle of a dummy or umbral symbol, *i.e.*, it is to be assigned in turn the n numerical values $(1, 2, \cdots n)$ and the n resulting expressions are to be added up. It will appear that, in the notation adopted, a Greek or summation letter will always appear twice in an expression, once as a subscript and once as a superscript.

It is an immediate consequence of the definition that

$$\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix} = \begin{bmatrix} s_1 \, s_2 \cdots s_m \\ r_1 \, r_2 \cdots r_m \end{bmatrix} \tag{2.1}$$

and also that

$$\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix} = \begin{bmatrix} r_1 \, r_2 \cdots r_m \\ \alpha_1 \, \alpha_2 \cdots \alpha_m \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ s_1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_m \\ s_m \end{bmatrix}. \tag{2.2}$$

The expression on the right indicates, according to our convention, m distinct summations. But $\begin{bmatrix} \alpha_1 \\ s_1 \end{bmatrix} = 0$ unless the summation symbol α_1 is assigned the particular value s_1 so that the summation with respect to α_1 yields

$$\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, \alpha_2 \cdots \alpha_m \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_m \\ s_m \end{bmatrix},$$

there being now $(m - 1)$ summations remaining. Continuing this argument we arrive at the result (2.2).

A similar argument gives the result

$$\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix} = \frac{1}{m!} \begin{bmatrix} r_1 \, r_2 \cdots r_m \\ \alpha_1 \, \alpha_2 \cdots \alpha_m \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \, \alpha_2 \cdots \alpha_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}. \tag{2.3}$$

For in the m summations on the right all the terms, for which $(\alpha_1, \alpha_2, \cdots \alpha_m)$ is not merely a rearrangement of *both* the groups of m distinct numbers $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$, vanish. All the terms for which $(\alpha_1, \alpha_2, \cdots \alpha_m)$ is such an arrangement have the same value ± 1; for an inversion in any particular arrangement $(a_1, a_2, \cdots a_m)$ of the summation symbols $(\alpha_1, \alpha_2, \cdots \alpha_m)$ changes the sign of each of the factors $\begin{bmatrix} r_1 \, r_2 \cdots r_m \\ a_1 \, a_2 \cdots a_m \end{bmatrix}$ and $\begin{bmatrix} a_1 \, a_2 \cdots a_m \\ s_1 \, s_2 \cdots s_m \end{bmatrix}$.

Since there are precisely $m!$ such arrangements $(a_1, a_2, \cdots a_m)$ when the groups $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$ of m distinct numbers are the same, the truth of (2.3) immediately follows.

We may show in a similar way that

$$\begin{bmatrix} r_1 r_2 \cdots r_n \\ s_1 s_2 \cdots s_n \end{bmatrix} = \frac{1}{m!}\frac{1}{(n-m)!}\begin{bmatrix} \alpha_1\, \alpha_2 \cdots \alpha_m \\ s_1\; s_2 \cdots s_m \end{bmatrix}\begin{bmatrix} \alpha_{m+1} \cdots \alpha_n \\ s_{m+1} \cdots s_n \end{bmatrix}\begin{bmatrix} r_1\; r_2 \cdots r_n \\ \alpha_1\, \alpha_2 \cdots \alpha_n \end{bmatrix} \tag{2.4}$$

or

$$\begin{bmatrix} r_1 r_2 \cdots r_n \\ s_1 s_2 \cdots s_n \end{bmatrix} = \frac{1}{m!}\frac{1}{(n-m)!}\begin{bmatrix} r_1\; r_2 \cdots r_m \\ \alpha_1\, \alpha_2 \cdots \alpha_m \end{bmatrix}\begin{bmatrix} r_{m+1} \cdots r_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix}\begin{bmatrix} \alpha_1\, \alpha_2 \cdots \alpha_n \\ s_1\; s_2 \cdots s_n \end{bmatrix}. \tag{2.4bis}$$

First of all it is evident that in the summations on the right-hand side of (2.4) $(\alpha_1, \alpha_2, \cdots \alpha_m)$, $(\alpha_{m+1}, \cdots \alpha_n)$ must be arrangements of $(s_1, s_2, \cdots s_m)$ and $(s_{m+1}, \cdots s_n)$ respectively and $(\alpha_1, \alpha_2, \cdots \alpha_n)$ must be an arrangement of $(1, 2, \cdots n)$ for the corresponding term to have a value different from zero. Hence $(s_1, s_2, \cdots s_n)$ must be an arrangement of the n numbers $(1, 2, \cdots n)$ for the expression on the right of (2.4) to have a non-zero value. An inversion in any arrangement $(a_1, a_2, \cdots a_m)$ of $(\alpha_1, \alpha_2, \cdots \alpha_m)$ changes the sign of both the factors $\begin{bmatrix} a_1\, a_2 \cdots a_m \\ s_1\; s_2 \cdots s_m \end{bmatrix}$ and $\begin{bmatrix} r_1\; r_2 \cdots r_n \\ a_1\, a_2 \cdots a_n \end{bmatrix}$ of the corresponding term in the summation and so the value of the term itself is unaltered. This explains the presence of the numerical factor $1/m!\,(n-m)!$.

3. The Definition of a Determinant. Let us consider the n quantities a_s^r where r and s run independently over the n numbers $(1, 2, \cdots n)$. We shall define the determinant of these n quantities as

$$D = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_1^{\alpha_1}\, a_2^{\alpha_2} \cdots a_n^{\alpha_n} \tag{3.1}$$

and this determinant will be said to be of the nth order. It is immediately apparent that if $(r_1, r_2, \cdots r_n)$ is any arrangement of the n numbers $(1, 2, \cdots n)$, we may write D in the equivalent form

$$D = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_{r_1}^{\alpha_1}\, a_{r_2}^{\alpha_2} \cdots a_{r_n}^{\alpha_n}. \tag{3.2}$$

For an inversion in the arrangement $(r_1, r_2, \cdots r_n)$ carries with it an inversion in the arrangement $(\alpha_1, \alpha_2, \cdots \alpha_n)$. We may, for example, in the case $n = 3$, write D or $\begin{bmatrix} 1 & 2 & 3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} a_1^{\alpha_1}\, a_2^{\alpha_2}\, a_3^{\alpha_3}$ in the equivalent form $\begin{bmatrix} 2 & 1 & 3 \\ \alpha_2 & \alpha_1 & \alpha_3 \end{bmatrix} a_2^{\alpha_2}\, a_1^{\alpha_1}\, a_3^{\alpha_3}$ and this, by a mere interchange of the summation symbols α_1 and α_2, is $\begin{bmatrix} 2 & 1 & 3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} a_2^{\alpha_1}\, a_1^{\alpha_2}\, a_3^{\alpha_3}$. We derive from (3.2) upon giving to the n labels $(r_1, r_2, \cdots r_n)$, each independently, the n values $(1, 2, \cdots n)$ and then summing the

resulting expressions the result

$$D = \frac{1}{n!}\begin{bmatrix}\rho_1 \, \rho_2 \cdots \rho_n \\ \alpha_1 \, \alpha_2 \cdots \alpha_n\end{bmatrix} a_{\rho_1}^{\alpha_1} \, a_{\rho_2}^{\alpha_2} \cdots a_{\rho_n}^{\alpha_n}. \tag{3.3}$$

This result is important as it shows that the superscripts and subscripts of the n^2 elements $a_s{}^r$ of the determinant play exactly the same rôles. In fact the determinant D might just as well have been defined by the equation

$$D' = \begin{bmatrix}\alpha_1 \, \alpha_2 \cdots \alpha_n \\ 1 \quad 2 \, \cdots \, n\end{bmatrix} a_{\alpha_1}^{1} \, a_{\alpha_2}^{2} \cdots a_{\alpha_n}^{n} \tag{3.1$^{\text{bis}}$}$$

as by the equation (3.1). For we derive as before

$$D' = \begin{bmatrix}\alpha_1 \, \alpha_2 \cdots \alpha_n \\ r_1 \, r_2 \cdots r_n\end{bmatrix} a_{\alpha_1}^{r_1} \, a_{\alpha_2}^{r_2} \cdots a_{\alpha_n}^{r_n} \tag{3.2$^{\text{bis}}$}$$

$$= \frac{1}{n!}\begin{bmatrix}\alpha_1 \, \alpha_2 \cdots \alpha_n \\ \rho_1 \, \rho_2 \cdots \rho_n\end{bmatrix} a_{\alpha_1}^{\rho_1} \, a_{\alpha_2}^{\rho_2} \cdots a_{\alpha_n}^{\rho_n}. \tag{3.3$^{\text{bis}}$}$$

Whence $D' = D$, *i.e.*, the value of the determinant of n^2 ordered elements $a_s{}^r$ is unaltered by the change of order corresponding to the interchange of subscript and superscript. Arranging the n^2 elements into n rows of n elements each in the usual way, this is the result that an interchange of rows and columns does not affect the value of a determinant.

It is somewhat more symmetrical to define the quantity $D_{s_1 s_2 \cdots s_n}^{r_1 r_2 \cdots r_n}$ by the equation

$$D_{s_1 s_2 \cdots s_n}^{r_1 r_2 \cdots r_n} = \begin{bmatrix}r_1 \, r_2 \cdots r_n \\ \alpha_1 \, \alpha_2 \cdots \alpha_n\end{bmatrix} a_{s_1}^{\alpha_1} \, a_{s_2}^{\alpha_2} \cdots a_{s_n}^{\alpha_n}. \tag{3.4}$$

Then $D_{s_1 s_2 \cdots s_n}^{r_1 r_2 \cdots r_n}$ is alternating in both the superscripts $(r_1, r_2, \cdots r_n)$ and the subscripts $(s_1, s_2, \cdots s_n)$. It has, accordingly, the value zero unless $(r_1, r_2, \cdots r_n)$ and $(s_1, s_2, \cdots s_n)$ are merely arrangements of the n distinct numbers $(1, 2, \cdots n)$. In this event $D_{s_1 s_2 \cdots s_n}^{r_1 r_2 \cdots r_n} = \pm D$ according as these arrangements are of the same class or not. Just as before we have a second expression for $D_{s_1 s_2 \cdots s_n}^{r_1 r_2 \cdots r_n}$, namely

$$D_{s_1 s_2 \cdots s_n}^{r_1 r_2 \cdots r_n} = \begin{bmatrix}\alpha_1 \, \alpha_2 \cdots \alpha_n \\ s_1 \, s_2 \cdots s_n\end{bmatrix} a_{\alpha_1}^{r_1} \, a_{\alpha_2}^{r_2} \cdots a_{\alpha_n}^{r_n}. \tag{3.4$^{\text{bis}}$}$$

It follows from (3.4) and (2.2) that the generalized Kronecker symbol $\begin{bmatrix}r_1 \, r_2 \cdots r_m \\ s_1 \, s_2 \cdots s_m\end{bmatrix}$ is a determinant of order m of which the elements are values of the ordinary Kronecker symbol $\begin{bmatrix}r \\ s\end{bmatrix}$. Thus

$$\begin{bmatrix} r_1 \, r_2 \\ s_1 \, s_2 \end{bmatrix} = \begin{vmatrix} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_1 \\ s_2 \end{bmatrix} \\ \begin{bmatrix} r_2 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_2 \\ s_2 \end{bmatrix} \end{vmatrix}, \qquad \begin{bmatrix} r_1 \, r_2 \, r_3 \\ s_1 \, s_2 \, s_3 \end{bmatrix} = \begin{vmatrix} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_1 \\ s_2 \end{bmatrix} & \begin{bmatrix} r_1 \\ s_3 \end{bmatrix} \\ \begin{bmatrix} r_2 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_2 \\ s_2 \end{bmatrix} & \begin{bmatrix} r_2 \\ s_3 \end{bmatrix} \\ \begin{bmatrix} r_3 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_3 \\ s_2 \end{bmatrix} & \begin{bmatrix} r_3 \\ s_3 \end{bmatrix} \end{vmatrix}$$

and so on.

4. Laplace's Expansion of a Determinant. If we use the result (2.4) in (3.4), we obtain

$$\begin{aligned} D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} &= \frac{1}{m!(n-m)!} \begin{bmatrix} \beta_1 \, \beta_2 \cdots \beta_m \\ \alpha_1 \, \alpha_2 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} \beta_{m+1} \cdots \beta_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix} \begin{bmatrix} r_1 \cdots r_n \\ \beta_1 \cdots \beta_n \end{bmatrix} a^{\alpha_1}_{s_1} a^{\alpha_2}_{s_2} \cdots a^{\alpha_n}_{s_n} \\ &= \frac{1}{m!(n-m)!} \begin{bmatrix} r_1 \, r_2 \cdots r_n \\ \beta_1 \, \beta_2 \cdots \beta_n \end{bmatrix} \left\{ \begin{bmatrix} \beta_1 \cdots \beta_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} a^{\alpha_1}_{s_1} \cdots a^{\alpha_m}_{s_m} \right\} \\ &\qquad \times \left\{ \begin{bmatrix} \beta_{m+1} \cdots \beta_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix} a^{\alpha_{m+1}}_{s_{m+1}} \cdots a^{\alpha_n}_{s_n} \right\}, \end{aligned} \tag{4.1}$$

where the face brackets indicate that the summations are performed first with respect to $(\alpha_{m+1}, \cdots \alpha_n)$, then with respect to $(\alpha_1, \cdots \alpha_m)$, and finally with respect to $(\beta_1, \cdots \beta_n)$. Although $(\alpha_{m+1}, \cdots \alpha_n)$ are to run over the n values $(1, 2, \cdots n)$, they may be regarded as running merely over the $(n - m)$ distinct values $(b_{m+1}, \cdots b_n)$ in the summation

$$\begin{bmatrix} b_{m+1} \cdots b_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix} a^{\alpha_{m+1}}_{s_{m+1}} \cdots a^{\alpha_n}_{s_n} \tag{4.2}$$

as the terms of this summation in which any one of the umbral symbols is assigned a value not in the set $(b_{m+1}, \cdots b_n)$ will all be zero from the definition of the symbol $\begin{bmatrix} b_{m+1}, \cdots b_n \\ \alpha_{m+1}, \cdots \alpha_n \end{bmatrix}$. The summation (4.2) is, therefore, a determinant of order $(n - m)$ which may conveniently be denoted by $D^{b_{m+1}, \cdots b_n}_{s_{m+1}, \cdots s_n}$. Dealing similarly with the second set of summations in (4.1), we obtain

$$D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \frac{1}{m!(n-m)!} \begin{bmatrix} r_1 \, r_2 \cdots r_n \\ \beta_1 \, \beta_2 \cdots \beta_n \end{bmatrix} D^{\beta_1 \cdots \beta_m}_{s_1 \cdots s_m} D^{\beta_{m+1} \cdots \beta_n}_{s_{m+1} \cdots s_n} \tag{4.3}$$

which is Laplace's expansion. We may avoid the numerical factor $1/m!(n-m)!$ by agreeing that instead of assigning to the summation symbols β the values 1 to n independently we understand that $(\beta_1, \cdots \beta_m)$ is a group of m out of the n symbols $(r_1, r_2, \cdots r_n)$, $(\beta_{m+1}, \cdots \beta_n)$ being the group of $(n - m)$ that is left, and that the same group is not repeated. We have then, for example,

$$D = D^{r_1 r_2 \cdots r_n}_{r_1 r_2 \cdots r_n} = \sum \begin{bmatrix} r_1 \cdots r_n \\ \beta_1 \cdots \beta_n \end{bmatrix} D^{\beta_1 \cdots \beta_m}_{r_1 \cdots r_m} D^{\beta_{m+1} \cdots \beta_n}_{r_{m+1} \cdots r_n} \tag{4.4}$$

there being $\binom{n}{m}$ terms in the summation on the right. In the special case when $m = 1$ we obtain the usual expansion in terms of any one row. It may be remarked that, owing to the symmetrical way in which the subscripts and superscripts enter the definition of a determinant, we have the alternative expression for $D^{r_1\cdots r_n}_{s_1\cdots s_n}$

$$D^{r_1 r_2\cdots r_n}_{s_1 s_2\cdots s_n} = \frac{1}{m!\,(n-m)!}\begin{bmatrix}\beta_1 & \cdots & \beta_n \\ s_1 & \cdots & s_n\end{bmatrix} D^{r_1\cdots r_m}_{\beta_1\cdots\beta_m} D^{r_{m+1}\cdots r_n}_{\beta_{m+1}\cdots\beta_n}. \tag{4.3$^{\text{bis}}$}$$

5. The Multiplication of Determinants. Let us consider two determinants D and $\overline{D}$ of order n whose elements may be denoted by $a_s{}^r$ and $b_s{}^r$ respectively. We have on multiplication of the two expressions

$$D^{r_1\cdots r_n}_{s_1\cdots s_n} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix} a^{\alpha_1}_{s_1}\cdots a^{\alpha_n}_{s_n} \qquad (Cf.\ 3.4)$$

$$\overline{D}^{s_1\cdots s_n}_{r_1\cdots r_n} = \begin{bmatrix}\beta_1 & \cdots & \beta_n \\ r_1 & \cdots & r_n\end{bmatrix} b^{s_1}_{\beta_1}\cdots b^{s_n}_{\beta_n} \qquad (Cf.\ 3.4^{\text{bis}})$$

by one another the result

$$D^{r_1\cdots r_n}_{s_1\cdots s_n}\cdot \overline{D}^{s_1\cdots s_n}_{r_1\cdots r_n} = \begin{bmatrix}\beta_1 & \cdots & \beta_n \\ r_1 & \cdots & r_n\end{bmatrix}\begin{bmatrix} r_1 & \cdots & r_n \\ \alpha_1 & \cdots & \alpha_n\end{bmatrix} a^{\alpha_1}_{s_1} b^{s_1}_{\beta_1}\cdots a^{\alpha_n}_{s_n} b^{s_n}_{\beta_n}. \tag{5.1}$$

This product is zero unless $(s_1, s_2, \cdots s_n)$ is an arrangement of the n numbers $(1, 2, \cdots n)$ and it is unaltered by any change in this arrangement for an inversion of any two of the labels $(s_1, s_2, \cdots s_n)$ merely changes the sign of both factors $D^{r_1\cdots r_n}_{s_1\cdots s_n}$ and $\overline{D}^{s_1\cdots s_n}_{r_1\cdots r_n}$. We may therefore give $(s_1, s_2, \cdots s_n)$, independently, the n values $(1, 2, \cdots n)$ and sum up the resulting expressions when we find

$$D^{r_1\cdots r_n}_{\sigma_1\cdots\sigma_n}\cdot \overline{D}^{\sigma_1\cdots\sigma_n}_{r_1\cdots r_n} = \begin{bmatrix} r_1 & \cdots & r_n \\ \alpha_1 & \cdots & \alpha_n\end{bmatrix}\begin{bmatrix}\beta_1 & \cdots & \beta_n \\ r_1 & \cdots & r_n\end{bmatrix} c^{\alpha_1}_{\beta_1}\cdots c^{\alpha_n}_{\beta_n}, \tag{5.2}$$

where

$$c_s{}^r = a_\sigma{}^r b_s{}^\sigma. \tag{5.3}$$

Just as the expression (5.1) was unaltered by a rearrangement of the labels $(s_1, s_2, \cdots s_n)$, it is unaltered by a rearrangement of the labels $(r_1, r_2, \cdots r_n)$. We have, then, on denoting $D^{1\,2\cdots n}_{1\,2\cdots n}$ and $\overline{D}^{1\,2\cdots n}_{1\,2\cdots n}$ simply by D and $\overline{D}$, respectively, as before

$$D\overline{D} = D^{r_1\cdots r_n}_{s_1\cdots s_n}\overline{D}^{s_1\cdots s_n}_{r_1\cdots r_n} \qquad \{(r_1, r_2, \cdots r_n) \text{ and } (s_1, s_2, \cdots s_n) \text{ being arrangements of the } n \text{ numbers } (1, 2, \cdots n)\}$$

$$= \frac{1}{n!} D^{r_1 r_2\cdots r_n}_{\sigma_1\sigma_2\cdots\sigma_n}\overline{D}^{\sigma_1\cdots\sigma_n}_{r_1\cdots r_n}$$

$$= \left(\frac{1}{n!}\right)^2\cdot D^{\rho_1\cdots\rho_n}_{\sigma_1\cdots\sigma_n}\overline{D}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n},$$

where the Greek letters are as always summation symbols. Hence the product $D\overline{D}$ of the two determinants of order n is given by

$$\begin{aligned} D\overline{D} &= \left(\frac{1}{n!}\right)^2 \left\{\begin{bmatrix} \rho_1 & \cdots & \rho_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix}\begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \rho_1 & \cdots & \rho_n \end{bmatrix}\right\} c_{\beta_1}^{\alpha_1} \cdots c_{\beta_n}^{\alpha_n} \\ &= \frac{1}{n!}\begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} c_{\beta_1}^{\alpha_1} \cdots c_{\beta_n}^{\alpha_n}, \qquad \text{by (2.3).} \end{aligned}$$

It is accordingly, by (3.3), the determinant of the elements c_s^r defined by (5.3). This is the usual law of the multiplication of determinants where we multiply "rows into columns."

If we glance again at the equation (5.1), we may derive at once a theorem of frequent application in algebra. Thus we have

$$\begin{aligned} D_{s_1 s_2 \cdots s_n}^{1\,2 \cdots n} \cdot \overline{D}_{1\,2 \cdots n}^{s_1 s_2 \cdots s_n} &= \frac{1}{n!} D_{s_1 \cdots s_n}^{\rho_1 \cdots \rho_n} \overline{D}_{\rho_1 \cdots \rho_n}^{s_1 \cdots s_n} \\ &= \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} (a_{s_1}^{\alpha_1} b_{\beta_1}^{s_1}) \cdots (a_{s_n}^{\alpha_n} b_{\beta_n}^{s_n}) \end{aligned}$$

as before. We may now let the labels $(s_1, s_2, \cdots s_n)$ run over a set of integers $(1, 2, \cdots p)$, where $p > n$, and we obtain on the left a summation of products of determinants of the nth order and on the right a determinant of the nth order whose elements are $c_s^r = a_\sigma^r b_s^\sigma$ (the summation symbol σ running over the numbers $1, 2, \cdots p$). Thus

$$D_{\sigma_1 \sigma_2 \cdots \sigma_n}^{1\,2 \cdots n} \cdot \overline{D}_{1 \cdots n}^{\sigma_1 \cdots \sigma_n} = \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} c_{\beta_1}^{\alpha_1} \cdots c_{\beta_n}^{\alpha_n}.$$

In the summation on the left a term $D_{s_1 s_2 \cdots s_n}^{1\,2 \cdots n}\ \overline{D}_{1\,2 \cdots n}^{s_1 s_2 \cdots s_n}$ is repeated $n!$ times and we may state our result as follows:

Given two sets of np ordered elements each a_t^r, and b_s^t, respectively (r, $s = 1, 2, \cdots n$; $t = 1, 2, \cdots p$), $p > n$, the determinant of order n whose elements c_s^r are given by $c_s^r = a_\tau^r\, b_s^\tau$ is equal to $\sum D_{\tau_1 \tau_2 \cdots \tau_n}^{1\,2 \cdots n}\ \overline{D}_{1\,2 \cdots n}^{\tau_1 \cdots \tau_n}$ where $(\tau_1, \cdots \tau_n)$ is any set of n distinct numbers out of the p numbers $(1, 2, \cdots p)$ and no set is repeated in the summation.

6. Conclusion. In the foregoing paragraphs we have endeavored to give a presentation of determinants in which the fundamental theorems would follow naturally from the definition. In the usual presentation the emphasis is laid, not on the explicit expression of a determinant as a function of its elements, but on the expansion theorem. This has obvious advantages in practical work with determinants but it seems to lack advantages for the theoretical discussion. Whilst the proof of such theorems as Laplace's expansion and the product theorem is not difficult, it has more the appearance of a verification of a known result than of the derivation of a new result. We hope to have shown above

that the consistent use of the explicit expression of a determinant as a function of its elements, given in (3.1), leads naturally to the theorems mentioned.

It may be pointed out that Kronecker [1] defined a determinant by means of the following characteristic properties:

(*a*) It is a function of the n^2 elements a_s^r (which we may suppose arranged in n rows of n elements each), which is linear and homogeneous in the elements of each row.

(*b*) It changes its sign when any two rows are interchanged.

(*c*) It has the value unity when the elements a_s^r take the values of Kronecker's symbol $\begin{bmatrix} r \\ s \end{bmatrix}$.

This postulational definition leads to the explicit expression (3.1). Again, however, the proof of the law of multiplication by means of these postulates has more the aspect of a verification than a derivation. The novelty in the present presentation lies essentially in the symmetrical use of subscripts and superscripts. Usually when an explicit formula for a determinant has been given in the past, a symbol $\epsilon_{r_1 \cdots r_n}$, which is essentially equivalent to $\begin{bmatrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix}$, has been used. As we hope to show elsewhere, the symbol $\begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ s_1 & s_2 & \cdots & s_n \end{bmatrix}$ seems to be really more fundamental than the symbol $\epsilon_{r_1 \cdots r_n}$.[2]

THE CIRCULAR CUBIC ON TWENTY-ONE POINTS OF A TRIANGLE.

By T. W. MOORE and J. H. NEELLEY, Graduate Students, Yale University.

1. **Introduction.** This article is concerned with the circular cubic on twenty-one notable points of a general triangle, first mentioned by Neuberg and referred to by Professor B. H. Brown in his article entitled "A Theorem on Isogonal Tetrahedra." [3]

While the theory of the circular cubic is well known and much has been written on it,[4] the equation of this particular curve as derived is of considerable interest and the results appear in particularly neat form. To serve our purpose, we shall take the given triangle as a triangle of reference and use a system of notation practically the same as that used by Clebsch in his article on the geometry of the triangle.[5]

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[1] Leopold Kronecker, *Vorlesungen über aie Theorie der Determinanten;* bearbeitet von K. Hensel; Erster Band, p. 291 et seq.

[2] Since writing the above paper we have read a paper by E. H. Moore, "A fundamental remark concerning Determinantal Notations," *Annals of Mathematics*, ser. 2, vol. 1, p. 177, which should be consulted by those interested in the subject of this paper.

[3] This MONTHLY, *1924*, 371–375. See also the note by Professor Brown on page 247 of this issue. EDITOR.

[4] Casey, *Transactions of the Royal Irish Academy*, volume 24, 1871. Loria, *Spezielle Algebraische und Transcendente Ebene Kurven*, pp. 31–35. Basset, *Elementary Treatise on Cubic and Quartic Curves*, chapters VI and IX. Hilton, *Plane Algebraic Curves*, chapter XIV. *Encyklopädie der Mathematischen Wissenschaften*, Band 3, Teil 2, 1ste Hälfte, p. 510.

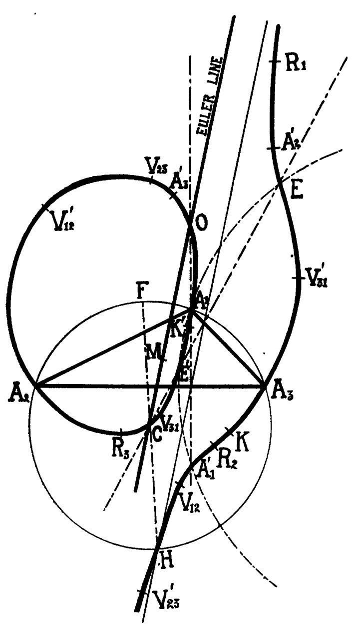
[5] Clebsch-Lindemann, *Vorlesungen über Geometrie*, pp. 312 ff.

In this notation the coördinates of points and lines and the coefficients in the equations involved are expressed as trigonometric functions of three angles ϕ_i, $i = 1, 2, 3$. Two of these angles are interior angles of the triangle and the remaining one is the opposite exterior angle given a negative sign, so that $\phi_1 + \phi_2 + \phi_3 = 0$. With Clebsch we shall place $s_i = \sin \phi_i$, $c_i = \cos \phi_i$ and use the symbols $\sigma_1, \sigma_2, \sigma_3$ for the sines of $\phi_2 - \phi_3$, $\phi_3 - \phi_1$, and $\phi_1 - \phi_2$ respectively.

2. The Equation of the Neuberg Cubic. In order to get the equation of the Neuberg cubic, we recall that the equation of any circular cubic may be written in the form

$$U \equiv QL_\infty + \lambda C_1 L_1 = 0, \tag{1}$$

where Q is a conic, L_∞ the line at infinity, C_1 a circle, and L_1 a line. Of the twenty-one points mentioned by Neuberg let us make use of the group given below:



(*a*) the vertices of the reference triangle, $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$,

(*b*) the reflections of the vertices in the opposite sides, A_1', A_2', A_3' respectively,

(*c*) the circumcenter C of the base triangle,

(*d*) the orthocenter O of the base triangle,

(*e*) the isodynamic centers, or equilateral poles, E and E', the two points of intersection of the three circles of Apollonius.

Together with these points which can be readily constructed, we shall use the circular points I and J.

All these points are mentioned by Clebsch except A_1', A_2', A_3'. The coordinates of C and O are (c_1, c_2, c_3) and $(1/c_1, 1/c_2, 1/c_3)$ respectively. By use of harmonic properties we obtain the coördinates of the points A_1', A_2', A_3' as $A_1'(\frac{1}{2}, c_3, c_2)$, $A_2'(c_3, \frac{1}{2}, c_1)$, $A_3'(c_2, c_1, \frac{1}{2})$. The circular points I and J are found by solving the equations of the circumcircle and the ideal line and are $(c_2 \pm is_2, c_1 \mp is_1, 1)$.

The equations of the three circles of Apollonius[1] may be written in the form

$$\begin{aligned} x_2^2 - x_3^2 + 2c_2x_3x_1 - 2c_3x_1x_2 &= 0, \\ x_3^2 - x_1^2 + 2c_3x_1x_2 - 2c_1x_2x_3 &= 0, \\ x_1^2 - x_2^2 + 2c_1x_2x_3 - 2c_2x_3x_1 &= 0, \end{aligned}$$

and, from inspection, the equation of the common chord of these three circles is

$$(\sigma x) \equiv \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3 = 0.$$

[1] The equations given in Clebsch-Lindemann, p. 319, are readily reduced to the forms given.

Now, of the eight points A_1, A_1', E, E', I, J, C, O, we see, either by definition or inspection, that the first six are on the Apollonius circle through the vertex A_1, and the remaining two are on the Euler line of the triangle, the line [1]

$$(sc\sigma x) \equiv s_1c_1\sigma_1x_1 + s_2c_2\sigma_2x_2 + s_3c_3\sigma_3x_3 = 0.$$

Again, three of these points, A_1, A_1' and O, are on the altitude through the vertex A_1, while the points E, E' and C are on the common chord of the three circles of Apollonius.

Hence the pencil of circular cubics on these eight points may be written in the form

$$L_2L_3L_\infty + \lambda C_1L_1 = 0, \tag{2}$$

where C_1 is the Apollonius circle through A_1, L_1 the Euler line, L_2 the altitude of the triangle on the vertex A_1, and L_3 the common chord of the circles of Apollonius. From the form of equation (2) it is obvious that the remaining point common to every member of the pencil is the intersection of the Euler line with the line at infinity. Therefore, we have the following

THEOREM: *Every member of the pencil of circular cubics on the eight points given above has its real asymptote parallel to the Euler line of the base triangle.*

To determine the Neuberg cubic itself from this pencil (2), the parameter λ must be such that the curve will pass through another vertex, say A_2. This means that the coefficient of x_2^3 in the expanded form of (2) must vanish. The equations of the various parts of (2) are:

$$\begin{aligned}
C_1 &\equiv x_2^2 - x_3^2 + 2c_2x_3x_1 - 2c_3x_1x_2 = 0,\\
L_1 &\equiv (sc\sigma x) \equiv s_1c_1\sigma_1x_1 + s_2c_2\sigma_2x_2 + s_3c_3\sigma_3x_3 = 0,\\
L_2 &\equiv c_2x_2 - c_3x_3 = 0,\\
L_3 &\equiv (\sigma x) \equiv \sigma_1x_1 + \sigma_2x_2 + \sigma_3x_3 = 0,\\
L_\infty &\equiv (sx) \equiv s_1x_1 + s_2x_2 + s_3x_3 = 0.
\end{aligned}$$

Now if the curve

$$(c_2x_2 - c_3x_3)(\sigma x)(sx) + \lambda(x_2^2 - x_3^2 + 2c_2x_3x_1 - 2c_3x_1x_2)(sc\sigma x) = 0 \tag{3}$$

is required to pass through $A_2(0, 1, 0)$, the result is $\lambda = -1$.

If equation (3) is expanded for this value of λ and simplified by trigonometric reductions, we find that the coefficient of the product term $x_1x_2x_3$ vanishes identically and that $s_1\sigma_1$ is a factor of the remaining terms. So the equation of the circular cubic of Neuberg reduces to the symmetrical form

$$(c_1+2c_2c_3)(x_2^2-x_3^2)x_1+(c_2+2c_3c_1)(x_3^2-x_1^2)x_2+(c_3+2c_1c_2)(x_1^2-x_2^2)x_3=0. \tag{4}$$

3. Points Associated with the Curve. Of the twenty-one points on the curve mentioned by Neuberg, or rather twenty-three, if we include the circular points,

[1] Clebsch-Lindemann, p. 334.

we have already the coördinates of ten. The coördinates of the isodynamic points E and E' are $(c_1\sqrt{3} \pm s_1,\ c_2\sqrt{3} \pm s_2,\ c_3\sqrt{3} \pm s_3)$. The other points mentioned by Neuberg are:

(*a*) the isogonal points, K and K', $[1/(c_1\sqrt{3}\pm s_1),\ 1/(c_2\sqrt{3}\pm s_2),\ 1/(c_3\sqrt{3}\pm s_3)]$, whose coördinates are the reciprocals of those of the isodynamic points,

(*b*) the three points R_i, symmetric to the vertices A_i in the lines $q_{jk}q_{kj}$, where q_{jk} is the point in which the side a_j of the reference triangle meets the perpendicular bisector of the side a_k. Their coördinates are $(2,\ 1/c_3,\ 1/c_2)$, $(1/c_3,\ 2,\ 1/c_1)$, $(1/c_2,\ 1/c_1,\ 2)$, or the reciprocal points to the reflections of the vertices in the opposite sides,

(*c*) the six vertices V_{ij} and V_{ij}' of the equilateral triangles formed on the sides of the base triangle are $(\sqrt{3},\ c_3\sqrt{3} \pm s_3,\ c_2\sqrt{3} \pm s_2)$, $(c_3\sqrt{3} \pm s_3,\ \sqrt{3},\ c_1\sqrt{3} \pm s_1)$, and $(c_2\sqrt{3} \pm s_2,\ c_1\sqrt{3} \pm s_1,\ \sqrt{3})$.

In addition to these, we have found a number of other points on the curve, which can be easily constructed, among which may be mentioned the remaining intersection of each side of the base triangle with the curve. These points are $(0,\ c_2 + 2c_3c_1,\ c_3 + 2c_1c_2)$, etc., and are the points where each side of the triangle meets the line through the opposite vertex parallel to the Euler line. Also the points $(1, \pm 1, \pm 1)$ are of particular interest as they are the centers of the inscribed and three escribed circles of the base triangle. Furthermore, the four points $(1, \pm 1, \pm 1)$ are the vertices of a complete quadrangle inscribed in the curve of which the vertices of the reference triangle are the diagonal points. (*Cf.* Clebsch-Lindemann, p. 327.)

If we write the equations of the tangents to the cubic at the points $(1, \pm 1, \pm 1)$, we find that they all meet in the point $(c_1 + 2c_2c_3,\ c_2 + 2c_3c_1,\ c_3 + 2c_1c_2)$, which is the point at infinity on the Euler line. Furthermore, the tangents to the cubic at the vertices of the base triangle, or the diagonal points of the quadrangle, meet on the cubic at the point $[1/(c_1 + 2c_2c_3),\ 1/(c_2 + 2c_3c_1),\ 1/(c_3 + 2c_1c_2)]$, the remaining point of intersection H of the cubic and the circumcircle.[1] Therefore the real asymptote of the curve is the line through this last point parallel to the Euler line. In addition the four points $(1, \pm 1, \pm 1)$ appear as the centers of inversion of this circular cubic.[2]

The double, or singular focus F of the curve is the opposite extremity of the diameter of the circumcircle $(s/x) = 0$ on the point where the asymptote cuts the circle and cubic,[3] and has coördinates $(1/\sigma_1,\ 1/\sigma_2,\ 1/\sigma_3)$. The circumcircle of the base triangle is the common Feuerbach circle of the four triangles formed by omitting in turn one of the points $(1, \pm 1, \pm 1)$.[4]

4. Special Cases of the Neuberg Cubic. If the base triangle becomes isosceles, with the vertex opposite the odd side at $(0, 0, 1)$, then $s_2 = s_1$, $s_3 = -2s_1c_1$, $c_2 = c_1$, and $c_3 = 2c_1^2 - 1$. The Apollonius circle through this vertex $A_3(0, 0, 1)$

[1] Salmon-Fiedler, *Höhere Ebene Kurven*, p. 168.
[2] Basset, *l.c.*, p. 157.
[3] Casey, *Transactions of the Royal Irish Academy*, vol. 24, p. 496.
[4] Basset, *Ibid.*, p. 158.

breaks up into $(x_1 - x_2)(x_1 + x_2 - 2c_1x_3) = 0$, representing the Euler line and the line at infinity. The cubic (4) reduces to the form

$$(x_1 - x_2)[c_1(x_3^2 + x_1x_2) - (x_1 + x_2)x_3] = 0,$$

which is the Euler line and a circle having its center at A_3, and passing through the remaining vertices of the reference triangle.

If the base triangle becomes equilateral, each coefficient of equation (4) vanishes identically, and as Professor Brown has pointed out, the cubic is not unique. Of the twenty-one points given by Neuberg, those which have determinate positions are located at one of the following seven points $(1, \pm 1, \pm 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and the others have arbitrary positions. These seven points, together with the two circular points, are the base points of a pencil of cubics which we shall consider.

5. Net of Cubics on Seven Points. While any four points in the plane, no three of which are collinear, may be given the projective coördinates $(1, \pm 1, \pm 1)$, it must be remembered that we are preserving metric properties and that the four centers of inversion of a circular cubic form an orthic set, that is, each point is the orthocenter of the triangle formed by the other three. In the notation used these four centers of inversion have coördinates $(1, \pm 1, \pm 1)$, the diagonal points being the reference points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, so any cubic on the seven points may be written in the form [1]

$$k_1(x_2^2 - x_3^2)x_1 + k_2(x_3^2 - x_1^2)x_2 + k_3(x_1^2 - x_2^2)x_3 = 0, \tag{6}$$

or

$$\begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ \dfrac{1}{x_1} & \dfrac{1}{x_2} & \dfrac{1}{x_3} \end{vmatrix} = 0. \tag{7}$$

Obviously, if the cubic (7) passes through any point (m_1, m_2, m_3), it will also pass through the point $(1/m_1, 1/m_2, 1/m_3)$. Hence the pencil of cubics on the seven given points and any eighth point (m) will have the remaining common intersection at $(1/m)$, and the coefficients of any member of this pencil, in terms of k_ι, will be the coördinates of a point on the line joining (m) and $(1/m)$. Therefore the Neuberg cubic may be considered as the circular cubic of that pencil of cubics in which two members have k_i equal to c_i and $1/c_i$ respectively. Since $(1/s)$, the point of intersection M of the medians, is also on the Euler line, this same pencil of cubics contains the seventeen-point cubic

$$\frac{1}{s_1}(x_2^2 - x_3^2)x_1 + \frac{1}{s_2}(x_3^2 - x_1^2)x_2 + \frac{1}{s_3}(x_1^2 - x_2^2)x_3 = 0 \tag{8}$$

mentioned by Casey.[2]

[1] *Cf.* Salmon-Fiedler, *Höhere Ebene Kurven*, p. 258.

[2] Casey, *Analytic Geometry*, p. 460.

The net of cubics (6) contains a pencil of circular cubics, the necessary and sufficient condition being $k_1s_1 + k_2s_2 + k_3s_3 = 0$. The cubics of this pencil are in an involution of pairs such that for each pair the real asymptotes are perpendicular to one another, and the double focus of each is the point where the other curve meets its asymptote. Thus the cubic (4) and the cubic

$$\sigma_1(x_2^2 - x_3^2)x_1 + \sigma_2(x_3^2 - x_1^2)x_2 + \sigma_3(x_1^2 - x_2^2)x_3 = 0 \tag{9}$$

are a pair in this involution.

6. Self-Inversion of the Net of Cubics. From the equation (7) we see that each cubic of the net is inverted into itself by the quadratic transformation $y_i = 1/x_i$, $i = 1, 2, 3$. The four centers of inversion remain fixed under this transformation, and every other point of any cubic of the net is sent into another point on the same curve with reciprocal coördinates. Hence, any pair of these reciprocal points on the Neuberg cubic lies on a line which is parallel to the Euler line.

Such pairs of points are the following:

(*a*) the isodynamic points and the isogonal points respectively,

(*b*) the points A_i' and the three points symmetric to the vertices A_i in the lines $q_{jk}q_{kj}$,

(*c*) the point at infinity on the Euler line and the sixth intersection H of the cubic and the circumcircle,

(*d*) the six vertices of the equilateral triangles constructed on the reference triangle and the six intersections of the lines joining the vertices A_i and the isodynamic points with the cubic,

(*e*) the three intersections of lines on the vertices A_i and parallel to the Euler line with the opposite sides of the reference triangle and the points A_i respectively.

This transformation also sends the circumcircle of the base triangle $(s/x) = 0$ into the ideal line $(sx) = 0$. Again, the diameters of the circumcircle are transformed into equilateral hyperbolas on the vertices of the reference triangle and its orthocenter.[1] In particular the common chord of the circles of Apollonius, $(\sigma x) = 0$, containing the isodynamic centers, is transformed into Kiepert's hyperbola, $(\sigma/x) = 0$, containing the isogonal centers and the point $(1/s)$.

7. An Unsolved Problem. Each one of the three circles of Apollonius of the reference triangle cuts the circumcircle in a second point. These three points form a triangle which is perspective with the base triangle through the symmedian point, the co-symmedian triangle. This triangle also possesses a Neuberg cubic, which has three points in common with the cubic of the first triangle in the circumcenter and the two isodynamic centers. Hence the remaining points of intersection must lie on a conic, which is necessarily a circle, since two of these six points are the circular points I and J. This circle apparently bears an interesting relation to the base triangle, and so far as known, its equation has not yet been found.

[1] Clebsch-Lindemann, *l.c.*, p. 338.

NOTE ON THE PRECEDING PAPER.

By B. H. BROWN, Dartmouth College.

The reader may have noted that some of the properties of the 21-point cubic proved by Messrs. Moore and Neelley were given in my recent article (*1925*, 110–115) on "The twenty-one point cubic." Happily there is no question of priority here, the two papers being independent and essentially simultaneous. The very neat attack in the Moore-Neelley paper is so different from mine that the duplication of results is at once instructive and amusing.

In one respect the papers are similar, namely, in the employment of pencils of cubics. I cannot refrain from commenting on the great utility of these pencils in the study of the elementary geometry of the triangle. A systematic search for sets of 9 notable points through which pass a pencil of cubics would undoubtedly prove profitable. I may be pardoned for mentioning again one such: the vertices, the in- and excenters, the orthocenter, and the circumcenter. The Neuberg cubic is one of this pencil. Another passes through the mid-points of the sides, the centroid, and the symmedian (Lemoine) point.

In this connection my pupil, Mr. L. S. Kennison, announces the following

THEOREM. *The diagonal points of the complete quadrangle whose vertices are the isogonal and the isodynamic centers are the centroid, the symmedian point, and the infinite point on the Euler line.*

The portion of the theorem relative to the point on the Euler line is given in the Moore-Neelley paper and in mine. That the isodynamic centers and the symmedian point are collinear was shown by Neuberg. The rest of this theorem is perhaps new. I have verified Mr. Kennison's proof, a straightforward analytic attack. It would be interesting to have a simple synthetic proof for this very neat theorem.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

I. NOTE ON CERTAIN PRACTICAL PROBLEMS WHICH REQUIRE THE USE OF EXTENDED VALUES OF MATHEMATICAL FUNCTIONS.[1]

BY C. E. VAN ORSTRAND, U. S. Geological Survey.

The increased use of computing machines tends to increase the demand for extended values of both functions and constants. This result is obvious when one consider the ease with which large numbers are handled on a computing

[1] Published with the permission of the Director of the U. S. Geological Survey.

machine. Obviously, there is no need of introducing slight inaccuracies in a final result when the labor involved in making the computation is almost the same as that involved in obtaining a lower degree of accuracy. For example, if one desires to obtain 5-place common logarithms from 5-place natural logarithms, it is preferable to use 6 or even 8 decimals of the modulus, and thus avoid the possibility of a slight accumulation of error in the product.

There is also an increasing demand for large numbers resulting from new developments in science. Consider for instance the problem of determining the number of particles in a cubic centimeter of gas which have velocities falling between certain limits. The total number of particles in a cubic centimeter of gas is approximately 2.705×10^{19}, or approximately 6.062×10^{23} in a gram molecule. The elaborate table of the probability integral by Burgess, a portion of which is tabulated to 15 places of decimals, barely suffices to give the desired information for large values of the argument.

Another very important use of extended values of mathematical functions arises from the demands imposed in the compilation or computation of mathematical tables. Anyone who has attempted this task soon realizes that the number of published papers dealing with extended values of even the most important functions is very limited, and furthermore, the labor of supplying this additional information may be appalling. Numerous interesting developments in series may be made, but such developments are not always easy of evaluation —the series frequently converges too slowly, and the accuracy of the resulting computation is not easily checked. To suggest the difficulty of the task, consider that

$$\sin 0.394 = 0.38388\ 49999\ 93636\ \ldots.$$

The correct tabulation of this value to 5 places of decimals requires the evaluation of the function to at least 12 places of decimals. This instance is somewhat exceptional, but experience proves that the calculations on which mathematical tables are based should be carried to 5 or more additional decimals. The actual error involved may be very small, but a mathematical table should be an absolute standard of reference, otherwise confusion arises in the inter-comparison of tables; and in the numerous practical applications to which the tables are put, small errors are introduced which slightly vitiate the results.

In conclusion, a curious result may be mentioned which arises from the use of computing machines in the evaluation of functions. Heretofore, in the evaluation of a series, the computation proceeded on a term by term basis which generally resulted in the addition of only 2 or 3 digits to the computed value of the function as the successive terms were added. With the computing machine, however, the steps may be increased in favorable cases to 6 or 8 digits, consequently the extra labor involved in obtaining several extra digits may not be of serious importance. It seems highly desirable, in these cases, to tabulate the extra digits; for no one can foresee the exceptional demands which mathematical tables may be required to meet.

II. On the Occasional Need of Very Accurate Logarithms.

By H. Bateman, California Institute of Technology.

Twelve-figure logarithms are sometimes used in computations where great accuracy is needed but, even when it is sufficient to know correctly only a few figures, cases may arise in which twelve- and even twenty-figure logarithms must be used.

To illustrate this, let us suppose that we need the value of the Legendre function [1]

$$Q_n(x) = \tfrac{1}{2}P_n(x) \log\frac{x+1}{x-1} - \sum_{s=1}^{n}\frac{1}{s}P_{n-s}(x)P_{s-1}(x) \quad (x > 1)$$

for values of x that are not far from unity so that a power series in $1/x$ cannot be used for computation.

Except when x is very nearly unity and n is small the value of $Q_n(x)$ is small while that of $P_n(x)$ is large. This is shown by the following figures.[2]

$P_{10}(1.1) = 19.79,$	$P_{10}(1.2) = 106.544,$	$P_{10}(2) = 96060.52,$
$Q_{10}(1.1) = 0.00528,$	$Q_{10}(1.2) = 0.0006756,$	$Q_{10}(2) = 0.0000002843,$
$P_{20}(1.1) = 1175.69,$	$P_{10}(3) = 8097453,$	
$Q_{20}(1.1) = 0.0000453,$	$Q_{10}(3) = 0.0000000021.$	

The quantity to be calculated is thus the small difference between two large quantities and so the logarithmic factor in the expression for the first of these must be known very accurately. Fortunately J. C. Adams has calculated the logarithms of the first few integers to a large number of decimal places [3] and his results can be used in tabulating functions such as $Q_n(x)$.

Cases analogous to the above may be of frequent occurrence because the solution of a linear differential equation often involves a logarithmic term multiplied by a particular solution which may become large while the solution needed is small and the expression involving the logarithm is the only one available.

There are similar cases of asymptotic expansions or exact representations in which the first term has a trigonometrical factor or a factor containing Gamma functions. Again this first term may be large though the quantity to be calculated is small and so great accuracy is needed. Suitable tables of trigonometrical functions and of $\log \Gamma(1 + x)$ have been published in the British Association Reports.[4]

[1] For this expansion see Whittaker and Watson, *Modern Analysis*, 3d edition, p. 333.

[2] *Messenger of Mathematics*, vol. 52 (1922), p. 71. Spheroidal harmonics are needed for the solution of many potential problems. See for instance *Technical Note 104*, National Advisory Committee for Aeronautics (1922).

[3] *Proc. Roy. Soc. London*, vol. 27 (1878), p. 88.

[4] Year (1916) for 11-figure tables of sines and cosines (J. R. Airey) and 10-figure tables of $\text{Log}_e\,\Gamma(1 + x)$ (G. N. Watson). Year (1923) for 15-figure tables of sines and cosines of angles in radians.

III. Solution of the General Biquadratic.

By E. J. Oglesby, New York University.

In a previous number of the Monthly (*1923*, 321) the author gave a solution of the general cubic equation. In the present note the same method is applied to the biquadratic.

Take the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0, \tag{1}$$

and assume as roots

$$\begin{aligned} x_1 &= a + b + c + d, \\ x_2 &= a + b - c - d, \\ x_3 &= a - b - c + d, \\ x_4 &= a - b + c - d, \end{aligned} \tag{2}$$

where a, b, c, and d are to be determined. Using the right-hand members of (2), we have the following identities:

$$x_1 + x_2 + x_3 + x_4 = 4a, \tag{3}$$

$$x_1x_2 + \ldots + x_3x_4 = 6a^2 - 2(b^2 + c^2 + d^2), \tag{4}$$

$$x_1x_2x_3 + \ldots + x_2x_3x_4 = 4a^3 - 4a(b^2 + c^2 + d^2) + 8bcd, \tag{5}$$

$$x_1x_2x_3x_4 = -4(ab - cd)^2 + (a^2 + b^2 - c^2 - d^2)^2; \tag{6}$$

whence

$$4a = -4a_1/a_0, \tag{7}$$

$$6a^2 - 2(b^2 + c^2 + d^2) = 6a_2/a_0, \tag{8}$$

$$4a^3 - 4a(b^2 + c^2 + d^2) + 8bcd = -4a_3/a_0, \tag{9}$$

$$-4(ab - cd)^2 + (a^2 + b^2 - c^2 - d^2)^2 = a_4/a_0. \tag{10}$$

Put

$$\begin{aligned} a_0a_2 - a_1^2 &= H, \\ a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 &= G, \\ a_0a_4 - 4a_1a_3 + 3a_2^2 &= I. \end{aligned}$$

From (7),

$$a = -a_1/a_0. \tag{11}$$

Substitute for a in (8), and we get

$$b^2 + c^2 + d^2 = -3H/a_0^2. \tag{12}$$

Substitute for a and for $b^2 + c^2 + d^2$ in (9), and we get

$$bcd = -G/2a_0^3. \tag{13}$$

From (12),

$$c^2 + d^2 = -(3H/a_0^2) - b^2. \tag{14}$$

From (13),

$$cd = -\ G/2a_0^3b. \tag{15}$$

Substitute from (11), (14), and (15) in (10), and we get

$$-4\left(-\frac{a_1b}{a_0}+\frac{G}{2a_0^3b}\right)^2+\left(\frac{a_1^2}{a_0^2}+2b^2+\frac{3H}{a_0^2}\right)^2=\frac{a_4}{a_0},$$
$$-(G-2a_0^2a_1b^2)^2+a_0^2(2a_0^2b^2+3H+a_1^2)^2b^2-a_0^5a_4b^2=0;$$

which reduces to

$$4a_0^6b^6+12Ha_0^4b^4+a_0^2(12H^2-a_0^2I)b^2-G^2=0, \tag{16}$$

our resolvent equation, a cubic in b^2.

The three roots of this equation are b^2, c^2, and d^2. The signs of b, c, and d are determined from (13). Fix any two of the signs and determine the third. Then substitute the values of a, b, c, and d in (2).

Example.

$$\begin{aligned}
&x^4-6x^3+12x^2-20x-12=0;\\
&a_0=1,\quad a_1=-3/2,\quad a_2=2,\quad a_3=-5,\quad a_4=-12;\\
&H=a_0a_2-a_1^2=-1/4;\\
&G=a_0^2a_3-3a_0a_1a_2+2a_1^3=-11/4;\\
&I=a_0a_4-4a_1a_3+3a_2^2=-30.
\end{aligned}$$

Hence (16) becomes

$$4b^6-3b^4+(123/4)b^2-121/16=0;$$
$$b^2=\tfrac{1}{4},\quad (1+2\sqrt{-30})/4,\quad (1-2\sqrt{-30})/4;$$

and

$$b=\pm\tfrac{1}{2},\quad \pm(\sqrt{6}+i\sqrt{5})/2,\quad \pm(\sqrt{6}-i\sqrt{5})/2.$$

Take

$$c=(\sqrt{6}+i\sqrt{5})/2,\qquad d=(\sqrt{6}-i\sqrt{5})/2.$$

Then, by (13), $bcd = 11/8$, so that $b = \frac{1}{2}$; whence

$$\begin{aligned}
x_1&=a+b+c+d=2+\sqrt{6},\\
x_2&=a+b-c-d=2-\sqrt{6},\\
x_3&=a-b-c+d=1-i\sqrt{5},\\
x_4&=a-b+c-d=1+i\sqrt{5}.
\end{aligned}$$

IV. Note on the Introduction of Integral Calculus into a College Course in Solid Geometry.

By J. P. Ballantine, Columbia University.

I believe that anyone who has taught solid geometry in college will agree that there are certain parts which are hardly up to college grade; such problems, for instance, as one in which the area of the base and the altitude of a prism are given, and the student is expected to apply the formula for the volume. Such volumes as the cone, sphere, etc., would be very easily computed by integral calculus, if there were only some easy way to give the student the necessary fundamentals. In the following discussion I propose a possible way in which this may be done.[1]

In the first place, we will need Cavalieri's principle. This may be taken as an unproved principle, or it may be justified by any degree of rigor which the instructor desires up to and including that which would be given in a course on integral calculus.

If h denotes the altitude of a particular solid, then any value of x between 0 and h will determine the cross-section at a distance x from the base. It is conceivable that in the case of a particular solid the area of this cross-section is given by a formula u, such as, for instance, $x^2 - 2x$, which will hold for every value of x between and including 0 and h.

By Cavalieri's principle, two solids whose cross-sections are given by the same formula must have the same volume. This volume is denoted by the symbol

$$\int_0^h u.$$

If, more generally, we are considering only the volume included between the sections $x = a$ and $x = b$, the following symbol $\int_a^b u$ is used.

The following three formulæ are established as consequences of the above two definitions:

$$\int_a^b u = \int_0^b u - \int_0^a u,$$

$$\int_a^b (u + v) = \int_a^b u + \int_a^b v,$$

$$\int_a^b (nu) = n \int_a^b u,$$

where n is any real number.

It is next necessary to construct sample solids with altitudes h and with cross-section formulæ 1, x, and x^2 respectively in order to evaluate the integrals of these three functions. The first is a prism, and its volume is h. The second

[1] Compare the discussion by W. R. Longley (*1924*, 196).

is also a prism whose base is an isosceles right triangle with equal sides h, and whose altitude is 1. The sections which are given by the formula x are taken parallel to one of the faces which is a rectangle h by 1. The volume is $h^2/2$. For the third solid, one starts with a cube $2h$ on an edge. Using the center as a vertex and each face of the cube as a base, six equal square pyramids may be formed, each with one sixth the volume of the cube. The pyramid formed on the upper face of the cube has its cross-section given by the formula $4x^2$. Hence we have the equation

$$\int_0^h 4x^2 = 4\int_0^h x^2 = \tfrac{1}{6}(2h)^3$$

from which the integral of x^2 may be obtained,

$$\int_0^h x^2 = h^3/3.$$

Thus, on the basis of the three formulæ for manipulating integrals, and the values of the three particular integrals, it is possible for the student in solid geometry to integrate any quadratic function. In particular, the volume of any pyramid can be found by this process. The volume of a sphere or of any spherical segment can be obtained by integrating the expression $\pi(r^2 - x^2)$.[1] In place of proving that the volume of every prismatoid is given by the prismatoid formula, one may prove the more interesting and more fundamental theorem that if the cross-section of a solid is given by a quadratic formula, then the volume is given by the prismatoid formula.

V. Two New Arctangent Relations for π.

By A. A. Bennett, University of Texas.

Among the numerous methods used to obtain successive numerical approximations to π, the most familiar is probably that which employs the arctangent function, profiting by the fact that the principal value of arctangent 1 is $\pi/4$. We have for example such familiar relations as the following:

$$\pi/4 = \arctan \tfrac{1}{2} + \arctan \tfrac{1}{3}, \tag{1}$$

$$\pi/4 = 2 \arctan \tfrac{1}{3} + \arctan \tfrac{1}{7}, \tag{2}$$

$$\pi/4 = \arctan \tfrac{1}{2} + \arctan \tfrac{1}{5} + \arctan \tfrac{1}{8}, \tag{3}$$

$$\pi/4 = 4 \arctan \tfrac{1}{5} + \arctan \tfrac{1}{239}. \tag{4}$$

Equation (1) gives rise to the series known as Euler's series. Equation (2) was used as a check by Clausen in 1847 in computing π to 248 places of decimals.

[1] The process of carrying out the suggested integration by use of the three formulæ referred to is exactly equivalent to the process, used for instance in Hawkes, Luby, and Touton's *Solid Geometry*, of considering the sphere as the difference between the circumscribing cylinder and a certain two-napped cone.

It had also been used previously by Vega as a check in computing π to 140 places. Equation (3) enabled Dase in 1844 to calculate π to 200 places of decimals. Equation (4) is the justly celebrated relation due to Machin (1706) and forms the basis for numerous computations of π, including the work of the computers mentioned above. In particular, W. Shanks computed π to 707 places by the use of Machin's formula.

All of these relations are suited for use in the power series, called Gregory's series,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \tag{5}$$

which is leisurely convergent even for $x = 1$. Another familiar series for arctangent x is the following:

$$\arctan x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{x^2}{1+x^2}\right)^3 + \cdots\right\} \tag{6}$$

given by Euler in 1755.

For (6) it is not necessarily most convenient to take x as the reciprocal of an integer. Indeed Euler's relation,

$$\pi/4 = 5 \arctan \tfrac{1}{7} + 2 \arctan \tfrac{3}{79}, \tag{7}$$

is in many ways more convenient for computation than any one of the relations (1), (2), (3), or (4).

The new equations here given are intended for use in Gregory's series. The first is not unsuitable for paper and pencil computations, as it converges much more rapidly than does Machin's formula. The second is selected with particular reference to computing machines. It satisfies the condition that no one of the denominators requires a setting of more than five successive digits on a machine, and moreover most of these denominators are particularly convenient for machine division, since the successive partial quotients are usually immediately obvious. These equations are:

$$\pi/4 = 12 \arctan \tfrac{1}{18} + 3 \arctan \tfrac{1}{70} + 5 \arctan \tfrac{1}{99} + 8 \arctan \tfrac{1}{307}, \tag{8}$$

$$\begin{aligned}\pi/4 = 160 \arctan \tfrac{1}{200} &- (\arctan \tfrac{1}{239} + 4 \arctan \tfrac{1}{515} + 8 \arctan \tfrac{1}{4030} \\ &+ 16 \arctan \tfrac{1}{50105} + 16 \arctan \tfrac{1}{62575} + 32 \arctan \tfrac{1}{500150} \\ &+ 80 \arctan \tfrac{1}{4000300}).\end{aligned} \tag{9}$$

It is easy to verify (8) by means of the sequence of relations,

$\operatorname{arccot} 5 = \operatorname{arccot} 7 + \operatorname{arccot} 18,$	$\operatorname{arccot} 7 = \operatorname{arccot} 12 + \operatorname{arccot} 17,$
$\operatorname{arccot} 12 = \operatorname{arccot} 17 + \operatorname{arccot} 41,$	$\operatorname{arccot} 17 = \operatorname{arccot} 18 + \operatorname{arccot} 307,$
$\operatorname{arccot} 41 = \operatorname{arccot} 70 + \operatorname{arccot} 99,$	$\operatorname{arccot} 70 = \operatorname{arccot} 99 + \operatorname{arccot} 239,$

together with (4). Similarly (9) is verified by means of the sequence of relations,

$$\begin{aligned}
\operatorname{arccot} 5 &= 2 \operatorname{arccot} 10 - \operatorname{arccot} 515,\\
\operatorname{arccot} 10 &= 2 \operatorname{arccot} 20 - \operatorname{arccot} 4030,\\
\operatorname{arccot} 20 &= \operatorname{arccot} 25 + \operatorname{arccot} 100 - \operatorname{arccot} 50105,\\
\operatorname{arccot} 25 &= 2 \operatorname{arccot} 50 - \operatorname{arccot} 62575,\\
\operatorname{arccot} 50 &= 2 \operatorname{arccot} 100 - \operatorname{arccot} 500150,\\
\operatorname{arccot} 100 &= 2 \operatorname{arccot} 200 - \operatorname{arccot} 4000300,
\end{aligned}$$

together with (4).

The extraordinarily rapid convergence of the several Gregory series arising from (9) makes this formula particularly useful unless only very few significant figures of π are desired.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

La "Thiende" de Simon Stevin. Fac-simile edition with an introduction by H. BOSMANS, S.J. Antwerp, Société des Bibliophiles Anversois, 1924. vi + 37 pages. Price 15 Belgian francs.

All who have interest in the epoch-making classics of mathematics will be glad to know of this recent reproduction in fac-simile of the first treatise ever published upon decimal fractions. The original work was issued in the Flemish language at Leyden, from the local branch of the Plantin Press, in 1585. It is now excessively rare, and it was only by good fortune that Père Bosmans happened to have the Louvain copy at his home when the library was burned during the World War. The French translation made in the same year, and also published by Plantin, is better known, particularly in the edition of *L'Arithmétique*.

Naturally it will be the French edition that will be generally consulted by students, unless they know of Miss Sanford's excellent translation which appeared in the *Mathematics Teacher* in October, 1921. It will, however, be of interest to many readers to have a fac-simile of the original edition. Thanks to the generous action of the Société des Bibliophiles Anversois (No. 22, Marché du Vendredi, Antwerp), it is possible to obtain a copy at a merely nominal price.

All students of the history of mathematics are indebted to Père Bosmans for his many valuable essays, and it will be of interest to readers to know that he has prefaced this edition with an introduction of much value to bibliophiles as well as to those who make a study of the origins of our science.

D. E. SMITH.

Fundamental Topics in the Differential and Integral Calculus. By GEORGE RUTLEDGE. New York, Ginn and Company, 1923. 252 pages. Price $2.00.

This book, which is intended for use in a liberal arts college where only one semester of mathematics is required, is not a unified course but simply an elementary course in the Calculus. The topics presented are variables; functions; graphs; continuity of a function and of its graph; derivative of a polynomial; slope of a graph; the sine function and its derivative; the square-root function and its derivative; differentials; the differential as an approximation to the increment; rates, velocity; acceleration; derivative of a product or quotient; derivative of a function of a function; differential of length; properties of linear, quadratic and cubic polynomial graphs; the square root of a quadratic polynomial; maxima and minima; definite integrals; area; length; volume; integration process in general; summation by substitution; the logarithmic function; the exponential function; the hyperbolic functions; inverse circular and hyperbolic functions; derivative equations; differential equations; what is calculus?

Thus if one believes in scrapping most of trigonometry, college algebra and analytic geometry and giving arts students in the minimum amount of time an insight into the Calculus, he will find this an admirable book. The book contains some excellent graphic illustrations and an abundance of well-graded exercises.

F. M. MORGAN.

Mathematical Theory of Life Insurance. By C. H. FORSYTH. New York, John Wiley and Sons, 1924. 74 pages. Price $1.25.

In this little book of 74 pages are presented the rudiments of life insurance in a delightful style. There is very little that is new; but what is contained in the text is brief, necessary, and well put.

Beginning with a short review of probabilities and interest, the author then gives the mortality table and expectations of life. Annuities are introduced by a study of the pure endowment ${}_nE_x$, and assurances are treated by means of the single net premium ${}_nI_x$; and in the chapter on commutation symbols the American N_x is used. There is a good chapter on valuation of policies, and finally one on the laws of Gompertz and Makeham.

A student who has mastered this little text will be well prepared for further work in the mathematics of life insurance.

C. T. BUMER.

Mathematical Analysis of Statistics. By C. H. FORSYTH. New York, John Wiley and Sons, 1924. 241 pages. Price $2.25.

There is at present a considerable interest on the part of many mathematicians in the problems of statistics, and any new book on the subject is sure to attract attention. At the present stage of the development the time is probably not ripe for putting the material of mathematical statistics in final form: and in the meantime, books such as this one satisfy a real need.

The book opens with an introductory chapter on errors and numerical computation, followed by two chapters on finite differences and applications to interpolation. Just why the Gamma function is introduced is not clear. The chapter is too difficult for the student who has not had a course in the calculus, and rather too brief for one who has had such a course. The work on probability, averages, dispersion, etc., covers the usual material. The method of moments has been applied to curve fitting and to the analysis of time series. In analyzing the time series the author uses the method originated by W. M. Persons for eliminating seasonal variation. Various types of statistical series are discussed after the method of Arne Fisher. The treatment of the normal curve, least squares, and probable error make up one chapter; and finally there is a short, sketchy lesson on the theory of correlation.

There is a certain abruptness in the book in places; but the impression arrived at, after using it in the classroom, is that the text is alive—something which cannot be said of most books on statistics. There are many problems throughout, some new and some old, which aid considerably in fixing in mind the theory explained.

C. T. Bumer.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Bulletin of the American Mathematical Society, volume 30, nos. 9–10, November –December, 1924: "On simple groups of low order" by F. N. Cole, 489–492; "Complete class number expansions for certain elliptic theta constants of the third degree" by E. T. Bell, 493–496; "Note on a special congruence" by M. C. Foster, 496–503; "Concerning relatively uniform convergence" by R. L. Moore, 504; "The theory of closure of Tchebycheff polynomials for an infinite interval" by J. A. Shohat, 505–510; "Nuclear and hypernuclear points in the theory of abstract sets" by E. W. Chittenden, 511–519; "Five axioms for point and translation in affine geometry" by A. A. Bennett, 520–526; "Some problems of closure connected with the Geiser transformation" by A. Emch, 527–535; "A symmetric coefficient of correlation for several variables" by D. Jackson, 536–542; "Methods for finding factors of large integers" by H. S. Vandiver, 542–553.

Proceedings of the National Academy of Sciences, volume 11, no. 1, January, 1925: "Notes on stellar statistics: The mathematical expression of the law of tangential velocities" by W. J. Luyten, 87–90; "Economics and the calculus of variations" by G. C. Evans, 90–95; "Separable quadratic differential forms and Einstein solutions" by E. Kasner, 95-96; "On a new method of factorization" by D. N. Lehmer, 97–98; "Functionals of curves admitting one-parameter groups of infinitesimal point transformations" by A. D. Michal, 98–101.

School Science and Mathematics, volume 25, no. 2, February, 1925: "Hurdle tests in algebra" by K. Wentz, 132–144; "Arithmetic in the junior high school" by L. W. Colwell, 171–178.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS.

1925 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS.

By W. C. EELLS, Whitman College, Walla Walla, Wash.

Program committees of college and high school mathematics clubs, teachers of courses in the history of mathematics, and others may be interested in having called to their attention important events in the history of mathematics for which 1925 is the centennial year.

Many of these events suggest timely topics for club programs in the form of centennial commemorations of occurrences which took place an even number of centuries preceding the present year.

Ample material in English for the development of most of the suggested topics may be found in Ball's *Short History;* Smith's *History;* Cajori's *History* (Revised), *Elementary History,* and *History of Mathematics in the United States;* Fink's *Brief History;* Heath's *History of Greek Mathematics,* and special treatises on *Euclid* and *Archimedes;* Allman's *Greek Geometry from Thales to Euclid;* and in general encyclopedias and other works of reference.

Below is given a list of such centennial dates, accompanied in some cases by brief suggestions of closely related topics for investigation.

375 B.C. Approximate date of birth of Menæchmus. Conic sections.

375 B.C. Theætetus flourished. Greek theory of irrationals; five regular solids.

275 B.C. Approximate date of death of Euclid, who for twenty-two centuries through the *Elements* with its more than two thousand editions "has exercised such profound influence on the teaching and knowledge of geometry."

275 B.C. Approximate date of birth of Eratosthenes. Construction of tables of prime numbers; modern factor tables; earliest measurement of arc of earth's surface.

75 B.C. Discovery by Cicero of tomb of Archimedes, "the great geometer," "the greatest mathematician of antiquity." Geometry of the sphere and cylinder.

125 A.D. First astronomical observations of Ptolemy. The Almagest; Greek Astronomy.

1025. Approximate date of Sridhara's *Trisatika.* Early Hindu mathematics.

1225. Publication of Leonardo of Pisa's *Liber Quadratorum.* (MONTHLY, *1919,* 1–8.)

1225. First "Mathematical Tournament." Other mathematical tournaments of history.

1225. Birth of Thomas Aquinas. Infinity and the linear continuum.
1525. Publication of Rudolff's *Die Coss,* first German text-book of algebra. Introduction of symbols for square root.
1525. Publication of Dürer's *Underweysung der Messung,* with derivation of the Epicycloid. History of the Epicycloid.
1625. (Mar. 5) Birth of Collins. Newton-Leibnitz controversy.
1625. (June 8) Birth of Cassini. Arc of earth's surface; the Cassinian Oval.
1625. Birth of de Witt. Projective generation of conics.
1725. Clairaut, at age of 12, writes important paper on "Four Geometric Curves." Other mathematical prodigies.
1725. (Sept. 5) Birth of Montucla. Historians and histories of mathematics.
1825. (Mar. 10) Death of Mollweide. "Gauss' Analogies."
1825. (Apr. 21) Death of Pfaff. Differential equations; "Pfaffian problem."
1825. Bolyai's development of the theory of parallels. Non-Euclidean Geometry.
1825. Publication of final volume of Laplace's "*Mécanique Céleste,*" "the translation of the *Principia* into the language of the differential calculus" (Ball), "a complete solution of the great mechanical problem presented by the solar system."
1825. Publication of first volume of Legendre's monumental treatise *Fonctions Elliptiques.*
1825. Publication in United States of Warren Colburn's *Algebra.* Colburn's other works; early American algebra.
1825. Founding of the *Mathematical Diary,* by Robert Adrain. Other early American mathematical journals.
1825. Benjamin Peirce, "father of American mathematics," enters Harvard University (MONTHLY, *1925,* p. 9).

CLUB ACTIVITIES.

THE MATHEMATICAL CLUB OF SMITH COLLEGE, Northampton, Massachusetts. [*1922,* 26.]

The officers for the year 1923–1924 were: President: Charlotte Nelson, '24; Secretary: Emily Wilson, '24; Treasurer: Dorothy Williams, '24.

The programs for the past two years are as follows:

October 22, 1923: "Extracts from the history of mathematics"—Charlotte Nelson, '24. Mathematical puzzles.

November 12, 1923: "Mathematical curricula in foreign countries"—Carolyn Waterbury, '24. Puzzles.

December 3, 1923: Christmas party at the home of Professor Wood. Games, puzzles, grinds, and presents.

February 18, 1924: "Demonstration of the Chinese method of calculating with the abacus"—Professor Cobb.

March 10, 1924: "Quadric surfaces"—Evelyn Winters, graduate student.

April 14, 1924: "Teaching mathematics"—Dorothy Lane, '25. Elizabeth Lane, '25, showed a hyperbolic paraboloid which she had made.

May 5, 1924: Reading from "Flatland"—Charlotte Nelson. "Hindu numerals"—Professor Benedict.

May 29, 1924: Farewell party of the year. A stunt was given showing the humorous side of our mathematics classes and teachers.

Officers for the year 1924–1925 are: President: Elizabeth Parkhurst, '25; Secretary: Mary Mangan, '25; Treasurer: Elizabeth Lane, '25.

October 20, 1924. Plans for the year were discussed. Members of the club volunteered to give papers through the year.

November 10, 1924: "History of mathematical education"—Hope Adams, '25.

December 1, 1924: Christmas party at Professor Wood's. Games were played and a stunt was given.

January 12, 1925: "Babylonian and Egyptian mathematics"—Doris Latimer, '25. "John Napier"—Rachel Lorrey, '26.

(Report by Miss Mangan, Secretary.)

Pi Mu Epsilon, University of Illinois, Urbana, Illinois.

The Illinois chapter of Pi Mu Epsilon is an organization which has for its aims, first, scholarship for individual members in all subjects and particularly in mathematics; second, the advancement of the science of mathematics; and last, the common and personal advancement of its members.

It is the custom of our chapter to initiate new members once each semester, the initiation for the first semester of the year 1924–1925 having taken place on December tenth, at which time seventeen new members were initiated into the organization, three members of the mathematics faculty, one graduate student, and thirteen others including engineers and mathematics majors. At the present time Illinois chapter has forty-six members, a body made up of juniors, seniors, and graduate students whose work has shown unusual interest and ability in mathematics or in the mathematical sciences, and members of the mathematics faculty among whom is Dr. E. J. Townsend, head of the mathematics department.

In the pursuance of its aims the chapter holds fortnightly meetings open to the public at each of which an address of some mathematical interest is given by some member of the organization. The open meetings of the first semester of the year 1924–1925 were as follows:

October 15: "Sturm's theorem," Dr. J. B. Shaw.

November 5: "Mathematical games," Dr. A. J. Kempner.

December 3: "The nine point circle," H. L. Black.

January 14: Report of National Convention of Pi Mu Epsilon, Dr. R. D. Carmichael.

The good attendance at these meetings shows that they have been of considerable interest both among faculty members and students.

(Report by Miss Waters, Secretary.)

The Irrational Club, University of Wyoming, Laramie, Wyoming.

The Irrational Club of the University of Wyoming, Laramie, Wyoming, started this year. Its membership is open to all who are interested in mathematics. The officers of the club are as follows:

Positive Square Root............Robert Burns.
Negative Square Root...........Oswald Seaverson.
Keeper of the Log and Bones.....Marcella Avery.
Custodian of the Indices.........The above fundamental elements of the club with the addition of Miss Gretna Neubauer and Clark Beisemeier.

The following subjects have been or will be discussed during the coming year: Magic squares, Flatland, Women and mathematics, Paper folding, History of mathematical symbols, History of logarithms and how to use them, Map drawing, Mathematics of astronomy, Fourth dimension.

A portion of each meeting is spent in a five-minute history sketch; a like amount of time is devoted to some mathematical recreation and some phase of computation; thirty minutes is given to the major topic. In addition to these items place is provided on the program for the solution of problems.

(Report by Professor H. C. Gossard.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. A. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3133. Proposed by A. A. BENNETT, University of Texas.

Show that in every set containing but a finite number of elements and admitting an associative rule of multiplication, there must be an idempotent element, that is, one which is equal to its square. Show that this does not continue to hold if the set contains an infinite number of elements.

3134. Proposed by J. ROSENBAUM, Milford, Connecticut.

In a given circle to inscribe a hexagon of assigned angles which shall have the maximum area.

3135. Proposed by N. P. PANDYA, Amreli, Kathiawad, India.

Form the equation of lowest degree with rational coefficients whose roots are $\sin A$, $\sin 4A$, $\sin 7A$, and $\sin 10A$, where $13A = \pi$.

3136. Proposed by PAUL CAPRON, U. S. Naval Academy.

If the sines of the half-sides of a spherical triangle are h, k, l, and the cosines of the halves of the corresponding opposite angles are H, K, L, and if p, P are the polar distances of the inscribed and circumscribed circles,

$$\sin P = \frac{hkl}{2\sqrt{\sigma(\sigma - h)(\sigma - k)(\sigma - l)}}; \qquad \cos p = \frac{HKL}{2\sqrt{\Sigma(\Sigma - H)(\Sigma - K)(\Sigma - L)}},$$

where $\sigma = (h + k + l)/2$ and $\Sigma = (H + K + L)/2$.

3137. Proposed by HARRY LANGMAN, New York City.

Show how to draw, using straight-edge only, a tangent to the circumference (or an arc) of a circle at a given point, without making use of Pascal's hexagon theorem.

3138. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The vertex of a triangle, whose base is fixed, moves along a straight line coplanar with the line of the base. Find (1) the locus of the orthocenter of the triangle; (2) the envelope of the line joining the feet of the two altitudes dropped from the two fixed vertices of the triangle.

3139. Proposed by C. N. MILLS, South Dakota State Normal School.

A rope weighs w pounds per foot. How many coils of the rope must be taken around a rough cylinder of radius r in order that a weight may support another n times as great, k being the coefficient of friction between the rope and the cylinder.

SOLUTIONS.

2850 [1920, 377]. Proposed by SARAH BEALL, U. S. Coast and Geodetic Survey.

An unknown star is observed at the altitudes h_1 and h_2 at the respective times t_1 and t_2, the latitude being known also. Obtain formulas for the right ascension and declination of the star: (1) when the time interval $t_1 - t_2$ is large; (2) when the time interval is so small that $(h_2 - h_1)/(t_2 - t_1)$ may be taken as the value of dh/dt corresponding to the mean altitude $(h_1 + h_2)/2$, and the mean time $(t_1 + t_2)/2$. This problem sometimes arises when a bright star is observed through the clouds.

SOLUTION BY C. C. WYLIE, University of Illinois.

Notation: Let z = zenith distance = $90° - h$; t = sidereal time; D = declination; A = right ascension = $t - P$; L = latitude; and P = hour angle.

Case 2. From one of the fundamental formulæ of spherical trigonometry we have

$$\cos z = \sin D \sin L + \cos D \cos L \cos P. \tag{1}$$

Differentiating,

$$\sin z\, dz = \cos D \cos L \sin P\, dP;$$

whence

$$\sin P = \frac{\sin z}{\cos D \cos L} \frac{dz}{dP}. \tag{2}$$

Equations (1) and (2) can be considered two equations in the two unknowns D and P. We eliminate P by obtaining $\cos P$ from (2).

$$\cos P = \frac{\sqrt{\cos^2 D \cos^2 L - \sin^2 z \left(\dfrac{dz}{dP}\right)^2}}{\cos D \cos L}. \tag{3}$$

Substituting in (1), and squaring, we have

$$\cos^2 z - 2 \sin D \sin L \cos z + \sin^2 D \sin^2 L = \cos^2 L(1 - \sin^2 D) - \sin^2 z \left(\frac{dz}{dP}\right)^2. \tag{4}$$

Rearranging,

$$(\sin^2 L + \cos^2 L) \sin^2 D + (-2 \sin L \cos z) \sin D + \left\{\cos^2 z + \sin^2 z \left(\frac{dz}{dP}\right)^2 - \cos^2 L\right\} = 0. \tag{5}$$

Denoting the coefficient of $\sin^2 D$ by a, that of $\sin D$ by b, and the other term by c, equation (5) becomes, since $a = 1$,

$$\sin^2 D + b \sin D + c = 0; \tag{6}$$

whence

$$D = \sin^{-1}\left\{\frac{-b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}\right\}. \tag{7}$$

Having D, $\cos D$ can be taken from tables or computed from $\sin D$, and we have from (2)

$$P = \sin^{-1}\left\{\frac{\sin z}{\cos D \cos L} \frac{dz}{dP}\right\},$$

and finally

$$A = t - P \tag{8}$$

which completes the solution of case (2).

Case 1. For case (1) we have $P_2 - P_1 = t_2 - t_1$; and the fundamental equation becomes

$$\begin{cases} \cos z_1 = \sin D \sin L + \cos D \cos L \cos P_1, \\ \cos z_2 = \sin D \sin L + \cos D \cos L \cos P_2. \end{cases} \tag{9}$$

Adding and subtracting we have

$$\begin{cases} \cos z_2 + \cos z_1 = 2 \sin D \sin L + \cos D \cos L\ (\cos P_2 + \cos P_1), \\ \cos z_2 - \cos z_1 = \cos D \cos L\ (\cos P_2 - \cos P_1). \end{cases} \tag{10}$$

From the well-known formulæ for the sum and difference of cosines we obtain

$$\cos \tfrac{1}{2}(z_2 - z_1) \cos \tfrac{1}{2}(z_2 + z_1) - \sin D \sin L = \cos D \cos L \cos \tfrac{1}{2}(P_2 - P_1) \cos \tfrac{1}{2}(P_2 + P_1); \tag{11}$$

$$\sin \tfrac{1}{2}(z_2 - z_1) \sin \tfrac{1}{2}(z_2 + z_1) = \cos D \cos L \sin \tfrac{1}{2}(P_2 - P_1) \sin \tfrac{1}{2}(P_2 + P_1). \tag{12}$$

From (12),

$$\sin \tfrac{1}{2}(P_2 + P_1) = \frac{\sin \frac{1}{2}(z_2 - z_1) \sin \frac{1}{2}(z_2 + z_1)}{\cos D \cos L \sin \frac{1}{2}(P_2 - P_1)}. \tag{13}$$

Whence,

$$\cos \tfrac{1}{2}(P_2 + P_1) = \frac{\sqrt{\cos^2 D \cos^2 L \sin^2 \frac{1}{2}(P_2 - P_1) - \sin^2 \frac{1}{2}(z_2 - z_1) \sin^2 \frac{1}{2}(z_2 + z_1)}}{\cos D \cos L \sin \frac{1}{2} (P_2 - P_1)}. \tag{14}$$

Substituting this value of $\cos \frac{1}{2}(P_2 + P_1)$ in (11) and squaring we have

$$\begin{aligned} &\cos^2 \tfrac{1}{2}(z_2 - z_1) \cos^2 \tfrac{1}{2}(z_2 + z_1) - 2 \sin D \sin L \cos \tfrac{1}{2}(z_2 - z_1) \cos \tfrac{1}{2}(z_2 + z_1) + \sin^2 D \sin^2 L \\ &= \cos^2 L(1 - \sin^2 D) \cos^2 \tfrac{1}{2}(P_2 - P_1) - \sin^2 \tfrac{1}{2}(z_2 - z_1) \sin^2 \tfrac{1}{2}(z_2 + z_1) \cot^2 \tfrac{1}{2}(P_2 - P_1). \end{aligned} \tag{15}$$

Rearranging,

$$\begin{aligned} &\{\sin^2 L + \cos^2 L \cos^2 \tfrac{1}{2}(P_2 - P_1)\} \sin^2 D + \{- 2 \sin L \cos \tfrac{1}{2}(z_2 - z_1) \cos \tfrac{1}{2}(z_2 + z_1)\} \sin D \\ &\quad + \cos^2 \tfrac{1}{2}(z_2 - z_1) \cos^2 \tfrac{1}{2}(z_2 + z_1) + \sin^2 \tfrac{1}{2}(z_2 - z_1) \sin^2 \tfrac{1}{2}(z_2 + z_1) \cot^2 \tfrac{1}{2}(P_2 + P_1) \\ &\quad - \cos^2 L \cos^2 \tfrac{1}{2}(P_2 - P_1) = 0. \end{aligned} \tag{16}$$

Using the same notation as in the preceding case, we can write (16) in the form

$$a' \sin^2 D + b' \sin D + c' = 0.$$

Whence,

$$D = \sin^{-1}\left\{\frac{- b'}{2a'} \pm \frac{\sqrt{b^2 - 4a'c'}}{2a'}\right\}. \tag{17}$$

With this value of D, obtain $\cos D$, and from (13) we have

$$\tfrac{1}{2}(P_2 + P_1) = \sin^{-1}\left\{\frac{\sin \frac{1}{2}(z_2 - z_1) \sin \frac{1}{2}(z_2 + z_1)}{\cos D \cos L \sin \frac{1}{2}(P_2 - P_1)}\right\}.$$

The remainder of the solution is

$$P_2 = \tfrac{1}{2}(P_2 + P_1) + \tfrac{1}{2}(P_2 - P_1) \quad \text{or} \quad P_1 = \tfrac{1}{2}(P_2 + P_1) - \tfrac{1}{2}(P_2 - P_1),$$
$$A = t_2 - P_2, \qquad A = t_1 - P_1.$$

2884 [1921, 139]. Proposed by E. H. MOORE, University of Chicago.

Consider an $m \times n$ array A of numbers a_{st} and an $n \times m$ array B of numbers b_{ts}. Show that the system of mn equations:

$$\sum_{tu} a_{st} b_{tu} a_{uv} = 0 \quad (sv),$$

implies the equation:

$$\sum_{ts} a_{st} b_{ts} = 0.$$

The suffixes s, u have the range $1, 2, \cdots, m$ and the suffixes t, v have the range $1, 2, \cdots, n$.

SOLUTION BY H. E. BRAY, The Rice Institute.

Let the rank of the matrix A be $r \leq m$. For simplicity, suppose that the first r rows of A are linearly independent—this situation can be brought about by shifting rows of A and at the same time shifting the corresponding columns of B without affecting the hypothesis or the conclusion.

It follows then, by the theory of linear equations, that r vectors

$$c_{l,g}, \qquad l = 1, 2, \cdots n; \; g = 1, 2, \cdots r,$$

can be determined in such a way that

$$\sum_{l=1}^{n} a_{il} c_{lg} = \delta_{i,g}, \qquad i, g = 1, 2, \cdots r; \tag{1}$$

where $\delta_{ig} = 1$ or 0 according as i is or is not equal to g. For the matrix of the coefficients of $c_{1g}, c_{2g}, \cdots c_{ng}$ contains at least one non-vanishing r-rowed determinant. And this is true for each g.

Again, since A is of rank r, every vector in the last $m - r$ rows of A can be expressed as a linear combination of the first r vectors. Hence every one of the m vectors of A can be expressed as a linear combination of the first r; *i.e.*, constants λ_{kp} can be determined so that

$$\sum_{p=1}^{r} \lambda_{kp} a_{pl} = a_{kl}, \qquad k = 1, 2, \cdots m; \; l = 1, 2, \cdots n. \tag{2}$$

By hypothesis,

$$\sum_{i=1}^{r} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{ij} b_{jk} a_{kl} c_{li} = 0.$$

But

$$\begin{aligned} \sum_{i=1}^{r} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{n} a_{ij} b_{jk} a_{kl} c_{li} &= \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{p=1}^{r} \sum_{i=1}^{r} \sum_{l=1}^{n} b_{jk} \lambda_{kp} a_{pl} c_{li} a_{ij} && \text{by (2)}, \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{p=1}^{r} \sum_{i=1}^{r} b_{jk} \lambda_{kp} \delta_{pi} a_{ij} && \text{by (1)}, \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{p=1}^{r} b_{jk} \lambda_{kp} a_{pj} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} b_{jk} a_{kj} && \text{by (2)}. \end{aligned}$$

Hence $\sum_{j=1}^{n} \sum_{k=1}^{m} b_{jk} a_{kj} = 0$.

2989 [1922, 356]. Proposed by L. M. HOSKINS, Stanford University.

How should the following questions be answered, assuming that the place referred to is in latitude 34° 8′?

A building twelve feet high has been erected 49 inches south of our lot line. We desire to erect a wall on our line six inches in thickness. (*a*) How high can we build the wall and have it wholly within the shadow cast by the building? (*b*) How high can we build the wall and have it within the shadow cast by the building during the winter months?

SOLUTION BY C. C. WYLIE, University of Illinois.

Case (*a*): As the problem is stated the answer is obviously "impossible." The latter part of June the sun shines on the north side of the building for some six hours each day and as the shadow is then to the south, a wall to the north could not be "wholly within the shadow cast by the building."

Case (*b*): Strictly speaking, the answer to this is also "impossible." Winter lasts until the sun reaches the Vernal Equinox, at which time it rises due east and nothing to the north of the building could be within its shadow at the time of sunrise.

The problem might be restated in several ways, but if we are concerned with any other time than local apparent noon, it is necessary to know something about the length and position of the wall as compared with the building. The solution for local apparent noon is quite simple and is as follows:

Case (*a*): The height in inches may be

$$144 - 55 \cot (34° 8' - 23° 30') = 144 - 293 = -149 \text{ inches.}$$

This means that if the wall were built 0 inches high the building would have to be 12 ft. 5 in. higher to have the wall in its shadow at the time of noon. We have seen before that the sun shines on the north side of the building for more than six hours each day about June 21, and it is now seen that the wall would be very much in the sunshine at the time of noon then.

Case (*b*): The height of the wall for this case may be

$$144 - 55 \cot (34° 8') = 144 - 81 = 63 \text{ inches.}$$

From this we see that the wall may be built about 5 ft. 3 in. in height and be in the shadow of the building at the time of local apparent noon, throughout the months of fall and winter.

Refraction, semidiameter, annual variation of sun's position, etc., have been neglected in the solution.

3059 [1924, 101, 502]. Proposed by DANIEL KRETH, Wellman, Iowa.

Given the perimeter and the radius of the inscribed and circumscribed circles, to construct the triangle and calculate the lengths of its sides.

NOTE BY J. ROSENBAUM, Milford, Conn., AND OTTO DUNKEL, Washington University.

This note disposes of the question in the Remark (*1924*, 502) regarding the case $R = 2r$ and completes the conditions to be placed upon s, r and R for the existence of a real solution of the problem. It is assumed that s, r and R are real and greater than zero. If the roots of the x equation, given in the solution, are all real, they must be greater than zero; this must also be true of the y equation given in the Remark. It then follows that the three roots of the first equation must be the sides of a triangle, since $s - a$, $s - b$, $s - c$ are each greater than zero.

Suppose then that r and R are two given positive quantities and that the positive quantity s is to be determined so that the roots of either equation are all real. The necessary and sufficient condition for real roots is

$$s^2(9r^2 + s^2 - 18rR)^2 - (s^2 - 3r^2 - 12rR)^3 \leqq 0. \tag{1}$$

Set

$$\begin{aligned} s_1^2 &= 2R^2 + 10rR - r^2 + 2\sqrt{R(R-2r)^3}, \\ s_2^2 &= 2R^2 + 10rR - r^2 - 2\sqrt{R(R-2r)^3}. \end{aligned} \tag{2}$$

Then (1) is true when and only when

$$s_2 \leqq s \leqq s_1, \qquad R \geqq 2r. \tag{3}$$

These results may also be obtained geometrically. If the two circles have a given position, it is known from geometry that if there exists one triangle inscribed in the R circle and circumscribing the r circle, then there exists an infinite number of such triangles including two isosceles triangles. It is easily shown by geometry that the necessary and sufficient condition for an isosceles triangle is

$$d^2 = R^2 - 2rR, \tag{4}$$

where d is the distance between the centers. This must therefore be the necessary and sufficient condition for the existence of any such triangle. If we examine the figure for the isosceles triangle, we obtain by aid of this relation the values of the equal sides,

$$\begin{aligned} x_1^2 &= 2R[R + r + \sqrt{R(R-2r)}], \\ x_2^2 &= 2R[R + r - \sqrt{R(R-2r)}], \end{aligned} \tag{5}$$

and also the corresponding values of s given in (1).

Considering now the cubic curve defined by the x equation, we see that if there is a value of s giving a real solution then a straight line parallel to the x axis at the distance s above cuts the cubic in three real points. The results (5) give the conditions that two values of x coincide in x_1 and x_2. Hence the derivative of s with respect to x is zero at these points. Moreover the x equation has three real and positive roots when and only when s lies in the interval defined by the corresponding s values (3), as may be seen from the form of the curve.

We have then three cases,

I. $R < 2r$. No real triangle.

II. $R = 2r$. Here $s = s_1 = s_2 = 3\sqrt{3}\,r$, and there are three equal roots $a = b = c = 2\sqrt{3}\,r$.

III. $R > 2r$. Here s may have any value within the limits set above and for each such value there exists a real triangle (also the congruent and symmetrically placed triangle). If s is equal to either extreme value, the triangle is isosceles.

If $2 < R/r < 1 + \sqrt{2}$, there is no right triangle; if $R/r = 1 + \sqrt{2}$, there is a single isosceles right triangle; if R/r is greater than this last number, there is a pair of congruent right triangles.

We have then the following theorem: If r and R are any two positive numbers such that $R > 2r$, and if two circles be drawn with these two numbers as radii, and with their centers at the distance $\sqrt{R(R - 2r)}$, there are an infinite number of triangles inscribed in the R circle

and circumscribing the r circle. The triangle with the maximum perimeter is isosceles with the perimeter $2s_1$, while the one with minimum perimeter is also isosceles with the perimeter $2s_2$, where the quantities s_1 and s_2 as well as the lengths of the equal sides x_1 and x_2 are defined above. See **3070** [*1924*, 206]. If $R = 2r$, there are an infinite number of triangles, but they are all congruent equilateral triangles.

3091 [1924, 353]. Proposed by PHILIP FITCH, Denver, Colorado.

The top of a grain hopper is a square whose side is 10 feet. The sides of the hopper are portions of right circular cylindrical surfaces, 200 feet in diameter. If the cylindrical surfaces meet the plane of the top at right angles and if the hole in the bottom of the hopper is one foot square, what is the volume of the hopper and how many bushels of grain will be taken out by lowering the level of the grain one foot?

SOLUTION BY G. A. LYLE, Lehigh University.

Take for the cylindrical surfaces which form the sides of the hopper (1) $x^2 + (y + 5)^2 = 100$; (2) $x^2 + (z + 5)^2 = 100$; (3) $x^2 + (y - 5)^2 = 100$; (4) $x^2 + (z - 5)^2 = 100$. The planes of the square top and of the square hole in the bottom are then $x = 0$ and $x = \sqrt{279}/2$ respectively. The required volume is four times the volume in the first octant enclosed by the coördinate planes, the cylindrical surfaces (1) and (2) and the planes of the top and bottom.

$$4\int_0^{\sqrt{279}/2} y^2dx = 4\int_0^{\sqrt{279}/2} (125 - x^2 - 10\sqrt{100 - x^2})dx$$
$$= (297\sqrt{297}/2) - 2000 \sin^{-1} (\sqrt{297}/20) = 503.6 \text{ cu. ft.}$$

Amount of grain taken out when the level of the grain is lowered one foot is

$$4\int_0^1 (125 - x^2 - 10\sqrt{100 - x^2})dx = 99.3 \text{ cu. ft.} = 79.8 \text{ bushels.}$$

3092 [1924, 353]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

What must be the relations between the coefficients of a cubic equation in order that its roots considered as lengths, shall form a triangle?

SOLUTION BY NINA MAY ALDERTON, Mills College, California.

Suppose a, b, c to be the real positive roots of the equation

$$x^3 - px^2 + qx - f = 0.$$

Then p, q and f are all positive since $a + b + c = p$; $ab + ac + bc = q$ and $abc = f$. Now the area S of the triangle whose sides are a, b and c is

$$S = \sqrt{\frac{p}{2}\left(\frac{p}{2} - a\right)\left(\frac{p}{2} - b\right)\left(\frac{p}{2} - c\right)} = \tfrac{1}{4}\sqrt{p(p - 2a)(p - 2b)(p - 2c)}$$
$$= \tfrac{1}{4}\sqrt{p[p^3 - 2(a + b + c)p^2 + 4(ab + ac + bc)p - 8abc]}$$
$$= \tfrac{1}{4}\sqrt{p[4pq - p^3 - 8f]}.$$

Now since p is positive, the condition that S be a real positive quantity is

$$4pq > p^3 + 8f.$$

This condition together with the condition that p, q and f are all positive are the conditions that the roots of the cubic $x^3 - px^2 + qx - f = 0$ may represent the lengths of the sides of a triangle.

NOTE ON THE ABOVE SOLUTION BY H. L. OLSON.

In order that a, b and c be real it is necessary and sufficient that the discriminant,

$$D = 18pqf - 4p^3f + p^2q^2 - 4q^3 - 27f^2,$$

of the cubic be positive. Then if p, q and f are positive, the roots a, b and c will, by Descartes' rule of signs, be all positive.

It remains to be proved that each factor under the radical sign is positive. p, of course, is positive, and if $p - 2a$ and $p - 2b$, for example, were negative, their sum, $2c$, would be negative, contrary to previous proof.

Thus the necessary and sufficient condition that a, b and c form a triangle is that p, q, f,

$$18pqf - 4p^3f + p^2q^2 - 4q^3 - 27f^2,$$

and

$$4pq - p^3 - 8f$$

be positive.

3093 [1924, 353]. Proposed by FRANK MORLEY, Johns Hopkins University.

Show that the equation

$$y = (a_0 + a_1x + \cdots + a_nx^n)/(b_0 + b_1x + \cdots + b_nx^n)$$

gives the differential equation

$$\begin{vmatrix} y_1, & y_2, & \cdots & y_{n+1} \\ y_2, & y_3, & \cdots & y_{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ y_{n+1}, & y_{n+2}, & \cdots & y_{2n+1} \end{vmatrix} = 0,$$

where $y_r = (D_x{}^r y)/r!$.

Solution by J. A. Bullard, U. S. Naval Academy.

If $A = a_0 + a_1x + \cdots + a_nx^n$ and $B = b_0 + b_1x + \cdots + b_nx^n$, we have $By = A$ and, by Leibniz's Formula,

$$\sum_{h=0}^{h=n} \frac{(n+i)!}{h!\,(n+i-h)!}(D_x{}^{n+i-h}y)(D_x{}^hB) = 0 \quad (i = 1, 2, \cdots, n+1),$$

since $D_x{}^kA = D_x{}^kB = 0$ for $k > n$. Dividing by $(n+i)!$ we may write using notation of problem

$$\sum_{h=0}^{h=n} y_{n+i-h}B_h = 0 \quad (i = 1, 2, \cdots, n+1).$$

Eliminating B_h $(h = 0, 1, 2, \cdots, n)$ from these equations, we obtain

$$\begin{vmatrix} y_1, & y_2, & \cdots & y_{n+1} \\ y_2, & y_3, & \cdots & y_{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ y_{n+1}, & y_{n+2}, & \cdots & y_{2n+1} \end{vmatrix} = 0.$$

3096 [1924, 402]. Proposed by W. J. SIDIS, New York City.

In a scale of numeration whose radix is prime,

(1) There cannot be four distinct digits whose cubes all end in the same digit.

(2) If the cubes of two distinct digits end alike (that is, in the same digit), there will always be a third such digit.

(3) Under the conditions of (2), given any digit except 0, there will be two digits whose cubes have the same last figure as the cube of the given digit.

An extension of (2) and (3) is possible. If n is an odd prime, then, if the nth powers of two distinct digits end alike, there will be a group of n distinct digits whose nth powers end alike (that is, in the same figure), and such a group may be made to include any given digit not 0.

As an extension of (1), we may say that, under the original conditions, there cannot be $n + 1$ digits whose nth powers end alike.

Solution by H. L. Olson, Michigan Agricultural College.

Let the radix be a prime p, and let r be a primitive root, mod. p. If $a \not\equiv 0$ (mod. p), there is an integer α such that $a \equiv r^\alpha$ (mod. p); then a number $x \equiv r^\xi$ (mod. p) satisfies the congruence $x^n \equiv a$ (mod. p) if and only if $n\xi \equiv \alpha$ (mod. $p - 1$). Thus if n is a prime and not a factor of $p - 1$, this congruence has one and only one solution for each value of α. If n is a prime factor

of $p - 1$, this congruence has n solutions or no solution according as α is or is not divisible by n. Hence, in a scale of numeration whose radix is a prime p:

If n is not a factor of $p - 1$, there is one and only one digit whose nth power ends in any given digit not 0. If n is a factor of $p - 1$, there are n digits or no digit whose nth powers end in a digit $a \equiv r^\alpha$ (mod. p) according as α is or is not divisible by n. 0 is the final digit of the nth power of 0 and of no other digit. We have now proved the extensions of (1), (2) and (3) to any prime exponent n.

To extend the theorems to a composite exponent n, we need only to observe that if d is the greatest common divisor of n and $p - 1$, the congruence $n\xi \equiv \alpha$ (mod. $p - 1$) has d roots or no root according as α is or is not divisible by d. In other words, in a scale of numeration whose radix is a prime p:

There are d (and only d) digits or no digit whose nth powers end in a given digit $a \equiv r^\alpha$ (mod. p) according as α is or is not divisible by d, the greatest common divisor of n and $p - 1$. As before, we see that 0 is the final digit of the nth power of 0 and of no other digit.

3098 [1924, 455]. Proposed by JEAN WINSTON, University of Cincinnati.

Find the involutes of the parabola $y = x^2$.

SOLUTION BY E. M. BERRY, Purdue University.

The involutes of a given curve are the orthogonal trajectories of the family of straight lines tangent to the given curve.

The differential equation of the family of straight lines tangent to the parabola $y = x^2$ is

$$p^2 - 4px + 4y = 0, \qquad p = \frac{dy}{dx}, \tag{1}$$

as is easily seen by differentiating and substituting in the point-slope form for the equation of a line.

Substituting $- 1/p$ for p in (1) we get

$$4p^2y + 4xp + 1 = 0 \tag{2}$$

as the differential equation of the orthogonal trajectories of the family of straight lines represented by (1).

Solving equation (2) for x we get

$$x = -\frac{1}{4p} - py. \tag{3}$$

On differentiating with respect to y this becomes

$$\frac{1}{p} = \frac{1}{4p^2}\frac{dp}{dy} - y\frac{dp}{dy} - p$$

or

$$\frac{dy}{dp} + \frac{p}{1 + p^2}y = \frac{1}{4p(p^2 + 1)},$$

a linear equation whose solution is

$$y = \frac{1}{\sqrt{p^2 + 1}}\left\{\int\frac{dp}{4p\sqrt{p^2 + 1}} + c\right\}. \tag{4}$$

Putting $p = \tan\varphi$ and integrating we have

$$y = \tfrac{1}{4}\cos\varphi[\log\tan(\varphi/2) + c]. \tag{5}$$

Substituting (5) in (3) we get

$$x = -\tfrac{1}{4}[\cot\varphi + \sin\varphi\log\tan(\varphi/2) + c\sin\varphi]. \tag{6}$$

Equations (5) and (6) are the equations of the involutes of the parabola $y = x^2$. If $c = 0$, we have $x = 0$ and $y = 0$ when $\varphi = 90°$, which is the particular one of these curves which passes through the vertex.

Also solved by THEODORE BENNETT and PAUL CAPRON.

3100 [1924, 455]. Proposed by H. E. TREFETHEN, Colby College.

Show that the segment between the axes tangent to the astroid at any point and the radius of the fixed circle to its point of contact with the generating circle bisect each other.

SOLUTION BY S. F. BIBB, University of North Dakota.

Let P be a point on the branch APB of the astroid in the first quadrant, and let the generating circle through P touch at Q the fixed circle of radius $OQ = a$ with its center at the origin O. The equations of the astroid are

$$x = a \cos^3 \theta, \qquad y = a \sin^3 \theta,$$

where $\theta = \angle AOQ$. The equation of the tangent at P, MPS, is

$$y + x \tan \theta = a \sin \theta.$$

The intercepts of the tangent on the x and y axes are, respectively, $OM = a \cos \theta$ and $OS = a \sin \theta$. Therefore OM and OS are the projections of OQ on the axes, and it then follows that $OMQS$ is a rectangle. Hence OQ and MPS bisect each other at N, where NQ is a diameter of the generating circle.

NOTE BY THE EDITORS. All of the above results are easily obtained from the geometric definition of the astroid without the use of equations or differentiation.

Also solved by THEODORE BENNETT, PAUL CAPRON, and the PROPOSER.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Among the grants in aid of research announced by the American Association for the Advancement of Science for 1925 are the following:

$250 to Dr. H. W. STAGER, of the University of Washington, for building a computing machine for use in constructing a list of prime numbers and a factor table; $300 to Professor A. O. LEUSCHNER, of the University of California, for use in the study of the perturbations of the minor planets.

On the occasion of the third Pan-American Scientific Congress, the University of San Marcos, at Lima, conferred the degree of Honorary Doctor in the Faculty of Sciences on Professor E. V. HUNTINGTON, of Harvard University. Professor Huntington attended the Congress as delegate of the American Mathematical Society, the American Academy of Arts and Sciences, and the Mathematical Association of America.

Assistant Professor V. B. HINSCH, of the Missouri School of Mines, has been promoted to an associate professorship of mathematics.

Assistant Professor MARSTON MORSE, of Cornell University, has been appointed associate professor of mathematics at Brown University.

Dr. H. L. OLSON, of the University of Michigan, has been appointed assistant professor of mathematics at Michigan Agricultural College.

Dr. H. M. STONE has been appointed to an instructorship in mathematics at Columbia University.

Mr. J. S. GOLD has been promoted to be assistant professor of mathematics at Bucknell University.

Assistant Professor C. A. GARABEDIAN, of Northwestern University, has been appointed to an associate professorship of mathematics in the University of Cincinnati.

Miss LAURA F. MCDONOUGH, of the University of Pennsylvania, has been appointed head of the department of mathematics at the Moravian College for Women, Bethlehem, Pa.

Associate Professor LAO G. SIMONS, of Hunter College, has published a memoir on the "Introduction of algebra into American schools in the eighteenth century"; it appears as Bulletin, 1924, No. 18, of the U. S. Bureau of Education and can be obtained from the Superintendent of Documents, Washington, D. C., for fifteen cents per copy. "It is the purpose of this study to show that algebra entered into the American education of the eighteenth century, and to show further that we must seek some other reason for its presence than a practical need for it." The final chapter contains a chronological list of American text-books in algebra to 1820 and a fairly extensive bibliography.

Mr. E. C. RHODES, who served for several years as assistant to KARL PEARSON, comes to America this summer from the School of Economics, University of London, to give two courses of lectures at the Summer Session of Northwestern University. One course is entitled "An introduction to statistical method," the other "Advanced mathematics of statistics"; for the latter a semester of calculus is a prerequisite.

After 37 years of service, Professor ALEXANDER ZIWET of the University of Michigan will retire at the end of this semester. At a dinner held recently at the Michigan Union, he was given the following testimonial signed by the 32 members of the Mathematics Club of the University:

"We, the members of the Mathematics Club of the University of Michigan, desire to express to you our deep appreciation not only of your long and sincere interest in the welfare of the Club and of mathematical science at this University but of your high scholarly attainments in many branches of learning.

"We hold you, Sir, in profound respect, but more than that, in love and affection. Our signatures as inscribed below attest our enduring sentiments toward you."

A DEVELOPMENT OF FIFTY YEARS IN DESIGN AND QUALITY

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 8–9, 1925;
Tenth Annual Meeting, Kansas City, Mo., December, 1925.

The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Peoria, May 9–10, 1925
INDIANA, Bloomington, May 8–9, 1925
IOWA, State Teachers College, Cedar Falls, May 1–2
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 20–21
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, December 6, 1924
MICHIGAN, Ann Arbor, April 1, 1926
MINNESOTA, St. Johns Univ., Collegeville, May 16
MISSOURI, Kansas City, December, 1925
NEBRASKA, Creighton Univ., Omaha, May 2
OHIO, Ohio State Univ., Columbus, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 28
TEXAS, Dallas, November 27–28, 1925

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MEETING OF THE SOUTHERN CALIFORNIA SECTION.

The first regular meeting of the Southern California Section of the Mathematical Association of America was held in Los Angeles, on Saturday, February 28, 1925, Professor E. R. Hedrick presiding.

There were forty-six present, including the following members of the Association: O. W. Albert, M. A. Basoco, H. Bateman, M. M. Beenken, G. E. Berry, W. N. Birchby, F. P. Brackett, J. R. Campbell, Mrs. T. Clark, M. Collier, Mae E. Conn, P. H. Daus, G. G. Entz, H. E. Glazier, F. C. Hall, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, Deca Lodwick, G. F. McEwen, W. E. Mason, F. R. Morris, W. A. Newlin, L. E. Reynolds, Boris Podolsky, W. P. Russell, G. E. F. Sherwood, M. Skarstedt, D. V. Steed, H. C. Van Buskirk, L. E. Wear, H. C. Willett, Clyde Wolfe, E. Worthington.

A constitution was adopted and the following officers elected: Professor HARRY BATEMAN, California Institute of Technology, *Chairman*; Professor H. C. WILLETT, University of Southern California, *Vice-Chairman*; Professor P. H. DAUS, University of California, Southern Branch, *Secretary-Treasurer*; Professor P. H. DAUS, Professor W. P. RUSSELL, Pomona College, and Mr. G. R. LIVINGSTON, San Diego Junior College, *Program Committee*.

Professor Bateman then took the chair. The following papers were presented. A short abstract appears below.

(1) "Contrast between the two methods of expressing symmetric functions in terms of the coefficients of the corresponding algebraic equations." Professor O. W. ALBERT, University of Redlands.

(2) "Note on a table of discounts." Professor P. H. DAUS, University of California, Southern Branch.

(3) "A solution of the quintic equation." Professor GLENN JAMES, University of California, Southern Branch.

(4) "Mathematics of fluid convection as the mechanism of heat conduction in large bodies of water." Professor G. F. MCEWEN, Scripps Institute, University of California.

(5) "The generalized Pellian equation." Professor CLYDE WOLFE, California Institute of Technology.

1. One method is based on the fundamental theorem for symmetric functions and expresses the function as a part of the product of elementary symmetric functions. The other is based on the value of

$$S_k \equiv \sum x_1^k,$$

which may be found by using Newton's identities or Waring's formula. The former method is useful when the degree of every root is small and the number of roots large, while the latter method is useful when the degree of any root is large and the number of roots is small.

2. Professor Daus discussed the construction of a discount table used by a second mortgage company.

3. Professor James expressed the roots of the general quintic equation as functions of the coefficients and indicated an algebraic method of evaluating these functions in numerical cases.

4. Professor McEwen discussed the conduction of heat in large bodies of water based upon a mathematical theory of fluid convection and the graphical solution of the resulting differential equations.

5. Professor Wolfe discussed the generalization of the Pellian equation

$$x^2 - Dy^2 = 1 \text{ to } x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1.$$

P. H. DAUS, *Secretary*.

THE SOLUTION OF EQUATIONS BY THE METHOD OF SUCCESSIVE APPROXIMATIONS.[1]

By L. R. FORD, Rice Institute.

1. Introduction. The use of successive approximations in the solution of ordinary equations leads to a method of great power and versatility. Unfortunately, the method has received little attention in text books, excepting some very special cases, such as the Newton-Raphson method.[2] Moreover, the general theory is exceedingly simple, and is—at least in the case of the equation in one variable—within the comprehension of a student equipped with a first course in the calculus. It is the purpose of this paper to present the theory in its simpler aspects.

We shall begin with the equation in one variable, passing subsequently to a brief consideration of equations in any number of variables. We assume at first that the equation studied has a solution, the theory being particularly simple in this case; later we prove the existence of a solution under suitable hypotheses. In the formulation of the various theorems the guiding principle has been simplicity in statement and in proof. All the theorems are capable of extension in various directions, a fact which may possibly be of interest to some readers of the MONTHLY.

2. Fundamental Theorem. We shall assume that the functions considered are real and continuous and possess derivatives of the first order in the intervals in which they are considered. We shall be concerned with real solutions only.

THEOREM I. *Let X be a solution of the equation*

$$x = f(x), \tag{1}$$

[1] Presented at the meeting of the Texas Section, San Antonio, Texas, November 28, 1924.

[2] Among elementary texts, Osgood's *Differential and Integral Calculus* should be noted as an exception. The method is treated at some length in Whittaker and Robinson's *The Calculus of Observations*.

and let

$$|f'(x)| < M < 1 \tag{2}$$

in the interval $R : (X - h \leq x \leq X + h)$. *If* x_0 *be in the interval* R *and if* x_1, x_2, $\cdots$ *be found successively from the equations*

$$x_1 = f(x_0), \qquad x_2 = f(x_1), \qquad \cdots, \qquad x_n = f(x_{n-1}), \qquad \cdots \tag{3}$$

then

$$\lim_{n \to \infty} x_n = X.$$

We show first that if x_{n-1} is in R then x_n is also in R. Since X is a solution of (1), we have

$$X = f(X).$$

Applying the law of the mean to the second member of the following equation, we have

$$x_n - X = f(x_{n-1}) - f(X) = f'(\xi_n)(x_{n-1} - X), \tag{4}$$

where ξ_n lies between x_{n-1} and X, and hence lies in R. Then

$$|x_n - X| < M|x_{n-1} - X| \leq Mh < h, \tag{5}$$

and x_n is in R. Since x_0 is in R, it follows that x_1, x_2, $\cdots$ are in R.

By a repeated application of the inequality (5) we have

$$|x_n - X| < M|x_{n-1} - X| < M^2|x_{n-2} - X| < \cdots < M^n|x_0 - X|; \tag{6}$$

whence, since $M < 1$,

$$\lim_{n \to \infty} |x_n - X| = 0,$$

which proves the theorem.

Remarks on the Character of the Convergence. A consideration of the preceding inequalities brings to light a number of facts concerning the manner in which x_n approaches X. Thus we note from (5) that x_n is nearer X than x_{n-1} is; that is, each approximation is nearer the root than was the preceding approximation.

Again, we see from (4) that if the derivative is *negative* in R then $x_n - X$ and $x_{n-1} - X$ differ in sign; that is, the successive approximations are alternately greater than and less than the root. Hence the root lies between any two successive approximations. On the other hand, if $f'(x)$ is *positive* in R, then $x_n - X$ and $x_{n-1} - X$ have the same sign, and the approximations are either all greater than X or all less than X, depending upon whether x_0 is chosen greater than or less than X.

What limit can be put on the error of the nth approximation? A first limit is given by (6),

$$|x_n - X| < M^n|x_0 - X| \leq M^n h.$$

A second limit can be got in terms of the difference between successive approximations. From (4)

$$x_n - X = f'(\xi_n)[x_n - X - (x_n - x_{n-1})],$$

whence

$$x_n - X = -\frac{f'(\xi_n)}{1 - f'(\xi_n)}(x_n - x_{n-1}).$$

This gives the inequality

$$|x_n - X| < \frac{M}{1 - M}|x_n - x_{n-1}|.$$

If, however, $f'(x)$ is negative in R, the denominator, $1 - f'(\xi_n)$, is greater than 1, and we have the simple and strong inequality

$$|x_n - X| < M|x_n - x_{n-1}|.$$

First Illustration: The Quadratic Equation. Any equation can be written in the form (1) in an infinite number of ways. The essential thing is so to write it that $f'(x)$ is less than 1 in absolute value in the neighborhood of the root. Consider the equation

$$x^2 + ax + b = 0.$$

We shall assume that the roots, X and Y, are real and unequal. We have

$$a = -(X + Y), \qquad b = XY.$$

(a) Write the equation in the form

$$x = -a - \frac{b}{x}.$$

In this and the following examples we shall represent the second member by $f(x)$. Then

$$f'(x) = \frac{b}{x^2},$$

and at the root X

$$f'(X) = \frac{b}{X^2} = \frac{Y}{X}.$$

This will be numerically less than 1 if $|X| > |Y|$. Hence, in the neighborhood of the numerically larger, if any, of the two roots the derivative is numerically less than 1, and the method of successive approximations, applied in a suitable interval about the larger root, will yield the larger root.

(b) Writing the equation in the form

$$x = -\frac{b}{x + a},$$

we have

$$f'(x) = \frac{b}{(x+a)^2}$$

and

$$f'(X) = \frac{b}{(X+a)^2} = \frac{X}{Y}.$$

This form of the equation is satisfactory in the neighborhood of the numerically smaller of the two roots.

(*c*) Putting the equation in the form

$$x = -\frac{b+x^2}{a},$$

we have

$$f'(x) = -\frac{2x}{a},$$

and

$$f'(X) = -\frac{2X}{a} = \frac{2X}{X+Y}.$$

This form is satisfactory for finding X provided X lies between Y and $-Y/3$.

(*d*) Let us put the equation in the form

$$x = \frac{(a+p)x+b}{p-x},$$

where p is any constant. Then

$$f'(x) = \frac{p^2+ap+b}{(p-x)^2},$$

and

$$f'(X) = \frac{p^2-(X+Y)p+XY}{(p-X)^2} = \frac{p-Y}{p-X}.$$

If p be chosen near the root Y, the derivative is small in the neighborhood of X, and the convergence is rapid.

In each of the preceding cases we have shown the existence of an interval about one of the roots within which the method of approximations is valid, the conditions of Theorem I being satisfied. The conditions of that theorem are, of course, sufficient and not necessary. The method may yield a root if x_0 lies entirely outside the interval within which $|f'(x)| < 1$. In fact, it is known from the theory of linear transformations that when a quadratic equation with real roots is written in the form $x = (Ax+B)/(Cx+D)$, as in (*a*), (*b*), and (*d*), a definite one of the roots, X, is yielded by the method whatever value, real or complex, is selected for x_0 (other than $x_0 = Y$, if $Y \neq X$).[1] If x_0 is

[1] Excepting the special case $A + D = 0$, as in $x = 1/x$.

outside an interval within which the conditions of Theorem I hold, it will not follow that x_n is nearer X than x_{n-1} is.

Second Illustration: Effective Rate of Interest on a Bond.[1] In the theory of investment the problem of finding the interest return on a bond which is not purchased at par can be solved only by some method of approximation. The equation of value gives the formula

$$A = v^n + r\frac{1 - v^n}{i},$$

where A is the price paid for a bond of face value unity, r is the nominal rate of interest paid, n is the number of years to maturity, i is the effective rate of interest, and $v = 1/(1 + i)$. It is required to find i when A, r, and n are known.

Following are a few forms in which the equation of value can be written. The identity of these equations with the original equation is easily verified.

$$(a)\quad i = \frac{1}{A}[r + (i - r)v^n]; \qquad (b)\quad i = r\frac{1 - v^n}{A - v^n};$$

$$(c)\quad i = r - (A - 1)\frac{i}{1 - v^n}; \qquad (d)\quad i = \frac{1}{A}\left[r + \frac{(1 - A)iv^n}{1 - v^n}\right].$$

The derivatives of the second members with respect to i are respectively

$$(a)\quad \frac{1}{A}[v^n + (r - i)nv^{n+1}]; \qquad (b)\quad r(A - 1)\frac{nv^{n+1}}{(A - v^n)^2};$$

$$(c)\quad (1 - A)\frac{1 - v^n - inv^{n+1}}{(1 - v^n)^2}; \qquad (d)\quad \frac{(1 - A)v^n(1 - v^n) + inv^{n+1}}{A(1 - v^n)^2}.$$

In a great many practical problems i and r are small, A is near 1, and v^n is considerably less than 1. The value of nv^{n+1}, which appears in all the derivatives, is not obvious. We find that nv^{n+1}, regarded as a function of n, has a maximum when $n = 1/\log(1 + i)$. This maximum value is $v/e \log(1 + i)$. If $i \leq .1$, so that $\log(1 + i) > i(1 - i/2) \geq .95i$, we see that

$$nv^{n+1} < \frac{1}{.95ei} < \frac{.388}{i}.$$

If n is fairly large, so that v^n is sufficiently small, a glance will usually show that each of the derivatives is less than 1 in absolute value. If n is small, or if A or r have unusual values, a special investigation of the derivative may be necessary.

[1] The usefulness of the method for the solution of problems in investment was emphasized by C. H. Forsyth, *The Solution of Certain Problems in Finance by the Method of Iteration*, in the March number of the MONTHLY.

Example. Find the return on a 5 per cent. bond due in 20 years bought at 90.

Here $A = .9$, $r = .05$, $n = 20$. Making the guess $i = 5.5$ per cent., and applying each of the four formulæ, we find the following successive approximations:

(*a*)	.055,	.0576,	.0583,	.0585,	.05858,	.05861;
(*b*)	.055,	.0596,	.0585,	.05863;		
(*c*)	.055,	.0580,	.0586,	.05862;		
(*d*)	.055,	.0587,	.05863.			

The solution, correct to six decimal places, is .058621.

3. The General Equation in One Variable. Let X be a root of the equation

$$F(x) = 0.$$

We shall suppose that $F'(x)$ does not change sign in the neighborhood of X. Without loss of generality we can suppose that $F'(x)$ is positive. Specifically, we shall assume that there exists an interval $R : (X - h \leq x \leq X + h)$ within which

$$0 < a < F'(x) < b.$$

It follows from the non-vanishing of the derivative that X is a simple root.

We can now write the equation in the form (1) in a great variety of ways.

A. If $0 < c < 2/b$, *where c is a constant, then*

$$x = x - cF(x)$$

satisfies the conditions of Theorem I in the interval R.

Clearly this equation is equivalent to the preceding. Representing the second member by $f(x)$, we have

$$f'(x) = 1 - cF'(x).$$

In the interval R

$$-1 < 1 - cb < f'(x) < 1 - ca < 1.$$

Taking M as the greater of the two quantities $|1 - cb|$ and $|1 - ca|$, we have

$$|f'(x)| < M < 1.$$

We can generalize the preceding form of the equation by replacing the constant c by a function of x. We shall require that this function shall not vanish in R, so that no extraneous roots are introduced.

B. If in R the function $\varphi(x)$ *satisfies the inequality* $0 < \varphi(x) < 2/b$ *and* $\varphi'(x)$ *is bounded,* $|\varphi'(x)| < Q$, *then there exists a sub-interval,* $R' : (X - k \leq x \leq X + k)$, *of R within which the equation*

$$x = x - \varphi(x)F(x)$$

satisfies the conditions of Theorem I.

Since the limits 0 and $2/b$ are not actually attained in the closed interval R, there exist constants c', c such that

$$0 < c' < \varphi(x) < c < 2/b.$$

We have

$$f'(x) = 1 - \varphi(x)F'(x) - \varphi'(x)F(x),$$

and

$$|f'(x)| \leq |1 - \varphi(x)F'(x)| + Q|F(x)|.$$

Now

$$-1 < 1 - cb < 1 - \varphi(x)F'(x) < 1 - c'a < 1.$$

Let M' be the larger of the quantities $|1 - cb|$ and $|1 - c'a|$; then $M' < 1$, and

$$|f'(x)| < M' + Q|F(x)|.$$

Since $F(x)$ vanishes at X an interval $R' : (X - k \leq x \leq X + k)$ can be found such that, when M is chosen between M' and 1,

$$|F(x)| < \frac{M - M'}{Q}$$

in R'. Then

$$|f'(x)| < M' + M - M' = M < 1.$$

4. The Newton-Raphson Method. This well-known method consists in using the equation

$$x = x - \frac{F(x)}{F'(x)}$$

as a basis for the successive approximations. By putting $\varphi(x) = 1/F'(x)$ this falls directly under the preceding treatment if we so restrict the original interval that $a > \frac{1}{2}b$ (so that $0 < 1/F'(x) < 2/b$), and if we assume the existence of a bounded second derivative, $F''(x)$.

It is a simple matter to investigate the equation directly. We have

$$f'(x) = \frac{F''(x)F(x)}{[F'(x)]^2}.$$

By taking a suitably small interval about X we can make $F(x)$ and hence $f'(x)$ as small as we please, thus satisfying the requirements of Theorem I.

This method has one great advantage not possessed, unless exceptionally, by the other methods treated. I refer to its rapidity of convergence. We have $f'(X) = 0$. In the equation

$$x_n - X = f'(\xi_n)(x_{n-1} - X)$$

as n increases without limit ξ_n approaches X and $f'(\xi_n)$ approaches zero. Hence, the ratio of the error of the nth approximation to that of the $(n - 1)$th approaches zero.

We can put this in another way if we admit the existence of a bounded third derivative, $F'''(x)$. Then $f''(x)$ exists and is bounded, and we have

$$x_n - X = f(x_{n-1}) - f(X) = (x_{n-1} - X)f'(X) + \tfrac{1}{2}(x_{n-1} - X)^2 f''(\eta_n),$$

where η_n lies between x_{n-1} and X. Then

$$|x_n - X| < C|x_{n-1} - X|^2,$$

where C is an upper bound of $\frac{1}{2}|f''(x)|$.

5. Extensions of Theorem I. A practical problem in the application of the method of successive approximations concerns the possibility of replacing the functions involved by simpler and more manageable expressions, at least in the earlier stages of the process. Can we, for example, replace a function by a few terms of its expansion in a power series, or by a simple interpolation formula, or by values read from a rough graph, the approximation used becoming more accurate as the work proceeds? It will appear that we can do so under suitable conditions. Two sets of sufficient conditions will be given in this section.

THEOREM II. *Let X be a solution of the equation*

$$x = f(x);$$

and in an interval $R : (X - h \leq x \leq X + h)$ let

$$|f'(x)| < M < 1.$$

Let $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ be a sequence of functions such that in R

$$|f_n(x) - f(x)| < (1 - M)h, \qquad n = 1, 2, 3, \cdots;$$

and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ uniformly in } R.$$

If x_0 is in R and if $x_1, x_2, \cdots$ be found successively from the equations

$$x_1 = f_1(x_0), \qquad x_2 = f_2(x_1), \qquad \cdots, \qquad x_n = f_n(x_{n-1}), \qquad \cdots,$$

then

$$\lim_{n\to\infty} x_n = X.$$

We show first that if x_{n-1} is in R so also is x_n. We have

$$\begin{aligned} x_n - X = f_n(x_{n-1}) - f(X) &= f_n(x_{n-1}) - f(x_{n-1}) + f(x_{n-1}) - f(X) \\ &= f_n(x_{n-1}) - f(x_{n-1}) + f'(\xi_n)(x_{n-1} - X), \end{aligned}$$

where ξ_n, lying between x_{n-1} and X, is in R. Then

$$\begin{aligned} |x_n - X| &< |f_n(x_{n-1}) - f(x_{n-1})| + M|x_{n-1} - X| \\ &< (1 - M)h + Mh = h. \end{aligned}$$

Therefore x_n is in R. Since x_0 is in R, it follows that $x_1, x_2, \cdots$ are in R.

We now prove the convergence. Let ϵ_n be the least upper bound of $|f_n(x) - f(x)|$ in R. Since the convergence is uniform, $\lim_{n\to\infty} \epsilon_n = 0$. We have

$$|x_n - X| < \epsilon_n + M|x_{n-1} - X|.$$

By a repeated application of this inequality we get

$$|x_n - X| < \epsilon_n + M\epsilon_{n-1} + M^2\epsilon_{n-2} + \cdots + M^{n-1}\epsilon_1 + M^n|x_0 - X|.$$

Let r_n be the greatest of the quantities $(\sqrt{M})^n|x_0 - X|$ and $(\sqrt{M})^{n-i}\epsilon_i$, $i = 1, 2, \cdots, n$; then

$$|x_n - X| < r_n[1 + \sqrt{M} + (\sqrt{M})^2 + \cdots + (\sqrt{M})^n] < \frac{r_n}{1 - \sqrt{M}}\cdot$$

As n increases without limit r_n approaches zero, and

$$\lim_{n\to\infty} |x_n - X| = 0,$$

which completes the proof.

It will be noted that no requirements are put on the derivative of $f_n(x)$, not even that it exist. In fact, $f_n(x)$ need not be continuous. It may be remarked that any sequence of functions approaching $f(x)$ uniformly in R can be used as the sequence $f_n(x)$, provided we discard such members of the sequence, if any, at the beginning for which the inequality $|f_n(x) - f(x)| < (1 - M)h$ fails to hold.

THEOREM III. *Let X be a solution of the equation*

$$x = f(x).$$

Let $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ be a sequence of functions with the following properties in an interval $R : (X - h \leq x \leq X + h)$:

$$(a) \qquad |f_n'(x)| < M < 1, \qquad n = 1, 2, \cdots;$$

$$(b) \qquad |f_n(X) - f(X)| < h(1 - M), \qquad n = 1, 2, \cdots;$$

$$(c) \qquad \lim_{n\to\infty} f_n(X) = f(X).$$

If x_0 is in R and if $x_1, x_2, \cdots$ be found successively from the equations

$$x_1 = f_1(x_0), \qquad x_2 = f_2(x_1), \qquad \cdots, \qquad x_n = f_n(x_{n-1}), \qquad \cdots,$$

then

$$\lim_{n\to\infty} x_n = X.$$

If x_{n-1} is in R, we have

$$\begin{aligned} x_n - X &= f_n(x_{n-1}) - f(X) = f_n(x_{n-1}) - f_n(X) + f_n(X) - f(X) \\ &= f_n'(\xi_n)(x_{n-1} - X) + f_n(X) - f(X). \end{aligned}$$

Then

$$\begin{aligned} |x_n - X| &< M|x_{n-1} - X| + |f_n(X) - f(X)| \\ &< Mh + (1 - M)h = h; \end{aligned}$$

and x_n is in R.

Putting $|f_n(X) - f(X)| = \epsilon_n$, we have

$$|x_n - X| < \epsilon_n + M|x_{n-1} - X|, \quad \text{where} \lim_{n\to\infty} \epsilon_n = 0.$$

From this point the proof is completed, word for word, as in the preceding theorem.

A useful case under this theorem is that in which $f_n(X) = f(X)$ for all values of n. Then obviously conditions (b) and (c) hold. We can now generalize the method of solution of $F(x) = 0$ when put in any of the forms given in Propositions A and B or in the Newton-Raphson method. For, whatever constant c or function $\varphi(x)$ be chosen the second members are all equal at X. Theorem III shows that the process will converge if we employ

$$x_n = f_n(x_{n-1}),$$

where

$$f_n(x) = x - K_n F(x),$$

where K_n is any constant satisfying the conditions of Proposition A, or any function of x satisfying the conditions of Proposition B, or $K_n = 1/F'(x)$ in a suitably restricted region about X.

In practice we can employ our freedom in the choice of K_n in various advantageous ways. For example, we can use constant values in the earlier stages of the process and use the Newton-Raphson method in the final stages, thus taking advantage of the rapid convergence of the latter method. Or, following a suggestion of Whittaker and Robinson,[1] we can intermingle constant values of K_n with the value $1/F'(x)$. Thus, in getting x_n by the Newton-Raphson method we must compute $F'(x_{n-1})$, and we can use the reciprocal of this for a number of succeeding values of K_n. This will usually effect a saving of labor, since the derivative need not be recomputed at every step.

6. The Existence of a Solution. Hitherto we have assumed the existence of a root in the interval considered. Under suitable conditions we can prove that a root exists. We shall prove first the following simple theorem.

THEOREM IV. *If in an interval* ($a < x < b$)

$$|f'(x)| < M < 1,$$

then there is not more than one solution of the equation

$$x = f(x)$$

in the interval together with its end points.

For, suppose there are two solutions, X and Y. Then

$$X - Y = f(X) - f(Y) = f'(\xi)(X - Y),$$

where ξ lies between X and Y and hence is in the open interval. Then

$$f'(\xi) = 1,$$

which is contrary to hypothesis.

[1] *The Calculus of Observations*, p. 91.

THEOREM V. *If in an interval $R' : (x' - h \leq x \leq x' + h)$ a number M exists such that*

$$|f'(x)| < M < 1 \quad \textit{in } R',$$

and

$$|f(x') - x'| < h(1 - M),$$

then there is one and only one solution of the equation

$$x = f(x)$$

in R', and the method of successive approximations with any initial value x_0 in R' converges to it.

That there is not more than one root follows from Theorem IV. We show next that there is at least one root. Suppose the contrary. Then $x - f(x)$ has the same sign throughout the interval.

Let $x - f(x)$ be positive in R'. Then at x' and $x' - h$ we have

$$\begin{aligned} 0 < x' - f(x') &< h(1 - M), \\ f(x' - h) - (x' - h) &< 0. \end{aligned}$$

Adding,

$$h + f(x' - h) - f(x') < h(1 - M).$$

Applying the law of the mean,

$$h - hf'(\xi) < h(1 - M),$$

where $x' - h < \xi < x'$. Then

$$1 - f'(\xi) < 1 - M, \quad \text{or} \quad f'(\xi) > M,$$

which is contrary to hypothesis.

Similarly, if $x - f(x)$ is negative in R', we have at x' and $x' + h$

$$\begin{aligned} 0 < f(x') - x' &< h(1 - M), \\ x' + h - f(x' + h) &< 0. \end{aligned}$$

Adding and applying the law of the mean,

$$\begin{aligned} h + f(x') - f(x' + h) &< h(1 - M), \\ h - hf'(\xi) &< h(1 - M). \end{aligned}$$

This leads, as before, to a contradiction. Hence there is one and only one root in the interval.

Let X be the root. Let x_0 lie in R' and apply the method of successive approximations. The reasoning used in the proof of Theorem I can be employed to prove that x_n approaches X, provided all the approximations lie in R'. For,

the first inequality in (5) holds; namely,

$$|x_n - X| < M|x_{n-1} - X|,$$

whence the inequalities (6) and the convergence follow.

It remains to show that x_n lies in R' for all values of n. It suffices to show that if x_{n-1} is in R' then x_n is in R'. We have

$$\begin{aligned} x_n - x' &= f(x_{n-1}) - x' = f(x_{n-1}) - f(x') + f(x') - x' \\ &= f'(\xi_n)(x_{n-1} - x') + f(x') - x', \end{aligned}$$

where ξ_n is in R'. Then

$$\begin{aligned} |x_n - x'| &< M|x_{n-1} - x'| + h(1 - M) \\ &\leq Mh + h(1 - M) = h; \end{aligned}$$

hence x_n is in R'. This completes the proof.

COROLLARY. *If $|f'(x)| < M < 1$ for all values of x, then there is one and only one solution of the equation $x = f(x)$, and the method of successive approximations with any initial value converges to it.*

For, letting x' be any value of x we can choose h so large that the inequality $|f(x') - x'| < h(1 - M)$ is satisfied and that any preassigned value x_0 is within the interval $(x' - h \leq x \leq x' + h)$. Theorem V then applies.

Examples to which the corollary applies are

$$x = \frac{1}{2}\cos x, \qquad x = \frac{1}{1 + x^2}\,.$$

Suppose that we lay aside all consideration of the properties of $f(x)$, choose an initial x_0, and apply the method of successive approximations, on the theory that the proof of the pudding is in the eating. If it turns out that we can find each approximation, x_n, and if the sequence of approximations converges, is the resulting limit a solution? With the slight condition of continuity such is the case.

THEOREM VI. *Let x_0 be chosen, and let x_1, x_2, $\cdots$ be found successively from the equations*

$$x_1 = f(x_0), \qquad x_2 = f(x_1), \qquad \cdots, \qquad x_n = f(x_{n-1}), \qquad \cdots;$$

If

$$\lim_{n \to \infty} x_n = X,$$

and if $f(x)$ is continuous at X, then X is a solution of the equation

$$x = f(x).$$

Subtracting and adding the equal values x_n and $f(x_{n-1})$, we have

$$X - f(X) = X - x_n + f(x_{n-1}) - f(X).$$

Given any $\epsilon > 0$ we can choose n so large that $|X - x_n| < \epsilon$ and, since $f(x)$ is continuous at X, so that also $|f(x_{n-1}) - f(X)| < \epsilon$. Hence, $|X - f(X)| < 2\epsilon$. Since 2ϵ can be made as small as we please, it follows that $X - f(X) = 0$, which was to be proved.

7. The Solution of Equations in Several Variables. In this section we shall make a brief discussion of the solution of m equations in m variables, the analysis being developed along the lines suggested by the case of the equation in one variable. The method of successive approximations is often very useful in this problem—frequently difficult—of solving simultaneous equations.

THEOREM VII. *Let* $(X_1, X_2, \cdots, X_m)$ *be a solution of the equations*

$$x_i = f_i(x_1, \cdots, x_m), \qquad i = 1, \cdots, m,$$

and in the region $R : (X_i - h \leq x_i \leq X_i + h)$, $i = 1, \cdots, m$, *let*

$$\left|\frac{\partial f_i}{\partial x_j}\right| < M_{ij}, \qquad i, j = 1, \cdots, m,$$

where

$$M_{i1} + M_{i2} + \cdots + M_{im} < r < 1, \qquad i = 1, \cdots, m.$$

Let $({}_0x_1, \cdots, {}_0x_m)$ *be in the region* R *and let* $({}_1x_1, \cdots, {}_1x_m)$, $({}_2x_1, \cdots, {}_2x_m)$, $\cdots$ *be found successively from the equations*

$$\begin{aligned} &{}_1x_i = f_i({}_0x_1, \cdots, {}_0x_m), \qquad {}_2x_i = f_i({}_1x_1, \cdots, {}_1x_m), \qquad \cdots, \\ &{}_nx_i = f_i({}_{n-1}x_1, \cdots, {}_{n-1}x_m), \qquad \cdots, \qquad i = 1, \cdots, m; \end{aligned}$$

then

$$\lim_{n \to \infty} {}_nx_i = X_i, \qquad i = 1, \cdots, m.$$

The proof follows the lines of the proof of Theorem I. We show first that all the approximations are in R. Since the first is chosen in R, it will suffice to show that if $({}_{n-1}x_1, \cdots, {}_{n-1}x_m)$ is in R so also is $({}_nx_1, \cdots, {}_nx_m)$. We have

$${}_nx_i - X_i = f_i({}_{n-1}x_i, \cdots, {}_{n-1}x_m) - f_i(X_1, \cdots, X_m),$$

and, applying the law of the mean for a function of m variables,

$$\begin{aligned} {}_nx_i - X_i = \left(\frac{\partial f_i}{\partial x_1}\right)_{x={}_n\xi_1, \cdots, x_m={}_n\xi_m} ({}_{n-1}x_1 - X_1) + \cdots \\ + \left(\frac{\partial f_i}{\partial x_m}\right)_{x={}_n\xi_1, \cdots, x_m={}_n\xi_m} ({}_{n-1}x_m - X_m), \end{aligned}$$

where ${}_n\xi_i$ lies between ${}_{n-1}x_i$ and X_i, so that $({}_n\xi_1, \cdots, {}_n\xi_m)$ is in R. Then, representing the largest of the quantities, $|{}_{n-1}x_i - X_i|$, by N_{n-1}, we have

$$|{}_nx_i - X_i| < (M_{i1} + \cdots + M_{im})N_{n-1} < rN_{n-1} \leq rh < h,$$

therefore $({}_nx_1, \cdots, {}_nx_m)$ is in R.

The convergence follows from these last inequalities. We have $N_n < rN_{n-1}$, whence

$$\left| {}_nx_i - X_i \right| \leq N_n < rN_{n-1} < r^2N_{n-2} < \cdots < r^nN_0.$$

As n approaches infinity r^n approaches zero, hence ${}_nx_i$ approaches X_i.

8. The General Equations. The equations $F_i(x_1, \cdots, x_m) = 0$ can be written in the form $x_i = f_i(x_1, \cdots, x_m)$ in an infinite variety of ways. If the Jacobian of the functions F_i is greater than some constant greater than zero in the neighborhood of the solution, then the conditions of Theorem VII can be satisfied.

9. Generalization of the Newton-Raphson Method. For simplicity we shall treat three equations in three variables,

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0, \qquad H(x, y, z) = 0.$$

We assume the existence of bounded second derivatives in a region R enclosing the solution (X, Y, Z), and we assume that

$$J = \begin{vmatrix} F_x & G_x & H_x \\ F_y & G_y & H_y \\ F_z & G_z & H_z \end{vmatrix} \neq 0 \quad \text{in } R.$$

THEOREM VIII. *There exists a region R' in R within which the equations*

$$\left.\begin{aligned} x = x - \frac{1}{J}\begin{vmatrix} F & G & H \\ F_y & G_y & H_y \\ F_z & G_z & H_z \end{vmatrix}, \qquad y = y - \frac{1}{J}\begin{vmatrix} F_x & G_x & H_x \\ F & G & H \\ F_z & G_z & H_z \end{vmatrix}, \\ z = z - \frac{1}{J}\begin{vmatrix} F_x & G_x & H_x \\ F_y & G_y & H_y \\ F & G & H \end{vmatrix}. \end{aligned}\right\} \tag{a}$$

satisfy the conditions of Theorem VII.

It is easily established that these equations have no solutions in R other than those of $F = G = H = 0$. (On simplification we have three linear equations in F, G, and H whose determinant is a power of J, which is not zero.)

We shall now examine the partial derivatives of the second members, which we shall represent by f, g, and h respectively. We have

$$f_x = 1 - \frac{1}{J}\left\{\begin{vmatrix} F_x & G_x & H_x \\ F_y & G_y & H_y \\ F_z & G_z & H_z \end{vmatrix} + \begin{vmatrix} F & G & H \\ F_{xy} & G_{xy} & H_{xy} \\ F_z & G_z & H_z \end{vmatrix} + \begin{vmatrix} F & G & H \\ F_y & G_y & H_y \\ F_{xz} & G_{xz} & H_{xz} \end{vmatrix}\right\}$$
$$+ \frac{J_x}{J^2}\begin{vmatrix} F & G & H \\ F_y & G_y & H_y \\ F_z & G_z & H_z \end{vmatrix}.$$

The first determinant is equal to J, so that the 1 is cancelled. The remainder can be put in the form

$$f_x = AF + BG + CH, \tag{b}$$

where A, B, C are functions bounded in R. Similarly g_y and h_z can be put in this form.

$$f_y = -\frac{1}{J}\left\{\begin{vmatrix} F_y & G_y & H_y \\ F_y & G_y & H_y \\ F_z & G_z & H_z \end{vmatrix} + \begin{vmatrix} F & G & H \\ F_{y^2} & G_{y^2} & H_{y^2} \\ F_z & G_z & H_z \end{vmatrix} + \begin{vmatrix} F & G & H \\ F_y & G_y & H_y \\ F_{yz} & G_{yz} & H_{yz} \end{vmatrix}\right\}$$
$$+ \frac{J_y}{J^2}\begin{vmatrix} F & G & H \\ F_y & G_y & H_y \\ F_z & G_z & H_z \end{vmatrix}.$$

The first determinant is zero, and f_y is of the form (b). The same is true of the remaining derivatives.

By taking a sufficiently small region about (X, Y, Z), F, G, and H can be made as small as desired; hence the partial derivatives can be made as small as desired. Hence, the conditions on the derivatives in Theorem VII can be satisfied.

The formulæ used here have the same advantage possessed by the Newton-Raphson formula in one variable; as the root is approached the partial derivatives all approach zero, and the convergence is increasingly rapid.

10. Analogues of Propositions A and B. Probably the simplest form for computation is the following:

$$x = x - a_1F - b_1G - c_1H, \qquad y = y - a_2F - b_2G - c_2H,$$
$$z = z - a_3F - b_3G - c_3H,$$

where $a_1, \cdots, c_3$ are constants. If here the constant coefficients are sufficiently near the variable coefficients of F, G, and H in (a), the conditions on the derivatives in Theorem VII are satisfied in a region about (X, Y, Z). For, by taking a_1, b_1, and c_1 sufficiently near to

$$\frac{1}{J}\begin{vmatrix} G_y & H_y \\ G_z & H_z \end{vmatrix}, \qquad \frac{1}{J}\begin{vmatrix} H_y & F_y \\ H_z & F_z \end{vmatrix}, \qquad \frac{1}{J}\begin{vmatrix} F_y & G_y \\ F_z & G_z \end{vmatrix},$$

f_x and f_y are arbitrarily near to

$$1 - \frac{1}{J}\left\{\begin{vmatrix} G_y & H_y \\ G_z & H_z \end{vmatrix} F_x + \begin{vmatrix} H_y & F_y \\ H_z & F_z \end{vmatrix} G_x + \begin{vmatrix} F_y & G_y \\ F_z & G_z \end{vmatrix} H_x\right\}, \quad \text{and}$$
$$-\frac{1}{J}\left\{\begin{vmatrix} G_y & H_y \\ G_z & H_z \end{vmatrix} F_y + \begin{vmatrix} H_y & F_y \\ H_z & F_z \end{vmatrix} G_y + \begin{vmatrix} F_y & G_y \\ F_z & G_z \end{vmatrix} H_y\right\},$$

both of which are zero. Similarly the other derivatives are in the neighborhood

of zero. Hence the conditions of Theorem VII can be satisfied. It will be noted that no second partial derivatives are used here, so that their existence, which was necessary in the Newton-Raphson method, is not required.

Again, the constants of the preceding paragraph may be replaced by functions of x, y, and z if these functions are sufficiently close to the coefficients in (a). Then

$$f_x = [1 - a_1F - b_1G - c_1H] - [a_{1x}F + b_{1x}G + c_{1x}H].$$

In this derivative, which is typical, the first bracket is small and the second can be made as small as desired by restricting the region. We can then satisfy the conditions of Theorem VII.

IN THE SURNAMED CHOSEN CHEST.

By DAVID EUGENE SMITH, Columbia University.

I. *Association Copies.*[1]

This cryptic title, from "A Death in the Desert," shows that Browning was a genuine bibliophile, one who loved to speak of the ancient "parchment, of my rolls the fifth," which "lies second in the surnamed Chosen Chest." The title is manifestly selected for this paper merely for the purpose of arousing at least a slight degree of curiosity, of securing the attention of those who love old books, and of tempting some friend, or friends, to taste the pleasures of knowing of other chosen chests than those which he, or they, may guard with jealous care.

The article simply mentions a few of the more important "association copies" in the writer's library. The purpose is to let others know that in this country there are slowly forming nuclei of bibliographic rarities such as we may properly confess to envy in the great collections of the Old World,—nuclei that will naturally have an influence upon our scientific collections in university, public, and private libraries as the years go on.

The books might have been listed as in one of those dreary catalogues whose soulless cards seem to warn a true booklover to abandon hope. This would have been the method of science, but not the method of art; the method of the book-stacks, not that which harmonizes with a chosen chest.

Few interesting association copies of the sixteenth century find place in modern private libraries. Such volumes have usually long since been lost or have been permanently acquired by the large public libraries abroad. One of the most interesting ones in this collection is the second edition of Tonstall's *De Arte Svppvtandi* (Paris, 1529) with the autograph of Thomas Digges,[2] who died in 1595 and whose contributions to mathematical literature were, considering the time, not without interest.

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[1] This article is the first of a series. EDITOR.

[2] He wrote, with his father Leonard, *An Arithmeticall Militare Treatise, named Stratioticos* (1572), and published his father's *Pantometria.* All such autographs and inscriptions have been carefully authenticated.

We know little of Rafael Bombelli, who wrote the first book to be printed under the single nominative of algebra.[1] This little is told in substance in any of our recent histories. One of the pleasant bits of information, which at least tells us that there were others of his family, lies in a copy of the first edition presented by his grandson Giovanni (Joannes) and bearing upon the title page the words "Dona˜ [*Donavit*] Exim˜ [*Eximius*] D. [*Dominus*] Doc˜ [*Doctor*] Jois [Joannes] Bombelli mihi Joi Bacialtio." The book therefore originally belonged to the grandson of Rafael Bombelli, the author, and he had written these words: "Doctoris Ioannis Bombelli huiusce Auctoris Nepotis ex filio," the word *nepos* evidently meaning in this case grandson instead of nephew. So it appears that the author had a son; that this son had a son, Doctor Giovanni (Joannes) Bombelli; and that this Giovanni had given the book to one Joannes Bacialtius (probably Giovanni Bacialcio). The book has numerous marginalia, some in the hand of Bacialtius, but most of them written by Giovanni Bombelli.

Vincenzo Viviani[2] took pride in speaking of himself as "Postremus Galilei Discipulus," having studied with the great Pisan master in the last days of the latter's life. Among his works one of the best known is his *De Maximis, / et Minimis / geometrica divinatio / In Qvintvm Conicarvm / Apollonii Pergaei / adhvc desideratvm* (Florence, 1659). The copy in this library was presented by him to Nicolaus Heinsius, the great Dutch bibliophile (1620–1681) at Leyden, and bears the inscription "Clarissimo, Eruditissimo, Doctiss°q· Viro D. Nicolao Heinsio Deditissimus Auctor D. D." (for *donum dat*). Beside it rests the vellum-bound *Bibliotheca Heinsiana / sive / Catalogvs / Librorum, / Quos, magno studio, & sumtu / dum viveret, collegit / vir illvstris / Nicolaus Heinsius, / Dan. Fil.*[3] (Leyden, 1682), the catalogue of the sale of this extensive library. It is a marked copy containing the prices received for the various books, and this particular work by Viviani then brought the equivalent of $1, which was rather high for a mathematical folio at that time, Regiomontanus's *De Triangulis* going for 60 cents and Bachet's *Diophantus* for only five cents more.

Another folio in the library was bought at about the same time and price, but not at this sale. It is the *Historia / Rervm in Orien-/te Gestarvm ab Exor-/ dio Mvndi et Orbe condito / ad nostra haec vsqve tempore* (Frankfort a. M., 1587). It bears the owner's autograph and the price he paid for it,—"Is. Newton, pret $4^s\ 6^d$."

When Joseph Raphson published his *Analysis Æquationum Universalis*, in 1690, he was already on friendly terms with the owner of the folio just men-

[1] *L'Algebra/ parte maggiore / dell' arimetica / divisa in tre libri / di Rafael Bombelli / da Bologna.* Bologna, 1572. In spite of the "Nouamente poste in luce," this was the first edition. Bombelli was born *c.* 1530. There were, however, various earlier works in which the name was embodied in the title page, such as Scheubel's algebra, Paris, 1551.

[2] Born at Florence in 1622; died there in 1703.

[3] The son of Daniel (1580–1655), who was the greatest Dutch scholar of his time.

tioned,—the man whose claims to the invention of the calculus he was later to champion in *The History of Fluxions* (London, 1715, posthumous). It was therefore natural that he should present a copy of his work to Newton. It was an essay of only forty-four pages, but was of quarto size and contained a considerable amount of material. He wrote on the fly leaf, "To Mr. Isaack Newton, with my most humble service. I. R."

Newton had also been receiving certain other publications in quarto size about the same time, among them the following: *Abregê des Observations & des Reflexions sur la Comete . . . de . . . 1680 . . . Par Mr Cassini* (Paris, 1681), *Observations sur la Comete qui a paru au mois de Decembre 1680. et en Janvier 1680,* also by Cassini (Paris, 1681), *Reglement ordonnê par le Roy pour l' Acadêmie Royale des Sciences* (Paris, 1699), De Lagny's work on roots (2d ed., Paris, "M. DC. LXCII," for 1692), and Huygens's *Astroscopia Compendiaria* (The Hague, 1684). On the wide margins of the second of the Cassini documents Newton made various notes, chiefly on the position of the comet as seen in England. He then wrote on the first fly leaf "Miscellanea Mathematica" and sent the lot to the binder. The latter bound them up with the title which Newton suggested, and then, with the usual lack of intelligence shown by the ordinary members of his guild, trimmed the edges, thereby cutting off portions of the writing.

The book is valuable, however, as evidence of the relations between Newton and certain of his contemporaries, and like all association copies of his it has come to rank among the rarities.

The *Synopsis Palmariorum Matheseos* by William Jones, published in London in 1706, is well known as being the first book in which π is distinctly used to represent the number 3.14159 . . . . The statements appear on page 263 in the form "3.14159, &c. = π," and "$d = c \div \pi$." The book is not of great rarity, but this particular copy is unique because of a large number of corrections and of manuscript pages interleaved. A written memorandum of the source of these notes is given on one of the fly leaves by an early owner, one T. Todd, in these words:

"The papers and notes interspersed in several places of this Book was copied by me from one of Mr. Jones Synopsis's which Mr. Reuben Burrow (now in India) lent me, and who told me they were wrote by a Gentleman [1] who took them from Mr. Jones's own Papers, and was intended for a new edition of this Book. There are some few notes besides, which are mentioned not his.—T. Todd."

From the standpoint of simple algebra the notes are interesting as showing the development of the author's views after his book was published, but their importance is only historical. The most extensive additions are concerned with fluxions, annuities, and the theory of quadratics.

Bishop Berkeley's *The Analyst; or, A Discourse Addressed to an Infidel Mathematician,* published anonymously "By the Author of The Minute Philoso-

[1] "I think this gentleman lived in the Tower, and his name Mr Campbell." [Note by Todd.]

pher" in London in 1734, is well known for the searching inquiry that it made into the bases of the Newtonian theory of fluxions. It has a special interest to Americans because of Dr. Berkeley's visit to this country and his influence upon the religious and philosophical thought of the early period of our national development. The book itself is not particularly rare, but this copy is interesting because it bears this inscription in the hand of Sir Thomas Hanmer (1677–1746), speaker of the House of Commons and editor of Shakespeare's works (1743–1744): "Written by Dean Berkley and by him Given to me." It afterward passed into the hands of Sir Henry Edward Bunbury (1778–1860), who wrote the *Correspondence of Sir Thomas Hanmer, Bart.* (London, 1838), and whose autograph statement of its history is on the fly leaf opposite his bookplate.

The Geometry of Curve Lines by Sir John Leslie (1766–1832) was published at Edinburgh in 1813. As the author says, "I have contented myself with printing a very limited number of copies, merely for the accommodation of my students; but as soon as I can find sufficient leisure, I propose to complete the task, and endeavour, by improving the arrangement, and expanding the contexture of the work, to render it more fit for the public eye." This essay, limited in number though it was, can hardly lay claim to any extreme rarity; but this copy, inscribed in Leslie's hand "A Monsieur Arago de la part de l'Auteur," is a pleasant reminder of those through whose hands it onetime passed.

The life of Charles, Baron Dupin (1784–1873) is well known, partly because of his contributions to mechanics and differential geometry, and partly because of the many activities of this one of the "trois Dupins," as the brilliant trio of brothers were called. He made frequent visits to England and was much interested in the life of the country and the achievements of her scholars. Among those with whom he came in contact was Archibald Constable (1774–1827) of Edinburgh, a man ten years his senior, first publisher of the *Edinburgh Review* (1802) and the one-time owner of the rights in the *Encyclopædia Britannica.* Constable's bankruptcy (1826) was connected with the severe financial embarrassment of Sir Walter Scott. Whenever, during a period of several years, Dupin published a new memoir he apparently sent a copy to Constable, sometimes with the inscription "Offert par l'auteur, Ch. Dupin," sometimes more affectionately "à mon frère, Ch. Dupin," and at other times in English or in a bilingual note like "offert par l'auteur to his friend Mr. A. Constable, Ch. Dupin." These memoirs Constable had bound in tooled calf, and these too are now in the company of others such as those already described.

Of all the rarities in the collection, no one has greater interest than the first volume of Libri's *Histoire des Science Mathématiques en Italie.* Libri relates that he had taken from the printing establishment "eight or ten" copies of the work, in unbound sheets, but that the day after the publication a fire consumed the

entire stock. He at once set to work to prepare the copy for the new edition, but it was three years before this so-called first edition appeared, in 1838. He evidently prepared two copies, one for himself and the other, with somewhat more extended corrections, for the printer. His own copy, with hundreds of addenda and corrigenda, he gave in his later years to his friend Professor Ferdinando Jacoli, of Venice, from whom I obtained it in 1904, and who, at my request, wrote a memorandum of the circumstances mentioned. This edition bears the date 1835, and was printed about the close of that year. It would be interesting to know how many of the "eight or ten" copies still survive, and whether the other one with Libri's notes has been preserved.

Of the works of the nineteenth century there are various presentation copies in the library. Among them is the *Letters of Euler . . . to a German Princess,* edited by Dr. David Brewster, with his autograph, "To Miss Jane Grant With Dr. Brewsters Comp^{ts}."

In the "Report on Mathematical Tables," in the *Report of the British Association* for 1873, page 31, there is a note on *A Table of the Circles arising from the Division of a Unit . . . by all the integers from 1 to 1024* (London, 1823). In this collection is a copy of the work with the inscription "W. George Horner, Bath," a man who is known to all algebraists because of Horner's Method (1819) of solving numerical higher equations. Horner (1786–1837) made a large number of notes in the book and was evidently much interested in the study.

The *Rara Mathematica* of James O. Halliwell (afterwards Halliwell-Phillips, 1820–1889), the well-known but unfortunate Shakespearean scholar and bibliophile, went through two editions. It is one of the most successful presentations of early texts by English writers on algorism, optics, astronomy, and kindred subjects, and includes also the "Carmen de Algorismo" of Alexandre de Villedieu. With respect to the latter, Chasles had contributed certain information to Halliwell, and it was therefore natural that the latter should present him with a copy of this work, which he does with the inscription "à Monsieur Chasles Offert par l'Editeur J. O. H." The book bears the bookplate "Ex Bibliotheca Michaelis Chasles Acad. Scientiar. Socii." with the monogram M. C.

The *Analytical View of Sir Isaac Newton's Principia,* by Lord Brougham and E. J. Routh (London, 1855), does not seem to have been received with much enthusiasm as a scientific contribution, but it is a worthy though rather prolix treatise, the mathematical part being due, no doubt, entirely to Mr. Routh, afterwards professor at the Woolwich Military Academy, and author of various standard treatises on dynamics.

The work is not rare, but a presentation copy with a four-page letter from Brougham naturally has considerable interest. In the letter, which was written

from Cannes in 1855 (probably December 31, but he wrote an execrable hand and it is always difficult to read his lines), he has added an interesting postscript relating to De Morgan and to Bulwer-Lytton: "Should you see our friend Prof. De.M. pray ask him if he has heard that Sir E. B. L. is writing a Novel on spirit manifestations? This rumor has reached me from Germany but I know no more."

In the collection is also a presentation copy "From the Author" of Lord Brougham's *Tracts, Mathematical and Physical* (London and Glasgow, 1860), besides some 350 of his autograph letters.

Lord Brougham's address on "The Monument to Sir Isaac Newton" first appeared in his little volume of *Addresses on Popular Literature* in 1858. The address was delivered at Grantham on September 21 of that year. Immediately after its publication (in fact, on October 27 of the same year), his lordship sent to Mr. Law, the publisher, a copy of the work with a large number of corrections (nearly seventy on the Newton essay alone), which corrections were to be embodied in the second edition. They were inclosed in an envelope addressed to Law and autographed "Brougham," and the former prized them so highly as to bind the original, the notes, and the envelope, and in this form the volume now has at least a temporary resting place in the company of other association copies mentioned in these notes.

When Prince Boncompagni learned of the existence of the manuscript of Chuquet (written in 1484; original Codex Colbert 2170 Regius 7482), in the Bibliothèque Nationale, he sent to Professor Aristide Marre, at that time (about 1880) one of the best-known scholars of Paris, asking him to transcribe the work for publication in the *Bullettino di Bibliografia e Storia delle Scienze matematiche e fisiche*. It appeared in Vol. XIII, page 555, supplemented by extensive notes. This was before the days of photostats (or rotographs), and so it needed a man of Marre's attainments to make a copy with sufficient precision for the purposes. This copy, of 147 folios (294 pages), was prepared with great care, the writing being somewhat in imitation of the original, and it bears numerous signed notes by Marre himself.

Boncompagni preserved the manuscript and at his death it came, with all the rest of his great library, to the auction room and finally found its present lodging in the company of these other rarities.

Three years after Sir William Rowan Hamilton published his *Lectures on Quaternions* (Dublin, 1853) he was moved to present a copy to the Earl of Burlington. The copy bears the inscription "Presented to The Earl of Burlington, &c &c &c with the respectful regards of the author. Observatory, Dublin, April 18th, 1856." The Earl, who had perhaps been visiting the observatory that day, later took the pains to open the first sheet, perhaps to see to whom the book was dedicated, but that was all; for the rest he had no use. As an association copy

it would not be quite proper for a later owner who is something of a bibliophile to go further, and so the book still remains with not a single page of the text opened by the paper knife.

At the time of the death of Augustus De Morgan (March 18, 1871) he had substantially completed his well-known *Budget of Paradoxes*. The work was made up in large part of numerous articles and letters of his which had already appeared in the *Athenæum*, but these he had carefully revised, supplementing them by many new pages of manuscript. Upon his death Mrs. De Morgan arranged the material for publication, wrote a preface, and saw the book through the press. The original manuscript and corrected articles, all in De Morgan's hand, and the manuscript of the preface, are now in this collection, as is also a copy of the first edition (1872), specially bound in vellum for the library of Mrs. De Morgan. After her death the manuscript and this copy passed to William De Morgan, the novelist, and at the latter's death they were acquired for this collection.

In the second edition of De Morgan's *Budget of Paradoxes* (Chicago, 1915) there is a note (Vol. I, p. 290) on the Sheepshanks-Babbage-De Morgan controversy. On the death of Sheepshanks in 1855, De Morgan read before the Royal Astronomical Society a memoir speaking of his life and work. This moved Sir James South, an astronomer of prominence, to attack De Morgan for his defense of Sheepshanks against a charge that he had defrauded the customs. All this stirred up a good deal of bitterness, correspondence, and newspaper controversy. This material De Morgan brought together in a folder and kept until he died, after which, in due time, it all came into this collection. It is now past history; otherwise its publication would be interesting.

Of other association copies of the last two generations it is not intended to speak. Such copies can rather easily be secured. Two exceptions will be made, however; one of them because of its American interest. When Professors Campbell and Garnett published the *Life of James Clerk Maxwell* (London, 1882) they referred (p. 41) to the fact that Maxwell was reported slow at learning and "that the tutor was rough. He [the tutor] was probably a raw lad, who having been drilled by harsh methods had no conception of any other, . . . He had, in fact, tried to coerce Clerk Maxwell."

The book found its way into the hands of this same tutor James Middlemiss, who had emigrated to Elora, Ontario, and who at once wrote to Professor Campbell a letter of some eighteen pages, giving his side of the case and describing the youth of his pupil. This letter the recipient placed in what was apparently his own copy of the work, and there it still remains, unpublished.

A single other one of the later items may be mentioned because it is unique. As is well known, the most important contribution to our knowledge of early

Hindu mathematics made in recent times is the Ganita-Sāra-Sangraha of Mahāvīra, transliterated from Canarese into Sanskrit characters by the late Professor Rangachārya of Madras, and by him translated into English. Among the valued items in the collection are two bound volumes of the proofsheets of the work with the corrections to the translations in the translator's own hand.

There are those who will look at such a list and ask the ancient question, *Cui bono?* For such the answer is that there is no good,—for them. Others will have a different feeling, one of personal interest in the makers of mathematics and in the evidences of their friendships and their possessions. Still others will go to their own "chosen chests" and take pleasure in the fact that they have also helped to bring to this country such bibliographic items, while (let it be hoped) planning to place them eventually in some great library where they will be preserved for the future use of all scholars, or at least of those who are glad to rank as biographers and bibliophiles.

A SOLUTION OF THE QUADRATIC CONGRUENCE MODULO p, $p = 8n + 1$, n ODD.

By P. A. CARIS, University of Pennsylvania.

1. Introduction. The form of explicit solutions of the quadratic congruence with prime modulus depends on the form of the modulus. If p be the prime used as modulus and if p be of the form $4q + 3$ or of the form $8n + 5$, explicit solutions can be and have been obtained. A summary of these cases is given below. The purpose of the discussion which follows will be to find explicit solutions in the case of $p = 8n + 1$ with n an odd number. For the sake of brevity, (p) will be used to represent (mod p).

If p is a prime of the form $4q + 3$, the congruence $x^2 \equiv a(p)$ has the solution $x \equiv \pm a^{q+1}(p)$.

If p is a prime of the form $4q + 1$, the solution of the above congruence becomes more difficult. If q is odd, let $q = 2n + 1$. Then $p = 8n + 5$. In this case if $a^{2n+1} \equiv 1(p)$, the solution is $x \equiv \pm a^{n+1}(p)$. But if $a^{2n+1} \equiv -1(p)$, the solution is $x \equiv \pm (4n + 2)!\,y^n(p)$, where $ay \equiv 1(p)$. The last result involves a solution of $x^2 \equiv -1(p)$ by Wilson's Theorem.

If q is even, let $q = 2n$. Then $p = 8n + 1$. In this case a solution (not explicit) may be obtained by the use of Gauss' "method of exclusion." [1]

2. Concerning Quadratic Non-residues of p. Let p be a prime of the form $8n + 1$ and let a be a quadratic residue of p. When n is odd it is proposed to find explicit expressions for x such that the congruence $x^2 \equiv a(p)$ may be satisfied.

[1] On these and related points the following may be consulted: Wertheim: *Elemente der Zahlentheorie*, pp. 207–216; Dickson: *History of the Theory of Numbers*, vol. I, p. 213; Matthews: *Theory of Numbers*, pp. 53–55; Smith, H. J. S.: *Coll. Math. Papers*, vol. I, pp. 146–149.

In the discussion of this problem much use will be made of a quadratic non-residue of p. It is advisable, therefore, to make some preliminary remarks on the question of finding a quadratic non-residue.

Theoretically a primitive root of p answers the question completely but the method of finding a primitive root is usually tedious enough to make the use of one of the following truths desirable:

1. The number 3 is a non-residue of p unless n is divisible by 3.

For any number n, $n \equiv 0, 1, 2 \pmod 3$; $8n \equiv 0, 2, 1 \pmod 3$; $8n + 1 \equiv 1, 0, 2 \pmod 3$. Now if $8n + 1$ be a prime, it is not congruent to 0 (mod 3). Hence $n \equiv 1 \pmod 3$ is impossible in the present discussion. If $n \equiv 2 \pmod 3$, $p \equiv 2 \pmod 3$. Now 2 is a non-residue of 3. Hence p is a non-residue of 3. Since p is of the form $4q + 1$, 3 is a non-residue of p. But if $n \equiv 0 \pmod 3$, $p \equiv 1 \pmod 3$ and 3 is a residue of p. The truth of the first statement is therefore apparent.

2. The number 5 is a non-residue of p if p ends with the digits 3 or 7.

For p is then congruent to 3 or 2 (mod 5) each of which is a non-residue of 5.

3. Any prime factor of the form $4h + 3$ of $p + z^2$, where z is an integer, is a non-residue of p.

By hypothesis $p \equiv - z^2 \pmod{\overline{4h + 3}}$ but z^2 is a residue of $4h + 3$ while $- 1$ is not. Hence p is congruent to a non-residue of $4h + 3$ and since p is of the form $4q + 1$, $4h + 3$ is a non-residue of p.

3. Proof of a Fundamental Lemma. Assuming then foreknowledge of a non-residue of p, consider first the following lemma: If two numbers u, v, where v is prime to p, can be found such that $u^2 - av^2 \equiv 0(p)$, then a solution of $x^2 \equiv a(p)$ can be found. Consider the linear congruence $\lambda p \equiv - u \pmod v$. By hypothesis, p and v are relatively prime. Hence this congruence is always solvable. Suppose λ_0 is a root and let

$$\lambda_0 p = - u + vx. \tag{1}$$

Then $u = vx - \lambda_0 p$. Now by hypothesis $u^2 \equiv av^2\ (p)$, that is,

$$v^2x^2 - 2\lambda_0 pvx + \lambda_0^2 p^2 \equiv av^2\ (p) \qquad \text{or} \qquad v^2x^2 \equiv av^2\ (p).$$

Now since v is prime to p, this last congruence implies $x^2 \equiv a\ (p)$.

4. Solution of $x^2 \equiv - 1(p)$. Let ξ be a non-residue of p. Then by Euler's criterion

$$\xi^{4n} \equiv - 1(p). \tag{2}$$

That is, $(\xi^{2n})^2 \equiv - 1(p)$ or $x \equiv \xi^{2n}\ (p)$ is a solution of $x^2 \equiv - 1(p)$ and hence

$$x \equiv \pm \xi^{2n}\ (p) \tag{3}$$

is the general solution.

5. Solution of $x^2 \equiv 2(p)$. From (2) $\xi^{4n} + 1 \equiv 0(p)$. That is,

$$(\xi^{2n} + 1)^2 - 2(\xi^n)^2 \equiv 0(p).$$

ξ^n is prime to p, for ξ is less than p. Hence the conditions of the lemma are satisfied and $u = \xi^{2n} + 1$, $v = \xi^n$.

Consider equation (1) of the proof of the lemma. It is not really necessary actually to find λ_0, for (1) may be written:

$$- vx \equiv - u(p). \tag{4}$$

That is, in this case $- \xi^n x \equiv - \xi^{2n} - 1(p)$. Multiplying by ξ^{3n} and replacing ξ^{4n} by $- 1$, this gives $x \equiv \xi^n - \xi^{3n}(p)$ which is $x \equiv - \xi^n(\xi^{2n} - 1)(p)$ and hence the general solution of $x^2 \equiv 2(p)$ is

$$x \equiv \pm \xi^n(\xi^{2n} - 1)(p). \tag{5}$$

Corollary: $x \equiv \pm 2n\xi^n(\xi^{2n} + 1)(p)$ is the general solution of $x^2 \equiv n(p)$. $(4n)^2 = 16n^2 \equiv - 2n(p)$. Replacing $- 1$ and 2 by the solutions obtained in (3) and (5), $(4n)^2 \equiv n[\xi^{3n}(\xi^{2n} - 1)]^2(p)$ or $(4n)^2 - n[\xi^{3n}(\xi^{2n} - 1)]^2 \equiv 0(p)$. Now ξ^{3n} is obviously prime to p. Suppose $\xi^{2n} - 1 \equiv 0(p)$. Then $\xi^{2n} \equiv 1(p)$ and $\xi^{4n} \equiv 1(p)$ which is contrary to the assumption that ξ is a non-residue of p. Hence $\xi^{2n} - 1$ is prime to p and the conditions of the lemma are satisfied. Here then $u = 4n$, $v = \xi^{3n}(\xi^{2n} - 1)$ and (4) becomes $- \xi^{3n}(\xi^{2n} - 1)x \equiv - 4n(p)$. Multiplying by $\xi^n(\xi^{2n} + 1)$ and replacing ξ^{4n} by $- 1$ as before, this gives $- 2x \equiv - 4n\xi^n(\xi^{2n} + 1)(p)$ or $x \equiv 2n\xi^n(\xi^{2n} + 1)(p)$ is a solution and hence $x \equiv \pm 2n\xi^n(\xi^{2n} + 1)(p)$ is the general solution of $x^2 \equiv n(p)$.

6. General Solution. The above results for $- 1$, 2 and n are true whether n be odd or even but from this point on suppose n is odd and equal to $2k + 1$.

With this condition on n, it is the purpose of the present section to find a general solution of $x^2 \equiv a(p)$. By Euler's criterion, $a^{4n} \equiv 1(p)$. That is, $a^{8k+4} \equiv 1(p)$. Then $a^{4k+2} \equiv 1$ or $- 1(p)$. If $a^{4k+2} \equiv 1(p)$, $a^{2k+1} \equiv 1$ or $- 1(p)$.

It is convenient to discuss these results under three cases: Case I: $a^{2k+1} \equiv 1(p)$. Case II: $a^{2k+1} \equiv - 1(p)$. Case III: $a^{4k+2} \equiv - 1(p)$.

Case I. Suppose $a^{2k+1} \equiv 1(p)$. Then $a^{2k+2} \equiv a(p)$ and

$$x \equiv \pm a^{k+1}(p) \tag{6}$$

is the general solution of $x^2 \equiv a(p)$ in this case.

Case II. Suppose $a^{2k+1} \equiv - 1(p)$. Then $a^{2k+1} \equiv \xi^{4n}(p)$ and $a^{2k+2} \equiv a\xi^{4n}(p)$. Here $u = a^{k+1}$, $v = \xi^{2n}$ and (4) becomes $- \xi^{2n}x \equiv - a^{k+1}(p)$. Multiplying by ξ^{2n} gives

$$x \equiv \pm a^{k+1}\xi^{2n}(p), \tag{7}$$

the general solution in this case.

Case III. Suppose $a^{4k+2} \equiv - 1(p)$. Then $a^{4k+2} + 1 \equiv 0(p)$, that is, $(a^{2k+1} + 1) - 2a^{2k+1} \equiv 0(p)$. Replacing 2 by a square congruent to it, from (5), this may be written:

$$(a^{2k+1} + 1)^2 - a[a^k\xi^n(\xi^{2n} - 1)]^2 \equiv 0(p).$$

Here $u = a^{2k+1} + 1$, $v = a^k\xi^n(\xi^{2n} - 1)$ and (4) becomes

$$-a^k\xi^n(\xi^{2n} - 1)x \equiv -(a^{2k+1} + 1)(p).$$

Multiplying by $a^{3k+2}\xi^{3n}(\xi^{2n} + 1)$, replacing ξ^{4n} by -1 and a^{4k+2} by -1 gives $2x \equiv -a^{3k+2}\xi^{3n}(\xi^{2n} + 1)(a^{2k+1} + 1)(p)$. Multiplying now by $4n$, the coefficient of x becomes congruent to $-1(p)$ and hence

$$x \equiv 4na^{3k+2}\xi^{3n}(\xi^{2n} + 1)(a^{2k+1} + 1)(p) \tag{8}$$

is a solution in this case.

Two of the powers in this result can be lowered as follows: $\xi^{3n}(\xi^{2n} + 1) = \xi^{5n} + \xi^{3n} \equiv -\xi^n + \xi^{3n}(p)$ which may be written:

$$\xi^{3n}(\xi^{2n} + 1) \equiv \xi^n(\xi^{2n} - 1)(p). \tag{9}$$

Again, $a^{3k+2}(a^{2k+1} + 1) = a^{5k+3} + a^{3k+2}$. Now since by hypothesis $a^{4k+2} \equiv -1(p)$, $a^{5k+3} + a^{3k+2} \equiv a^{3k+2} - a^{k+1}(p)$ and

$$a^{3k+2}(a^{2k+1} + 1) \equiv a^{k+1}(a^{2k+1} - 1)(p). \tag{10}$$

Substituting in (8) from (9) and (10) gives $x \equiv 4n\xi^n(\xi^{2n} - 1)a^{k+1}(a^{2k+1} - 1)(p)$ and hence

$$x \equiv \pm 4n\xi^n(\xi^{2n} - 1)a^{k+1}(a^{2k+1} - 1)(p) \tag{11}$$

is the general solution for this final case.

Formulas (6), (7) and (11) are therefore the required explicit expressions for x in $x^2 \equiv a(p)$ when n is odd.

THE CONTINUITY OF A FUNCTION DEFINED BY A DEFINITE INTEGRAL.

By R. L. JEFFERY, Acadia University, Nova Scotia.

The function $f(x, y)$ defined on the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$ is bounded and an integrable function of x for each y. A function of y is thus defined by $\int_a^b f(x, y)dx \equiv F(y)$. If it is further assumed that $f(x, y)$ is a continuous function of the two variables, or is continuous in y uniformly[1] with respect to x, it is readily shown that $F(y)$ is continuous. Theorems to this effect have become the common property of the various treatises on analysis.[2] W. H. Young[3] has shown $F(y)$

[1] This implies that for a given number $\epsilon > 0$ there exists a number $\delta > 0$ and independent of x, such that $|f(x, y_0) - f(x, y)| < \epsilon$ when $|y - y_0| < \delta$.

[2] Osgood: *Functionentheorie*, p. 167. Goursat-Hedrick: *Math. Analysis*, vol. I, p. 97. Hobson: *Theory of Functions of a Real Variable*, 1st ed., p. 594. Pierpont: *Theory of Functions of a Real Variable*, vol. I, p. 389.

[3] *Monat. für Math. und Phys.*, vol. II, 1910.

continuous by merely assuming, in addition to its being bounded and integrable in x for each y, that $f(x, y)$ is continuous in y for each x. In the theorem of this note, the conditions on $f(x, y)$ are slightly less stringent than those imposed by Young. Furthermore, the method used affords a simple illustration of the beauty and power of the point set theory.

THEOREM: *Let the bounded function $f(x, y)$ defined on $a \leqq x \leqq b$, $c \leqq y \leqq d$ be a summable function of x for each y, and a continuous function of y at y_0 for each x except a set E of x-points of zero measure; then $F(y) = L\int_a^b f(x, y)dx$ is continuous at y_0 ($c \leqq y_0 \leqq d$).* Here $L\int$ denotes Lebesgue integral.

PROOF. Let $\epsilon > 0$ be given; then choose $y_1, y_2, \cdots y_n, \cdots$, an arbitrary sequence of values of y approaching y_0 as a limit, and $\delta_1, \delta_2, \cdots \delta_n, \cdots$, an infinite sequence of positive numbers with zero as a limit. We can assume, with no loss of generality, that both limits are approached monotonically. Now consider the set of x-points G_i for which

$$|f(x, y_0) - f(x, y_n)| \leqq \epsilon,$$

where y_n is all points of the above chosen sequence which satisfy the inequality $|y_0 - y_n| < \delta_i$.

The set G_i is measurable. To show this we denote by y_{n_1} the first number of the y-sequence for which the inequality $|y_0 - y_n| < \delta_i$ holds, and then consider the set of x-points S_1 for which

$$|f(x, y_0) - f(x, y_{n_1})| \leqq \epsilon.$$

The functions $f(x, y_0)$ and $f(x, y_{n_1})$ are, by hypothesis, both measurable functions of x, and from this it readily follows that S_1 is a measurable set.[1] Similarly if y_j is the jth number of the y-sequence for which the inequality $|y_0 - y_n| < \delta_i$ holds, the set S_j defined in a manner similar to S_1 is measurable. Continuing this process, we get an infinite sequence of measurable sets, $S_1, S_2, \cdots S_j, \cdots$, and since each set of the sequence is measurable, $D(S_1, S_2, \cdots S_j, \cdots)$, the set common to all the sets of the sequence is measurable.[2] But this is clearly the set G_i.

Thus, by means of the chosen sequence, $\delta_1, \delta_2, \cdots \delta_i, \cdots$, we have defined an infinite sequence of measurable sets, $G_1, G_2, \cdots G_i, \cdots$, and consequently the set $M(G_1, G_2, \cdots G_i, \cdots) \equiv G$, which consists of all points belonging to any G_i, is measurable.[3] Furthermore, it is readily seen that G contains all the x-points for which $f(x, y)$ is continuous in y at y_0; hence mG, the measure of G, is $b - a$. It is also evident that G_i contains G_{i-1} ($i = 1, 2, \cdots$); hence $\lim_{i\to\infty} mG_i = mG$ [4]

[1] From the definition of a measurable function, and the fact that the difference of two measurable functions is a measurable function.

[2] Hobson: *Theory of Functions of a Real Variable*, 2d ed., vol. I, § 131.

[3] Hobson: *Ibid.*, 130.

[4] Hobson: 2d ed., vol. I, § 131.

$= b - a$. If we now denote by $C(G_i)$ the set complementary to G_i on (a, b), there exists a value k of i sufficiently large to insure that

$$mC(G_k) < \epsilon. \tag{1}$$

With the set G_k and the corresponding δ_k determined in accordance with (1), we next denote the least upper bound and greatest lower bound of $f(x, y)$ by M and m respectively, and proceed to show that

$$|F(y_0) - F(y_n)| \leqq \epsilon(b - a) - \epsilon(M - m)$$

when $|y_0 - y_n| < \delta_k$. To this end let us consider the inequality

$$\begin{aligned} |F(y_0) - F(y_n)| &\leqq L \int_a^b |f(x, y_0) - f(x, y_n)|\,dx \\ &\leqq \int_{G_k} |f(x, y_0) - f(x, y_n)|\,dx + \int_{C(G_k)} |f(x, y_0) - f(x, y_n)|\,dx. \end{aligned}$$

Over the set G_k, $|f(x, y_0) - f(x, y_n)| \leqq \epsilon$; hence the first term on the right of the above inequality is $\leqq \epsilon(b - a)$. Over the set $C(G_k)$, $|f(x, y_0) - f(x, y_n)| \leqq M - m$; it then follows from (1) that the second term on the right is $\leqq \epsilon(M - m)$. The positive number ϵ and the sequence $y_1, y_2, \cdots y_n, \cdots$ being arbitrary, we can now conclude that $F(y)$ is continuous at y_0.

In conclusion we note that $F(y)$ has been proved continuous in conformity with the Heine [2] sequence definition of continuity. This remark also applies to the work of Young. For the general case, the Heine definition is proved equivalent to that of Cauchy [2] only through the medium of the Zermelo axiom.[1] The question then arises: Is it possible, without resorting to the Zermelo axiom, to show the special function $F(y)$ defined above continuous in accordance with Cauchy's conception of continuity? One method of attack would be as follows: With the δ-sequence chosen as above, define G_i' to be the set of x-points for which

$$|f(x, y_0) - f(x, y)| \leqq \epsilon$$

when $|y_0 - y| < \delta_i$, and then show G_i' measurable. With this accomplished, an argument proceeding as in the foregoing [2] from the point where G_i was proved measurable would lead to the desired result. As yet, however, I have been able to carry through this idea only in the special case where $f(x, y)$ is continuous in each variable separately. Then this set G_i' is readily shown to be closed, and hence measurable.

[1] See Hobson: 2d ed., vol. I, §§ 197, 198, 211, where these points are discussed at length, and where references are given to the original articles.

[2] The only difference being that G_i is replaced by G_i' and y_n by y.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

In Question **53** Professor H. S. Uhler states a theorem in determinants which he has obtained by a long process. He would be glad to see in these columns either a reference to where it may be found in the literature or a neat rigorous proof.

53. Let $-N$ denote the bordered symmetrical determinant

$$\begin{vmatrix} 0 & b_1 & b_2 & \cdots & b_k \\ b_1 & c_{11} & c_{12} & \cdots & c_{1k} \\ b_2 & c_{21} & c_{22} & \cdots & c_{2k} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_k & c_{k1} & c_{k2} & \cdots & c_{kk} \end{vmatrix} \text{ where } c_{r,\,s} \equiv \sum_{j=1}^{j=n} (a_{r,\,j}a_{s,\,j}) = c_{s,\,r} \text{ and the } a\text{'s belong}$$

$$\text{to the matrix } M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix};\ k \leqq (n-1),\ \text{rank} = k,\ a_{i,\,j} \neq a_{j,\,i}.$$

Is there a simple proof for the following theorem? *N equals the arithmetical sum of the squares of* $\binom{n}{k-1}$ *determinants of order k in every one of which the first column read downward consists of* $b_1, b_2, \cdots, b_k$ *and the remaining* $k-1$ *monomial columns constitute collectively one of the combinations of the n columns of the matrix M taken* $k-1$ *at a time.*

DISCUSSIONS.

I. CONCERNING CUBIC POLYNOMIALS.

BY LOUIS WEISNER, University of Rochester.

To illustrate the importance of finding maximum and minimum points in curve tracing, most writers of calculus text-books propose that students trace the curve $y = x^3 + dx^2 + ex + f$. To avoid needless calculations the coefficients are usually chosen so that the abscissas of the maximum and minimum points are rational. To further minimize the computations, the coefficients may be chosen so that the intercepts on the x-axis are rational. However, I find that in the proposed problems of this type with the stated conditions, two of the intercepts are usually coincident; that is, the equation may be put in the form $y = (x - a)^2(x - b)$.

The following problem therefore arises: To determine all triads of distinct rational numbers a, b, c, such that the zeros of the derivative of $(x-a)(x-b)(x-c)$ are rational. It is desired that the equation

$$3x^2 - 2(a + b + c)x + ab + ac + bc = 0$$

have rational roots. This will be the case if, and only if, its discriminant is a perfect square; that is, if the equation

$$a^2 + b^2 + c^2 - ab - ac - bc = t^2$$

has a rational solution.

To solve this Diophantine equation, substitute

$$b = \frac{qa}{p}, \qquad c = \frac{ra + pt}{p}.$$

Then

$$\frac{a}{t} = \frac{p(p + q - 2r)}{p^2 + q^2 + r^2 - pq - pr - qr}.$$

Since the Diophantine equation is homogeneous in a, b, c, t, let

$$t = p^2 + q^2 + r^2 - pq - pr - qr.$$

Then

$$\begin{aligned} a &= p(p + q - 2r), \\ b &= q(p + q - 2r), \\ c &= p^2 + q^2 - r^2 - pq. \end{aligned}$$

These four equations furnish a three-parameter solution of the Diophantine equation.

If p, q, r are distinct rational numbers, none of which is half the sum of the other two, then a, b, c are distinct; and conversely. The problem is solved.

The following are examples of cubic polynomials with relatively small integral coefficients, having rational zeros, and whose derivatives have rational zeros: $(x + 1)(x - 2)(x - 7)$, $(x - 4)(x + 4)(x + 1)$, $x(x + 3)(x - 5)$.[1]

[1] The problem may also be solved as follows. Let m, n be the zeros of the derivative. These are connected with the zeros a, b, c of the cubic by the equations $3(m + n) = 2(a + b + c)$, $3mn = ab + ac + bc$, which are linear in m and a, and readily determine these two quantities in terms of b, c, and n, unless $b + c = 2n$. It follows that b, c, n may be any three rational numbers not in arithmetic progression.

Since the problem is solved when the ratio $(b - a)/(c - b)$ is suitably determined, it is convenient to assume that $b = 0$. The equations in m and a then give

$$m = c(3n - 2c)/(6n - 3c), \qquad a = n(3n - 2c)/(2n - c).$$

Now $3n - 2c$, $2n - c$, n, and c are in arithmetic progression, and may be taken to be any four rational quantities in arithmetic progression, α, β, γ, δ. The expressions for m and a then become $\alpha\delta/(3\beta)$ and $\alpha\gamma/\beta$. In other words, a, c, m, and n are to be rational multiples of $\alpha\gamma$, $\beta\delta$, $\alpha\delta/3$, and $\beta\gamma$.

When b is not zero, we have instead

$$a = b + k\alpha\gamma, \qquad c = b + k\beta\delta.$$

Moreover, it will be found that every admissible case is uniquely obtained by taking for α and β two mutually prime integers, of which α is negative and β is positive, b and k being any rational numbers. This enables one easily to pick out the simplest cases. Thus all of Dr. Weisner's examples correspond to $\alpha = -1$, $\beta = 1$, $\gamma = 3$, $\delta = 5$. The next simplest cases are obtained from $\alpha = -2$, $\beta = 1$, $\gamma = 4$, $\delta = 7$.

EDITOR.

II. A Digit for Negative One.

By J. P. Ballantine, Columbia University.

Mathematical historians will tell you how many years mathematics was held back for want of a digit 0. Though not comparing in importance with that digit, there is a certain advantage in having a digit to represent negative one. For this purpose we will use the digit for 1 inverted: thus $\rotatebox{180}{1}$.

We have as a matter of fact no well-accepted way of writing a negative number, except by the ambiguous minus sign which usually denotes subtraction. It is no wonder that students do not grasp the logical difference between the problem in subtraction $0 - 7$ and the number negative seven which we now may denote $\rotatebox{180}{1}3$.

In numerical work with logarithms, one runs across such numbers as $9.69897 - 10$. How much simpler to write $\rotatebox{180}{1}9.69897$ or $\rotatebox{180}{1}.69897$? It may even be written $\rotatebox{180}{1}.70\rotatebox{180}{1}\rotatebox{180}{1}7$.

The laws of operation of the new digit are easily mastered. In the new multiplication table, such entries as $\rotatebox{180}{1} \times 7 = \rotatebox{180}{1}3$ are easily memorized. Such identities as $\rotatebox{180}{1}3 = \rotatebox{180}{1}93 = \rotatebox{180}{1}993 = \rotatebox{180}{1}99993$ are also obvious. This latter remark is of significance in connection with computing machines. It is commonly understood that a big string of digits 9 extending to the left on the machine is a negative number, and is commonly explained by use of the words "complementary number." The whole thing becomes clear immediately if we place $\rotatebox{180}{1}$ at the head of these nines.[1]

III. On the Kinematic Construction of Certain Higher Plane Curves.

By R. E. Moritz, University of Washington.

1. If a point has simple harmonic motion along a straight line, while at the same time the line moves with constant velocity in the direction of the perpendicular, the locus of the point is a simple harmonic curve. The simple harmonic curve is, therefore, the locus of the motion resulting from the composition of two simple motions, a simple harmonic motion and a uniform linear motion. The equations of these component motions are the parametric equations of the resultant simple harmonic curve. Thus, if t is the time, y the distance of the point from a fixed point in the line of vibration, a the amplitude and $2\pi/p$ the period of vibration, k the distance of the mean point of vibration from the fixed

[1] Possibly some readers have not tried the somewhat similar experiment of using the digits $\bar{5}$, $\bar{4}$, $\bar{3}$, $\bar{2}$, $\bar{1}$, 0, 1, 2, 3, 4, 5, instead of the usual set of 0, 1, $\cdots$, 9. Here $\bar{5}$ stands for -5, etc. On this plan 3.1415926 becomes $3.142\bar{4}\bar{1}3\bar{4}$, and the negative of this $\bar{3}.\bar{1}\bar{4}\bar{2}41\bar{3}4$. The merchant who marks his goods at \$9.98 would unfortunately be obliged to make it \$$10.0\bar{2}$. When some of the figures at the end of a decimal fraction were omitted, no correction of the last remaining figure would now be necessary, though there would be the usual doubt as to what to do when the next figure was exactly 5; and indeed numbers ending in 5 could be written in two ways, as $25 = 3\bar{5}$. This last difficulty could however be remedied, if we also made the slightly revolutionary change of replacing the decimal scale by that of eleven.

Editor.

point, and q the uniform velocity of the moving line, the equations of the component motions are

$$y = a \cos pt + k \quad (1), \qquad x = qt \quad (2),$$

and these two equations are the parametric equations of the resultant simple harmonic curve

$$y = a \cos \frac{p}{q} x + k. \tag{3}$$

It is the purpose of this paper to give a brief sketch of the derivation of the equations of those curves which result from a composition of the simplest motions, —uniform linear motion, uniform rotatory motion, and simple harmonic motion. Since these motions may be readily effected and compounded mechanically, the curves here considered may all be traced kinematically.[1] In each case the equations of the component motions are parametric equations of the resultant curve. In the case of the trochoids and cycloids these equations surpass in simplicity those in common use.

2. Let a point move with uniform velocity along a line, while at the same time this line moves with a uniform angular velocity about one of its points. If k denotes the distance of the initial position of the moving point from this fixed point, p and q the linear and angular velocities respectively, the equations of the component motions are obviously

$$\rho = pt + k \quad (4), \qquad \theta = qt \quad (5),$$

and these are the parametric equations of conchoids of the Archimedean spiral

$$\rho = \frac{p}{q}\theta + k. \tag{6}$$

3. Let a point oscillate along a straight line while at the same time this line rotates uniformly about one of its fixed points. With the proper meaning assigned to the constants, the equations of the component motions are obviously

$$\rho = a \cos pt + k \quad (7), \qquad \theta = qt \quad (8),$$

and these are the parametric equations of the resultant cyclic-harmonic curves

$$\rho = a \cos \frac{p}{q}\theta + k. \tag{9}$$

9) includes a number of well-known special curves,—the limaçon ($p = q$), the nephroid ($2p = q$, $a = 2k$), the double egg curve ($p = 2q$, $a = k$), the roses ($p = nq$, $k = 0$, n integral), etc.

[1] Wieleitner goes so far as to say that every curve, no matter how defined, may be described kinematically (*Spezielle Ebene Kurven*, Leipzig, 1908, sect. 72).

4. Let a point oscillate in a straight line while at the same time the line along which the oscillation takes place itself oscillates in a perpendicular direction. The equations of the component motions are obviously

$$x = a \cos (pt + \epsilon) \quad (10), \qquad y = b \cos qt \quad (11).$$

These are the parametric equations of the resultant Lissajou's curves

$$p \cos^{-1} (y/b) = q[\cos^{-1} (x/a) - \epsilon]. \tag{12}$$

(12) includes as special cases the straight line ($p = q$, $\epsilon = 0$), the parabola ($q = 2p$, $a = b$, $\epsilon = 0$), the ellipse ($p = q$, $\epsilon = \pi/2$), a lemniscate ($q = 2p$, $a = b$, $\epsilon = \pi/4$).

5. Let a point move with uniform velocity along the circumference of a circle, while at the same time the center of the circle moves with a constant velocity along a straight line. The parametric equations of the component motions are obviously

$$x' = a \cos pt, \qquad y = a \sin pt \quad (13), \qquad x'' = qt \quad (14).$$

The parametric equations of the resultant trochoids are therefore

$$x = a \cos pt + qt, \qquad y = a \sin pt. \tag{15}$$

When $q < ap$ the curves are prolate cycloids,
when $q = ap$ the curves are common cycloids, and
when $q > ap$ the curves are curtate cycloids.

6. Let a point move uniformly along the circumference of a circle, while at the same time the center of this circle moves uniformly along the circumference of a second circle. If we denote the radius of the first circle by a, that of the second circle by b, and the uniform velocities by p and q respectively, the resultant angular velocity of the moving point is obviously $(p + q)t$ and the parametric equations of the component motions are therefore

$$x' = a \cos (p + q)t, \qquad y' = a \sin (p + q)t; \tag{16}$$

$$x'' = b \cos qt, \qquad y'' = b \sin qt. \tag{17}$$

The parametric equations of the resultant epi- and hypotrochoids are therefore

$$x = a \cos (p + q)t + b \cos qt, \qquad y = a \sin (p + q)t + b \sin qt. \tag{18}$$

These curves are epitrochoids if p and q have like signs, hypotrochoids if p and q have unlike signs.

If $a(p + q) = bq$, the curves are epicycloids, if $a(p - q) = bq$, they are hypocycloids.

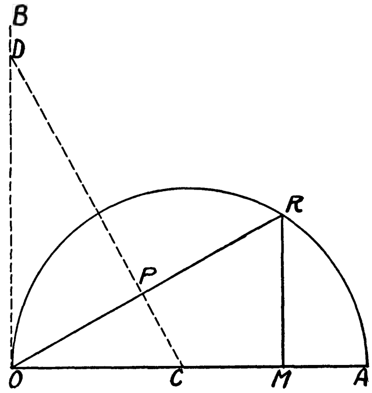
7. Aside from their simplicity of form, ease of derivation, and comprehensive character, the equations (18) offer the added advantage of extension to epicycles of any order.

Let a point P move with uniform angular velocity p_1 on a circle C_1 with radius a_1 whose center moves with uniform angular velocity p_2 on a circle C_2 with radius a_2 whose center in turn moves with uniform angular velocity p_3 on a circle C_3 with radius a_3, and so on. The parametric equations of the resultant motion of P are obviously

$$x=a_1 \cos (p_1+p_2+\cdots+p_n)t+a_2 \cos (p_2+p_3+\cdots+p_n)t+\cdots+a_n \cos (p_n)t,$$
$$y=a_1 \sin (p_1+p_2+\cdots+p_n)t+a_2 \sin (p_2+p_3+\cdots+p_n)t+\cdots+a_n \sin (p_n)t.$$

IV. Note on the Four-Leaf Rose.

By C. H. Chepmell, Hove, England.



With reference to "Geometrical Construction of Points on a Four-Leaf Rose," by H. H. Downing, in this Monthly (*1924*, 340), an alternative construction is possible, as indicated by the dotted lines BO (tangent to circle at O) and CPD (perpendicular to OR at P).

For clearly $CD = OA = 2a$ (a constant); and the curve can be described as the locus of P, the foot of the perpendicular from O on a line CD of constant length moving in the legs of the right angle BOA—thus dispensing with the rest of the figure.

RECENT PUBLICATIONS.

Edited by W. B. Carver, Cornell University, to whom books and communications should be sent.

REVIEWS.

Hand-book of Mathematical Statistics. By H. L. Rietz, Editor-in-Chief, H. C. Carver, A. R. Crathorne, W. L. Crum, J. W. Glover, E. V. Huntington, T. L. Kelley, W. M. Persons, A. A. Young. Boston, Houghton Mifflin Co., 1924. viii + 221 pages. Price $4.00.

This "Hand-book" was prepared by a group of mathematicians and statisticians, assisted by an appropriation from the National Research Council. It deals with the general problems of the mathematical analysis of statistical data, as distinguished from the problems which arise in the collection of the data or in the interpretation of the results. In general the method of treatment is, as the name hand-book would imply, to present a digest of results under each topic, with some discussion of their significance and of the limitations of the methods of analysis. While the mathematician will regret the omission of proofs, and the practical statistician will wish that more attention might be given to showing how these methods tie in with common-sense analysis, there can be

no question that the book as it stands has a real value, and will further the progress of mathematical statistics.

Nevertheless, in view of the opportunity and the special fitness of the authors for their tasks, the book is somewhat disappointing. Methods based on higher mathematics are being found increasingly valuable in business statistics, economics, public health work, and investigation of social and educational problems. Many statisticians with little formal mathematical training are much more willing than formerly to use mathematical formulas or terminology, and even to try to follow mathematical reasoning, if presented so that it seems related to their problems and not too complicated. The coöperating authors are unquestionably leaders in the field of statistics, most of them in mathematical statistics. The book seems to the reviewer to fail in many places to meet the need as it might have been met by these authors, and therefore, with a view to improving the next attempt, this review will be devoted principally to distributing brick-bats rather than posies.

In order to have a fair basis for criticism, let us pause to sum up the virtues which ought to be found in a hand-book, and then examine to what extent certain individual chapters seem to us to have these virtues. The reviewer expects a hand-book to have the following characteristics:

1. It should include the material which the user is likely to need frequently, and exclude material which will be wanted only once or twice in a decade.

2. It should be written in a way that will be comprehensible to the readers for whom it is intended, who are presumably not merely the experts, but also those only partially familiar with the points of view and methods of mathematical statistics.

3. It should be authoritative, that is to say, should either settle disputed or doubtful points by the considered judgment of experts, such as the group of authors, or state the different methods or points of view, if no one is to be preferred.

4. It should be convenient for reference.

In the opinion of the reviewer, several chapters fall decidedly short under one or more of these heads.

Chapter I, for instance, seems rather beyond the understanding of those unfamiliar with mathematics, and it is therefore unfortunate that it is the first thing which confronts the reader. The introduction to Chapter II would serve excellently as an introduction to the whole book. Chapter I contains certain material—elementary discussion of averages, preliminary treatment of probability and the normal curve, and other topics which would come in appropriately in later chapters; in fact, these later chapters would be strengthened and the material would be more convenient for reference if this rearrangement were made. The discussion of numerical computation in the first four pages of Chapter I might well be omitted entirely, or else expanded enough really to analyze the points raised. As presented it hardly seems adequate or authoritative. For instance, statements on page three imply that consideration of the maximum

possible error is a proper guide in deciding as to the number of significant figures to be retained in the sum of a large number of items. The real analysis of this question seems to involve an application of the multinomial theorem; it is undesirable to suggest a rule which would upset common practice without having the matter worked out more carefully. One thing in Chapter I that might well be excluded, as needed only once in a decade, is the approximate formulas for the Bernoulli numbers for large values of n.

Chapter V, on random sampling, covers the ground in a fairly satisfactory manner. The main criticisms are as follows:

1. The importance of having the sample really random, in order that these theorems may apply, is not emphasized. For a discussion of the great difficulties which this requirement produces, in a practical application, the reader may be referred to the July 1924 issue of the *Journal of the Royal Statistical Society*, pp. 544–570.

2. Neither here nor in the first chapter is it brought out that Bernoulli's theorem justifies, if conditions remain substantially unchanged, using observed ratios as if they were a priori probabilities, and provides a measure of how safe such use is.

3. It would be well to note that statistical prediction and estimation really depend on the assumption that the observed past provides a fair random sample of all similar events past and future. As, in the opinion of the reviewer, these three points are more important as a basis for applications than anything now in Chapters V and VI, it is unfortunate that they were omitted. The difficulty is partly due to the inclusion of parts of the theory of probability in Chapter I instead of in this chapter.

Chapter VII, on frequency curves, comes nearer to reducing that subject to order than anything else that has been published. It is to be wished, however, that the author had introduced his subject more as a continuation of Chapter II, with some discussion of the advantages of fitting an equation at all. A person might gather from the text as it is that the only scientific way to proceed with a frequency distribution is to fit an equation. But unless we are interested in completely describing things as they are, or want to smooth an empirical distribution in some systematic way, why find an equation? No mention is made of a definite relation of the equations to probability except in the case of the normal curve. In the opinion of the reviewer, the use of these equations, other than that of the normal curve, is an interesting and valuable occupation for a certain number of specialists who want to find out how the world wags, but is not a necessary part of statistical practice for most people.

Chapter IX on multiple and partial correlation will seem rather complicated to the non-mathematical; the use of the determinant notation will repel. A complete statement of the coefficients of regression and multiple correlation for the three variable case would have shown that the subject is not really so complicated. The most unfortunate point in this chapter, however, is the selection of the illustrative example. One of the variables expresses merely classification

in one of three categories, to which are arbitrarily assigned the values 1, 2, and 3. The coefficients are then computed to four (4) significant figures!

Chapter X, on the correlation of time series, is, like Chapter XII on index numbers, written from the point of view of those primarily interested in economics rather than mathematics. As an exposition of methods, with a discussion of what it is all about, this chapter is superior to most of the others; but there are several points to criticize from the scientific standpoint. The discussion of the elimination of seasonal variation, for instance, can hardly be considered authoritative; it selects one of the various methods which are actually used and describes it as "the" procedure, with only incidental and incomplete discussion of one other method. The author does not mention at all the fact that Professor Hart has shown that for a certain time series "the" procedure explained here leads to seriously erroneous results. Where, as in this case, it is still a subject of discussion which of several methods is preferable, the proper attitude for a hand-book would seem to be either to explain all unexploded methods, or to decide in favor of one of them on the basis of proof submitted to the scientific world in this book or elsewhere. It is surprising to find in this book statements that "The theory of probability does not apply to our data" and that "the specific relationships between time series are much more adequately set forth by charts than by numerical measures." These statements express exactly the argument of those "low brow" statisticians who deny the use of the theory of probability and of real mathematical analysis in economic statistics; a good discussion of these statements would have been a valuable addition to the hand-book, but that is not given here.

It seems fair to say, then, in general comment on the hand-book, as represented by these chapters, that some important points have been omitted or inadequately stressed, and that the treatment of others is not authoritative. The main criticism, however, is that the book is proving difficult to understand outside the circle of mathematical specialists. This difficulty is due in a large measure to the neglect of discussion of fundamental principles as compared with formulas. The reviewer can not admit that telling what it is all about is outside the sphere of mathematics. The practical statistician has justification for saying, "We asked for bread, and you have given us a stone—a whetstone, to be sure, with which, if we are resolute enough, we may sharpen our wits, but nevertheless, a stone."

In spite of the defects noted, the book is a real contribution to the development of mathematical statistics. There are many conspicuous merits in the chapters mentioned above and in other chapters. Chapter XII on index numbers, for instance, is both a good introduction to the subject for those of little experience and a summary of essential principles which will be valuable to the experienced economist or business statistician. Classes in mathematical statistics will find the book essential, at least as a reference book, and in many cases, supplemented by lectures and problems, as the best available text. If a second edition, incorporating a thorough revision, can be prepared before very long, it will have a wide and continually increasing field of usefulness.

R. W. BURGESS.

Leçons d'analyse fonctionelle. By PAUL LEVY with preface by J. HADAMARD. Paris, Gauthier-Villars, 1923. vi + 442 pages. Price 35 francs.

No doubt one of the largest fields in analysis to be opened up during the last three decades has come through the discovery that many of the results which can be generalized from functions of a single variable, *i.e.*, a linear space, to functions of n variables, or n-space, can also be extended to the function space, *i.e.*, the space in which the variable is a function of a continuous variable. The impetus for this development came from the study of the linear integral equation, with the deduction of its solution from the solution of algebraic equations in n variables, the work of Volterra and Fredholm. The influence of the former has been potent for further development, and one feels something of his influence in the work under discussion, a work which is calculated to open up new vistas and possibilities in the field of functionals, *i.e.*, functions of functions.

The purpose of the volume as stated in the introduction is to extend to function space (1) the elementary results of the differential calculus, (2) the theory of partial differential equations, and (3) the theory of multiple integrals. A closer examination of the work reveals the fact that the author's primary interest is in the second of these two topics, the theory of partial differential equations in function space, and this has influenced to a large extent the choice of material. As a consequence, one finds that many matters which one might expect to be treated in one of the first two books [1] on this subject are touched upon only lightly, and other matters which may not seem to be of major importance in the field of functionals are treated extensively. Such however is the privilege of authorship.

The contents of the book fall, as indicated above, into three parts. In the first part, the foundations for the later developments are laid. The range on which functionals operate is defined to be that of functions of a single continuous variable (as a rule), and the author oscillates between two definitions of distance as between functions, one based on the maximum of the absolute value of the difference of two functions (which is the more natural definition for continuous functions and uniform convergence) and the other, the integral of the square of the difference, the latter connected particularly with functions of Lebesgue integrable square.

The major portion of this part is devoted to differentials of functionals of the first and higher orders. Differentials involve the notion of linear and multilinear operations, so we find a discussion of the expressions for linear functionals in function space. A contrast is made between what is called the logical and the practical point of view, the logical point of view being that to which one is led if he seeks the general expression for a linear functional on continuous functions, while the practical leads to a more special form, a form which is the more convenient from the point of view of the generalizations to be made, since in its application to the notion of differentials it gives a ready generalization of partial

[1] A brief treatment of some of the same material is to be found in the *Cambridge Colloquium Lectures* by G. C. Evans on Functionals.

derivatives. Whether this limitation is satisfying depends upon the point of view. From the point of view of completeness it is to be hoped that it may be possible at some future time to treat properties of differentials on the basis of a linear functional on continuous functions defined as a Stieltjes integral, instead of the integral of the product of an integrable function into the variable function.

Attention might well be called to the discussion of definitions of differentials. There are contrasted the Fréchet definition, in which the differential of a functional U is linear in the difference function and satisfies the condition

$$\lim_{m\Delta f \to 0} \frac{u(f + \Delta f) - u(f) - \delta u(\Delta f)}{m\Delta f} = 0,$$

where $m\Delta f$ is the distance of Δf from the zero function; the definition of Gateaux, *viz.*,

$$\delta u(\delta f) = \frac{d}{d\lambda} u(f + \lambda \delta f)\,|_{\lambda=0},$$

and the author's, which is the Gateaux definition in case the differential is linear in δf. The Gateaux type of definition is frequently encountered in mathematical physics, while the Fréchet type is a generalization of one which has shown itself to be very elegant and satisfactory for functions of two or more variables.

The second part of the book is devoted to a discussion of partial differential equations in function space. There is considered first of all an equation which is a generalization of the total differential equation in n-space. We find here integrability conditions similar to those obtained in ordinary space. As an application of this theory, the equation of variation of Green's function which was found by Hadamard is considered rather extensively. There is considered next a partial differential equation which is the generalization of a system of n partial differential equations in $2n$ independent variables; and in this theory we find, for instance, an investigation of characteristics as in ordinary space. By way of application, a functional equation which arises from the problem of minimizing an area integral is treated. The final chapter of this part is devoted to equations which are legitimate generalizations of linear partial differential equations of the first order, systems of such, and non-linear partial differential equations. It is really very fascinating to see how well the theory of partial differential equations of the first order carries over into the functional field.

The third part of the book has as its object the extension of the theory of the solution of the Laplace differential equation to the field of functionals. In order to carry this through, it is found desirable to develop in detail a generalization of the integration process. Now it turns out that in infinitely many variables most volumes are infinite, and so the ordinary processes of integration do not carry over. Gateaux had the happy inspiration to note that the integral is very closely related to the mean value, and that the mean value over a line, or surface or volume, could be carried over to space of infinitely many dimensions. The major portion of this part of the book is therefore concerned with the development of the mean value integral over a functional field, and the proof that a definition

which is similar to that of the Lebesgue integral gives an existent mean value for functions of a large class, the extension of the class of continuous functions, when the mean value is taken over the interior of what is the generalization of a sphere. There is also a development of the differential geometry type of properties of surfaces, surfaces being taken in the functional sense. Much of this work seems philosophic in character. It will remain for the future to determine the usefulness and applicability thereof.

There is an enormous amount of material in this volume, and most of it is not of an elementary character, although the author has tried to give it an elementary tone by introducing a chapter on Lebesgue integration and linear integral equations in the first part. In order to follow the author easily in his deliberations, it is desirable to have a considerable acquaintance with the literature on functionals, and a thorough knowledge of partial differential equations of the first order in n-space, as well as a thorough acquaintance with the solution of Laplace differential equations. From the point of view of the reviewer the author's style is not lucid; it is not what one usually expects to find in the French mathematical presentations, and particularly in most of the books of the Borel series of monographs. And there are misprints, some of them rather puzzling. But much of the contents of the book is worth making the effort to understand, and it is true that it points the way towards much that is of value in the direction of new research in the functional field.

T. H. HILDEBRANDT.

(*a*) *Sidelights on Relativity.* By ALBERT EINSTEIN. Translated by G. B. JEFFERY and W. PERRETT. New York, E. P. Dutton and Co. 56 pages. Price $1.50.

(*b*) *The Principle of Relativity with Applications to Physical Science.* By A. N. WHITEHEAD. Cambridge, The University Press, 1922. xii + 190 pages.

(*a*) This is a translation of two lectures given by Einstein, the first "Ether and the Theory of Relativity" on May 5, 1920, in Leyden, and the second "Geometry and Experience" on January 27, 1921, in Berlin. Both lectures are "popular" in the best sense of the word and the book may be recommended to anyone seeking to "understand" relativity theory. It is, unfortunately, too easy for the mathematician to focus his attention on the tensor theory which is after all merely the skeleton upon which the theory of relativity is built, and it should not be forgotten that the great contribution made by Einstein is that he clothed the skeleton and gave to it the breath of life. Just how it is that tensor analysis finds its applications in explaining the external physical world constitutes the theory of relativity, and two aspects of this question are ably treated in the little book under review. The first lecture deals with the ether which seems, at first sight, to lose, by the special relativity theory, the one remaining property of matter with which it had been left by the pre-relativists, namely, the property of being able to move. The second lecture clearly distinguishes between "purely axiomatic geometry" and "practical geometry."

It is interesting after reading these lectures to reflect on the progress which has been made in the five-year period which has elapsed since they were delivered. We think that most will agree that by far the greatest advances have been along the purely mathematical side. Our attention has been directed to a study of the articulation and the nature of the bones of the skeleton, and we have not very much more to say about how the living being may be expected to operate than we had five years ago. However, it seems that we are advancing toward an understanding of the connection between the electromagnetic and gravitational fields, so that the remark on p. 21 of the book under review ". . . the formal nature of the electromagnetic field being as yet in no way determined by that of the gravitational ether" may have to be reconsidered.[1]

(*b*) Professor Whitehead's book gives an alternative rendering of the theory of relativity. The tensor skeleton is the same and we may say at once that it is as an anatomist that the author most appeals to us. The third part of the book, in which an account of the elementary theory of tensors is given, is excellent. We may, in particular, mention the emphasis laid on the fact that absolute differentiation may be defined with respect to any symmetric covariant tensor of the second rank, not necessarily the metrical tensor. Similarly we may speak of tensors reciprocal to one another with respect to a given symmetrical covariant tensor of the second rank. Once, however, the bearing space has been endowed with a metrical quadratic form with respect to which all reciprocation is supposed to be performed, any tensor may be presented in a covariant, contravariant, or mixed form. No matter what the form of presentation is, it is the same tensor,[2] and it seems unfortunate to point so significantly to the dress, thus distracting attention from the personality underneath. We would suggest, therefore, the phraseology "covariant presentation" of a tensor and so on. Of course when the reciprocation is done with respect to some tensor other than the metrical one, reciprocal tensors are essentially different and are not merely different presentations of the same thing.

Part 1 of the book is mainly philosophical and we may as well confess at once our inability to pass a worthwhile judgment upon it. We are like the mathematician, to whom the author refers in his prefatory explanations, who advised the "cutting out of the philosophy," and we are therefore able to sympathize with the philosopher who advised the "cutting out of the mathematics." It is no doubt a fault of our training, but when we read such a group of sentences as "Fact is a relationship of factors. Every factor of fact essentially refers to its relationships within fact. Apart from this reference it is not itself" (p. 16), we feel like Scott's covenanter who sat patiently for hours upon hours listening to his preacher proceeding from "firstly" to "forty-ninthly" and at the end carried away in his heart nothing but a proper humility for his inability to comprehend such learning.

[1] *Cf.* G. Rainich, *Proc. Nat. Ac. Sci.*, vol. 10, pp. 124, 194. *Trans. Amer. Math. Soc.*, vol. 27, Jan. 1925.

[2] The fact that not only the tensor but also its various presentations are the same in a system of "normal coördinates" is what makes these coördinates so useful.

Chapter 4 of the book is a reproduction of the lecture delivered at Bryn Mawr on April 18, 1922, at the celebration in honor of Professor C. A. Scott. It gives an admirable presentation of the nature of Whitehead's difference with Einstein as to how the tensor skeleton has its being and lives in the external physical world. In a word, Whitehead's philosophy leads him to the conclusion that the physical space-time continuum is of necessity homogeneous and he endows it accordingly with an Euclidean metrical character (involving apparently an appeal to the existence of similar figures in addition to the homogeneous character of the space). The fundamental law of inertia is then stated in the form that the paths of a particle shall be extremals for an integral called the "realised impetus along the path." This integral has for its integrand a combination of the square root of a quadratic form with a linear form (to take care of the gravitational and the electromagnetic fields). In a sense, then, Whitehead considers a geometry of paths, and he is able in this way to give a satisfactory explanation of such matters as the motion of Mercury's perihelion and the bending of a ray of light in a gravitational field. Many interesting physical applications are given in Part 2 of the book.

It will be surmised from the foregoing remarks that Whitehead's and Einstein's theories are like two geometries each self-contained and consistent and each capable of being applied to the physical world. Is one theory "better" than the other? The answer to this question must rest largely with the individual since better merely means more convenient or more pleasing to one's æsthetic sense. To many, the Einstein theory has the stronger appeal since it fixes the inertial paths by means of the underlying metrics of the bearing space. While we freely admit that an expounder of the Whitehead point of view is under no logical obligation to explain why his inertial paths minimize the impetus integral, we suspect that he must secretly ponder over this matter in the privacy of his study.

F. D. MURNAGHAN.

Mathematical Principles of Finance. By FREDERICK CHARLES KENT. New York, McGraw-Hill Book Company, 1924. xii + 253 pages. Price $3.00.

The special field covered by this text, that of the accumulation and investment of funds treated from the point of view of mathematics, has had quite an offering of books, including revisions, in the last few years and consequently it is natural to ask the *why* for this book. Certainly not the least important of the reasons is that it is offered by a prosperous firm which is ever expanding its technical field. The text proper, which is generally well arranged, contains a chapter on the Federal farm loan project in addition to the topics customarily treated by a book of this character. Also the tables, which are easy on the eyes, offer a wider than ordinary range in the interest rates employed, including the three useful rates of 7/24, 1/3, and 7/8 of a per cent.

The pages present a pleasant reaction to the eye because of their open non-crowded arrangement. A large number of illustrative examples are worked out. On the other hand, the text is marred by unfortunate bits of carelessness, some

being due to set-up of the type and to proofreading. Thus the author writes

$$1/6 \log \cdot 1 = -1.8333$$

instead of the much more common forms of $\bar{1}.8333$ or 9.8333–10. There appears (p. 46)

$$\text{antilog} - 1 = 1.99667$$

and three lines later

$$\text{antilog} - 1 = .056409$$

from which by subtraction the careless student might deduce

$$0 = 1.940261.$$

On page 62 is found

$$1/i \cdot 1/s_{\overline{x}|}$$

which has a doubtful meaning to say the least. Similar lack of definiteness abounds. In the chapter on life insurance there is used in its customary significance the symbol A_x whereas the same symbol rather than the better one $A_{\overline{x}|}$ has been used earlier in a different sense. In Chapter X there occurs as formula (80) the expansion by the binomial theorem of $(1 + b)^n$ for n a positive integer, including explicitly the *last* term b^n. The next sentence is "If n is a negative number or a fraction, we may still expand $(1 + b)^n$ by formula (80); but in all such cases the series will contain an infinite number of terms"! The given value of $e = 2.7182812 \cdots$ is incorrect. In the geometric progression

$$1 + r + r^2 + r^3 + \cdots$$

the all too common error of requiring for convergence that r be a *proper fraction* rather than that $-1 < r < +1$ is repeated. In the explanation of permutations the word *independent* is sadly missing.

Most if not all of these as well as other bits of unfortunate carelessness can easily be corrected in the second printing of the book. We feel that such correction ought to be made before the careful teacher can put the book into the hands of students who are so immature mathematically as to have had only one unit of high school algebra, for which class of students the book seems in part to be arranged. The inclusion of interpolation by second differences rather than the extensive use of logarithms would seem to the reviewer to be a desideratum.

C. F. Craig.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 11, no. 2, February, 1925: "On the intersection invariants of a manifold" by J. W. Alexander, 143–146; "The rotating disc" by P. Franklin, 147–149; "Transitive groups involving direct products of lower degree" by G. A. Miller, 150–152.

SCHOOL SCIENCE AND MATHEMATICS, volume 25, no. 3, March, 1925: "Algebra in the junior high school" by E. C. Hinkle, 271–286; "The mathematics involved in solving high school physics problems" by G. W. Reagan, 292–298.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3140. Proposed by C. C. MACDUFFEE, Ohio State University.

Let f be any algebraic form of total degree $m > 1$ in n variables, and $H(f)$ its Hessian. Let φ be any analytic function. Prove that

$$H[\varphi(f)] = H(f)\cdot\left[\left(\frac{\partial\varphi}{\partial f}\right)^n + \frac{mf}{m-1}\frac{\partial^2\varphi}{\partial f^2}\left(\frac{\partial\varphi}{\partial f}\right)^{n-1}\right].$$

In particular when $\varphi(f)$ is f^r, we have Mrs. Ballantine's generalization of Problem **2908** proposed by Professor Dickson [*1921*, 326; *1923*, 41].

Again, when $\varphi(f) = \log f$, we have

$$H(f) = (1-m)f^nH[\log f],$$

which is a generalization of Exercise 22 of Sir Thomas Muir's "*Budget of Exercises on Determinants*" printed in this MONTHLY [*1922*, 10].

3141. Proposed by JOHN BIGGERSTAFF, University of Washington.

Prove that the domain $\Omega(N)$ is identical with the domain $\Omega(a)$, N being primitive in $\Omega(a)$.

3142. Proposed by HARRY LANGMAN, New York City.

Cut a rectangle 1 x 2 into three pieces which will fit into a Maltese cross.

3143. Proposed by EDWARD CONDON, University of California.

Prove that

$$\frac{\sin(n-1)\alpha}{\sin n\alpha} = \frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\cdots,$$

in which the continued fraction terminates when $2\cos\alpha$ has appeared $n-1$ times. Prove also the corresponding formula for the hyperbolic sines and cosines.

UNSOLVED PROBLEMS.

There is now a somewhat large accumulation of unsolved problems. Below a few of these are reprinted and others will appear in later issues. Readers are urged to solve these and send in their solutions.

2662 [1918, 19]. Proposed by JOHN LOUKE, New York City.

Assume that we have two piles of gold bars. The dimensions of the bars in the first pile are $2.643 \times 5.286 \times 10.573$ and the dimensions of the bars in the second are $2.13 \times 6.53 \times 10.573$. If possible, arrange the bars from the first pile and from the second pile so as to form perfect cubes, the bars from the piles to be taken separately or in combination.

2706 [1918, 216]. Proposed by H. F. MACNEISH, New York City.

Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the given point and the given line may be equal to the part within the given circle.

2717 [1918, 260]. Proposed by E. W. WITMER, Sophomore in Franklin and Marshall College.

Determine the integral values of m and n for which the equation $x^4 + mx^2y^2 + ny^4 = z^2$ has non-trivial solutions [Carmichael, *Diophantine Analysis*, Prob. 13, p. 53].

2720 [1918, 302]. Proposed by A. A. BENNETT, Austin, Texas.

Given three points A, B, C, in a plane, draw from an arbitrary fourth point D the segments AD, BD, CD. Also draw rays AA', BB', CC' making equal (small) angles respectively with segments AD, BD, CD. The triangle determined by the three rays does or does not contain the point D according as the original triangle ABC does or does not contain D.

Prove the theorem, considering also the case in which A, B, C, D are concyclic.

SOLUTIONS.

3101 [1924, 498]. Proposed by R. E. GAINES, University of Richmond, Va.

A variable conic passes through four fixed points. Show that the locus of its center is a conic. Determine the arrangement of the four points so that this locus shall be a circle and discuss the result. This problem is treated in texts on projective geometry, but a simple analytical treatment is desired.

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

Abbreviate the equation

$$A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0$$

by letting L_1 stand for the left-hand member, so that we may write the equation as $L_1 = 0$. Then the coördinates of the center of this conic will satisfy the equations $\partial L_1/\partial x = 0$ and $\partial L_1/\partial y = 0$, where we note that the left-hand member of each of these equations is a polynomial of the first degree in x and y.

If the pencil of conics passing through the four given points has the equation $L_1 + \lambda L_2 = 0$, where λ is a variable parameter, then the coördinates of the center of any one of the pencil will satisfy the equations

$$\frac{\partial(L_1 + \lambda L_2)}{\partial x} = 0, \ (A) \qquad \text{and} \qquad \frac{\partial(L_1 + \lambda L_2)}{\partial y} = 0, \ (B),$$

for the appropriate value of λ. Hence we get the equation of the locus of the center by eliminating λ. Since the above equations are linear in x and y and also in λ, the result of the elimination is an equation of the second degree, which represents a conic. This shows that the center of any conic of the pencil lies on this (locus) conic. Conversely, if we take any point on the (locus) conic, it will be the center of a conic of the pencil; for the result of the elimination may be written

$$\left(\frac{\partial L_1}{\partial x} + \lambda\frac{\partial L_2}{\partial x}\right)\frac{\partial L_2}{\partial y} - \left(\frac{\partial L_1}{\partial y} + \lambda\frac{\partial L_2}{\partial y}\right)\frac{\partial L_2}{\partial x} = 0. \qquad (C)$$

Take a point on this curve. If its coördinates satisfy both $\partial L_2/\partial x = 0$ and $\partial L_2/\partial y = 0$, it is the center of $L_2 = 0$; if not, then we can determine λ so that the coördinates satisfy one of the equations (A) or (B) above, and since they satisfy (C), they must satisfy the other of (A) and (B) also and are consequently the coördinates of the center of a conic of the pencil.

To find the conditions that the locus be a circle, we need to write out the equation of the locus. It is

$$(A_1x + B_1y + D_1)(B_2x + C_2y + E_2) - (B_1x + C_1y + E_1)(A_2x + B_2y + D_2) = 0. \qquad (D)$$

(We can see here directly that this will be of the second degree unless $A_1 : B_1 : C_1 = A_2 : B_2 : C_2$). For this to be the equation of a circle it is necessary that

$$A_1B_2 - B_1A_2 = B_1C_2 - C_1B_2$$

and

$$A_1C_2 + B_1B_2 = B_1B_2 + C_1A_2.$$

These equations may be written

$$B_2(A_1 + C_1) = B_1(A_2 + C_2),$$
$$A_2(A_1 + C_1) = A_1(A_2 + C_2),$$
$$C_2(A_1 + C_1) = C_1(A_2 + C_2).$$

But from these it follows at once that either $A_1 + C_1 = 0$ and $A_2 + C_2 = 0$ or else that $A_1 : B_1 : C_1 = A_2 : B_2 : C_2$. But in the second case the two conics either would be identical or would not have four points in common, as we assumed. The first case is the condition that L_1 and L_2, and consequently all conics of the pencil, be equilateral hyperbolas. Substituting in (D), we see that this condition is sufficient. The circle will obviously be real, as each hyperbola has a real center.

It remains to investigate the position of the four points. The degenerate conics through the four points must be pairs of perpendicular lines. (This follows analytically from the fact that the equation of any one of them has $A + C = 0$.) Taking one of these as our axes, we find that the points must have coördinates $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$, $(0, y_2)$, where $x_1x_2 = - y_1y_2$. This may be interpreted geometrically to mean that if we pass a circle through three of the four points A, B, C, this circle will cut one of the axes in a fourth point, E, so that $OE = DO$ and D and E lie on the same axis. Another way of making the statement is that if we draw a line through any two of the four points A, B, C, D, and a second line through the other two, the lines will be perpendicular. In other words, any one of these points is the orthocenter of the triangle formed by the other three.

II. Solution by the Proposer.

Let the four fixed points be A, B, C, D, no three of which lie on a straight line. The three pairs of lines through these points are the degenerate conics of the system and the locus passes through the intersection of each pair. There must be at least one pair intersecting in a finite point; let such a pair be OAC and OBD. Using these two lines as axes the equation of any curve of the system may be written

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right)\left(\frac{x}{c} + \frac{y}{d} - 1\right) - \lambda xy = 0, \qquad (1)$$

where $abcd \neq 0$. Eliminating λ from the two linear equations which give the center, we find as the locus of that center the conic

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{c} + \frac{y}{d} - 1\right) + \left(\frac{x}{c} - \frac{y}{d}\right)\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0. \qquad (2)$$

This equation shows that the conic passes through four of the intersections of the four lines obtained by equating each of the four parentheses to zero. These points are easily found to be I and J, the mid-points of AB and CD, O and the intersection, M, of AB and CD. We could have used the pair of lines AD and BC in place of AB and CD for writing the system of conics; also we could have chosen any one of the two other pairs of lines for axes provided that they meet in a finite point. In this way, or by direct verification, it follows that (2) also passes through K, H, R, S, the mid-points of BD, AC, BC, AD, and N the intersection of BC and AD. The locus is the eleven-point conic of the line at infinity; nine of the points have been found above and the other two are the double points of the involution determined by (1) on the line at infinity.

A figure shows that the six mid-points form three parallelograms with sides parallel, respectively, to AC and BD, AD and BC, AB and CD. The center of (2) is the common point of the diagonals of these parallelograms. The necessary and sufficient condition that (2) is a circle is

that each of the last three pairs of lines are perpendicular. From this follows that the axes are rectangular and then that $ac + bd = 0$. By use of this last relation the terms of the second degree in (1) may be reduced to the form $x^2 - y^2 + kxy$; and this shows that (1) represents now a system of rectangular hyperbolas. Any three of the given four points form a triangle with the fourth point as its orthocenter, and it has been shown above that the circle passes through nine points of the triangle. The circle is the nine-point circle for any one of the four triangles.

Also solved by NINA M. ALDERTON and E. F. ALLEN.

3102 [1924, 498]. Proposed by R. E. GAINES, University of Richmond, Va.

For the cardioid, $\rho = 1 + \cos\theta$, (*a*) find the maximum chord; (*b*) find the maximum chord parallel to the axis.

I. SOLUTION BY SIDNEY ZABARO, California Inst. of Technology.

(*a*) If (ρ_1, θ_1) and (ρ_2, θ_2) are two points on the cardioid, the distance l between them is given by

$$l^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2). \tag{1}$$

Setting the derivatives of l^2 with respect to θ_1 and θ_2 to zero, there result the two necessary conditions

$$\rho_1 \sin\theta_1 - \rho_2 \sin\theta_1 \cos(\theta_1 - \theta_2) - \rho_1\rho_2 \sin(\theta_1 - \theta_2) = 0, \tag{2}$$

$$\rho_2 \sin\theta_2 - \rho_1 \sin\theta_2 \cos(\theta_1 - \theta_2) + \rho_1\rho_2 \sin(\theta_1 - \theta_2) = 0. \tag{3}$$

The addition of these two equations and certain simple reductions give the result

$$(\cos\theta_2 - \cos\theta_1)\sin(\theta_1 - \theta_2) = 0. \tag{4}$$

From this follows that either

$$\theta_1 - \theta_2 = 0; \quad \text{or} \quad \theta_1 - \theta_2 = \pi; \quad \text{or} \quad \theta_1 + \theta_2 = 0. \tag{5}$$

The first gives obviously a minimum; the second taken with (2) gives the two angles 0 and π for which the value of l is 2. The curve shows that this is also a minimum. The third with (2) gives

$$2\sin\theta_1(1 + \cos\theta_1)(2\cos^2\theta_1 + \cos\theta_1 - 1) = 0.$$

The first two factors give $\theta_1 = \theta_2 = 0$ and $\theta_1 = -\theta_2 = \pi$, which again yield minima. The third factor gives $\cos\theta_1 = -1$ and $\cos\theta_1 = \frac{1}{2}$. The first of these has already been treated. Since there is obviously a maximum, the last result must give it. Hence the maximum chord is given by $\theta_1 = -\theta_2 = \pi/3$, and then $l = 3\sqrt{3}/2 = 2.598$.

(*b*) Here we have the two equations

$$l = \rho_1 \cos\theta_1 - \rho_2 \cos\theta_2, \qquad \rho_1 \sin\theta_1 = \rho_2 \sin\theta_2. \tag{1}$$

From these we obtain after certain reductions

$$\frac{dl}{d\theta_1} = -\frac{2\cos\dfrac{\theta_1}{2}\cos\dfrac{\theta_2}{2}\sin\dfrac{3}{2}(\theta_1 - \theta_2)}{\cos\dfrac{\theta_2}{2}\cos\dfrac{3\theta_2}{2}} = 0. \tag{2}$$

The first two factors in (2) give the angles 0 and π corresponding to a minimum. The third gives $\theta_1 - \theta_2 = 0$ (minimum), and $\theta_1 - \theta_2 = 2\pi/3$ [this is equivalent to $\theta_1 - \theta_2 = 4\pi/3$].

The second equation in (1) reduces to

$$2\sin\frac{\theta_1 - \theta_2}{2}\left[\cos\frac{\theta_1 + \theta_2}{2} + \cos(\theta_1 + \theta_2)\cos\frac{\theta_1 - \theta_2}{2}\right] = 0.$$

Inserting in this equation $\theta_1 - \theta_2 = 2\pi/3$, there results

$$2\cos^2\frac{\theta_1 + \theta_2}{2} + 2\cos\frac{\theta_1 + \theta_2}{2} - 1 = 0.$$

The only admissible root is

$$\cos\left(\frac{\theta_1+\theta_2}{2}\right) = (\sqrt{3}-1)/2,$$

or $\theta_1+\theta_2 = 137° \; 4'$, considering only the upper half of the figure. Hence $\theta_1 = 128° \; 32'$, $\theta_2 = 8° \; 32'$. The value of l for the maximum chord reduces to

$$l = \frac{(1+\sqrt{3})\sqrt{6\sqrt{3}}}{4} = 2.20.$$

II. Solution by the Proposer.

If any one of a set of parallel chords has a maximum length, then the tangents to the curve at its extremities are parallel. If τ is the angle which the tangent at any point of the cardioid makes with the positive direction of the axis, then

$$\tau = \frac{\pi}{2} + \frac{3\theta}{2}, \tag{1}$$

This result may be easily obtained geometrically by considering the curve as generated by a circle rolling upon an equal circle of radius $\frac{1}{2}$. If θ_1 and θ_2 give the maximum chord for any set; then

$$\theta_1 - \theta_2 = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}\cdot \tag{2}$$

The length of the chord is then given in either case by

$$l^2 = \rho_1^2 + \rho_2^2 + \rho_1\rho_2. \tag{3}$$

By the use of (2) the equation (3) reduces after a trigonometric reduction to

$$l^2 = \frac{15}{4} \pm 3\cos\frac{\theta_1+\theta_2}{2}, \tag{4}$$

where the upper sign is for the first case and the lower sign for the second in (2). In either case the maximum value is $l = 3\sqrt{3}/2$. In the first case $\theta_1 = 420°$, $\theta_2 = 300°$, and in the second $\theta_1 = 300°$, $\theta_2 = 60°$. The position of the chord is the same in both, *i.e.*, perpendicular to the axis.

For (b) we consider the upper half of the curve; and hence we have the first equation in (2), the equation (4) with the $+$ sign, and the additional equation

$$\sin\theta_1(1+\cos\theta_1) - \sin\theta_2(1+\cos\theta_2) = 0.$$

This equation reduces by aid of (2) to

$$2\cos^2\frac{\theta_1+\theta_2}{2} + 2\cos\frac{\theta_1+\theta_2}{2} - 1 = 0.$$

Hence

$$\cos\left(\frac{\theta_1+\theta_2}{2}\right) = (\sqrt{3}-1)/2$$

and then from (4)

$$l = \frac{\sqrt{9+6\sqrt{3}}}{2} = 2.2018.$$

The values of θ_1 and θ_2 are easily computed from the equations above.

3104 [1924, 499]. Proposed by OTTO DUNKEL, Washington University.

If $f(x)$ is a single-valued and continuous function of x in the interval $a \leq x \leq b$ which is not identically zero and which satisfies the inequality $0 \leq f(x) \leq M$, show that

$$0 < \left[\int_a^b f(x)dx\right]^2 - \left[\int_a^b f(x)\cos x\,dx\right]^2 - \left[\int_a^b f(x)\sin x\,dx\right]^2 \leq M^2(b-a)^4/12$$

SOLUTION AND GENERALIZATION BY A. A. BENNETT, University of Texas.

This interesting problem suggests the following rather extensive generalization. Except for the added weight of notation, this more general problem is here solved with only minor additional complications.

HYPOTHESIS. Consider a space of n non-homogeneous coördinates $(x_1, x_2, \cdots, x_n)$, and consider a continuously turning curve located at finite distance in this space. Instead of the familiar ds as differential of arc, we may take t as parameter, where $Q(dx_i) = (dt)^2$, Q denoting a non-singular positive definite quadratic form in the n arguments. Thus $x_1(t), x_2(t), \cdots, x_n(t)$ are the coördinates of an arbitrary (differentiable) curve, and are one-valued functions of t for all real finite values of t, and are such that $Q(x_i') \equiv 1$ for all points on the curve. We shall also assume that the first four derivatives of each $x_i(t)$ exist and are continuous. We shall utilize the function Q also in a second sense. Indeed $Q(y_i) = 1$ represents a closed non-singular quadric surface, symmetrical about the origin. By a suitable transformation, Q could be reduced to a sum of squares but we shall find it practicable to carry through the work in the more general form. We shall now restrict our curve to be one lying on this quadric surface. Let x_i' be denoted also by u_i. Let $B(y_i, z_j)$ denote the symmetric bilinear form which reduces to $Q(x_i)$ for $y_i = x_i$, $z_j = x_j$. A final condition that we impose upon the curve which is the locus of the x's is that $B[x_i^{\text{iv}}(t), x_j(a)]$ regarded as a function of t shall have an absolute maximum for $t = a$, where $x^{\text{iv}}(t)$ is the fourth derivative with respect to t. Let $H(f, g)$ stand for the expression

$$\left[\int_a^b f(t)dt\right]\left[\int_a^b g(t)dt\right] - B\left[\int_a^b f(t)u_i(t)dt, \int_a^b g(t)u_j(t)dt\right],$$

so that $H(f, f)$ represents

$$\left[\int_a^b f(t)dt\right]^2 - Q\left[\int_a^b f(t)u_i(t)dt\right],$$

where f and g are continuous, non-negative and not identically zero, real one-valued functions of t, such that $0 \leq f(t) \leq M$, $0 \leq g(t) \leq M$, M being a fixed positive constant.

THEOREM. $0 < H(f, f) \leq M^2(b-a)^4 Q[u_i'(a)]/12$. Also more obviously $H(f, f) \leq M^2(b-a)^2$, which for large values of $b - a$ is ordinarily the more stringent condition.

EXAMPLE. The conditions are clearly satisfied by the special case stated in the problem, where $Q(x_i)$ is $x_1^2 + x_2^2$, and where $x_1 = \sin t$, and $x_2 = \cos t$. Here $B(y_i, z_j)$ is merely $y_1z_1 + y_2z_2$, and $B[x_i^{\text{iv}}(t), x_j(a)]$ becomes $\sin t \sin a + \cos t \cos a = \cos(t - a)$, which has an absolute maximum at $t = a$.

PROOF. We shall approximate the expression $H(f, g)$ by replacing each integral by its approximate value as a finite sum. Let the interval a to b be divided into equal sub-intervals and let $f(t)$ in a representative sub-interval be represented by f_r, and $g(t)$ by g_s. Similarly $u_i(t)$ at the same point as f_r will be denoted by u_{ir}, and $u_j(t)$ at the same point as g_s by u_{js}. We have, omitting mention of the common length of the equal sub-intervals, the following sum:

$$\Sigma f_r \Sigma g_s - B(\Sigma f_r u_{ir}, \Sigma g_s u_{js}),$$

or utilizing the bilinear homogeneous character of B,

$$\Sigma f_r g_s [1 - B(u_{ir}, u_{js})].$$

Since $B(u_i, u_j) = Q(u_i) \equiv 1$, and B is symmetrical, we may write this last expression as

$$\tfrac{1}{2}\Sigma f_r g_s [B(u_{ir}, u_{jr}) - B(u_{ir}, u_{js}) - B(u_{is}, u_{jr}) + B(u_{is}, u_{js})]$$

or

$$\tfrac{1}{2}\Sigma f_r g_s B(u_{ir} - u_{is}, u_{jr} - u_{js}) = \tfrac{1}{2}\Sigma f_r g_s Q(u_{ir} - u_{is}).$$

Since f and g are continuous, non-negative and not identically zero, there is at least one pair of distinct subscripts, r, s, for which both f_r and g_s are positive. Then the product $f_r g_s$ is positive, and since Q is positive definite, and $u_{ir} \neq u_{is}$, we conclude that for this term $f_r g_s Q(u_{ir} - u_{is}) > 0$. Thus passing to the limit, we have

$$H(f, g) > 0.$$

Now we have also that

$$H(f + g, f + g) = H(f, f) + 2H(f, g) + H(g, g),$$

owing to the additive character of integration and the bilinear character of B.

Suppose that f be different from the upper bound M. Then $M - f$ is a possible choice for the function g. Thus since $H(f, g) > 0$ and $H(g, g) > 0$, it follows that $H(M, M) > H(f, f)$. Thus the maximum value of $H(f, f)$ is attained when f coincides with the upper bound M. Thus we have that

$$0 < H(f, f) \leqq H(M, M) = M^2 H(1, 1).$$

Let us obtain an expression for this upper bound $H(1, 1)$ also expressible as $(b - a)^2 - Q[x_i(b) - x_i(a)]$. We shall write $b = a + r$, and expand

$$Q[x_i(a + r) - x_i(a)] = B[x_i(a + r) - x_i(a), \quad x_j(a + r) - x_j(a)]$$

in powers of r to four terms. We may readily derive in succession the following relations which are identities in t:

$$\begin{aligned}
&B[x_i(t), x_j(t)] = 1,\\
&B[x_i'(t), x_j(t)] = 0,\\
&B[x_i''(t), x_j(t)] + B[x_i'(t), x_j'(t)] = 0,\\
&B[x_i'(t), x_j'(t)] = B[u_i(t), u_j(t)] = 1,\\
&B[x_i''(t), x_j'(t)] = 0,\\
&B[x_i'''(t), x_j'(t)] + B[x_i''(t), x_j''(t)] = 0,\\
&B[x_i''(t), x_j(t)] = -1,\\
&B[x_i'''(t), x_j(t)] + B[x_i''(t), x_j'(t)] = 0,\\
&B[x_i'''(t), x_j(t)] = 0,\\
&B[x_i^{\text{iv}}(t), x_j(t)] + B[x_i'''(t), x_j'(t)] = 0,\\
&B[x_i^{\text{iv}}(t), x_j(t)] = B[x_i''(t), x_j''(t)] = Q[x_i''(t)].
\end{aligned}$$

Now $dQ[x_i(a + r) - x_i(a)]/dr = 2B[x_i'(a + r), x_j(a + r) - x_j(a)] = 2B[x_i'(a + r), x_j(a + r)] - 2B[x_i'(a + r), x_j(a)] = - 2B[x_i'(a + r), x_j(a)]$. Hence $d^kQ[x_i(a + r) - x_i(a)]/dr^k = - 2B[x_i^{(k)}(a + r), x_j(a)]$, for $k \geqq 1$. Expanding, we have for $r = 0$, $Q = 0$, $Q' = 0$, $Q'' = 2$, $Q''' = 0$. Thus the finite expansion of $H(1, 1)$, $= r^2 - Q[x_i(a + r) - x_i(a)]$ reduces to $2B[x_i^{\text{iv}}(a + \xi), x_i(a)]r^4/4! \leqq 2B[x_i^{\text{iv}}(a), x_i(a)]r^4/4! = Q[x_i''(a)]r^4/12$, where ξ is a mean value of r. This suffices to establish the theorem.

Also solved by L. V. ROBINSON.

3105 [1924, 499]. Proposed by W. A. GRANVILLE, Chicago, Illinois.

Find the length of the arc of the cissoid $\rho = 2a \tan \theta \sin \theta$, from $\theta = 0$ to $\theta = \pi/4$.

SOLUTION BY J. S. GEORGES, University of Chicago.

The formula for the length of the arc gives

$$\begin{aligned}
S &= 2a \int_0^{\pi/4} (4 + \tan^2 \theta)^{1/2} \tan \theta \, d\theta = 2a \int_1^{\sqrt{2}} \frac{(3 + y^2)^{1/2}}{y} dy, \quad y = \sec \theta,\\
&= 2a[\sqrt{5} - 2 - \sqrt{3} \log (\sqrt{3} + \sqrt{5})(2 + \sqrt{3})/\sqrt{2}].
\end{aligned}$$

Also solved by S. F. BIBB, J. A. BULLARD, A. G. CLARK, W. H. HILL, G. A. LYLE, E. I. WATSON, W. W. WALLACE, and T. O. WALTON.

3107 [1925, 46]. Proposed by A. S. WIENER, Brooklyn, New York.

A certain city is divided into rectangular blocks by two systems of parallel streets, one system running north and south. A man standing at the intersection of two streets wishes to reach the intersection of two other streets m blocks north and n blocks east. In how many ways can he do so?

Also what is the probability that he will pass the corner which is i blocks north and j blocks east ($i \leqq m$, $j \leqq n$), assuming that he is just as likely to take one route as another?

SOLUTION BY W. A. JENKINS, University of Michigan.

Let $f(m, n)$ be the number of ways a man can travel from one intersection to another intersection m blocks north and n blocks east. Lest this be infinite, we must assume that at no time does the man proceed southward or westward. Then, if either argument is negative, we set $f(m, n) = 0$; also if one argument is zero and the other is not negative, $f(m, n) = 1$. This function satisfies the law

$$f(m, n) = f(m - 1, n) + f(m, n - 1).$$

Developing $f(m - 1, n)$ and $f(m, n - 1)$ by the above law, we find

$$f(m, n) = f(m - 2, n) + 2f(m - 1, n - 1) + f(m, n - 2).$$

Developing again each term on the right, we obtain another similar result with binomial coefficients; and continuing in this way we have

$$f(m, n) = \sum_{s=0}^{s=k} {}_kC_s f(m - k + s, n - s).$$

Now set $k = m + n$, then in the general term on the right, $f(-n + s, n - s)$, one argument is negative and the term drops out unless $s = n$. Hence

$$f(m, n) = {}_{m+n}C_n = \frac{(m + n)!}{m!n!}.$$

The probability is therefore

$$\frac{f(i, j)f(m - i, n - j)}{f(m, n)} = \frac{(i + j)!(m + n - i - j)!\, m!\, n!}{i!j!\,(m - i)!\,(n - j)!\,(m + n)!}.$$

Also solved by A. G. CLARK and H. O. HANSON.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

At Brown University, Assistant Professor R. W. BURGESS has resigned and will continue the work with the general statistical department of the Western Electric Company which he has been doing during the current year while on leave of absence. Dr. C. R. ADAMS has been promoted to an assistant professorship of mathematics.

F. S. BALDWIN, one of the pioneers in the development of the calculating machine, died at Morristown, N. Y., on April 8th at the age of 87. Possessing a thorough knowledge of mathematics and gifted with unusual inventive ability, he was almost continuously engaged in perfecting machines for calculation from 1870 until within a few years of his death. In 1875 he patented the first practical reversible-cycle calculator. In 1911, in coöperation with Mr. J. R. MONROE, the Monroe calculating machine was developed.

At Harvard University, Dr. H. W. BRINKMANN has been appointed instructor and member of the faculty. Professor E. T. BELL, of the University of Washington, has been appointed visiting lecturer, and Dr. L. M. GRAVES, now National Research Fellow, instructor, for the first half-year. Mr. B. O. KOOPMAN has

been appointed Benjamin Peirce instructor, and Mr. H. L. GARABEDIAN, Mr. H. B. HAMMATT, Mr. W. A. JENKINS, Mr. MALCOLM MACCLAREN, Jr., Mr. MORRIS MARDEN, Mr. F. W. PERKINS, Mr. H. P. STABLER, and Mr. D. E. WHITFORD instructors, for the year 1925–1926.

Through the benefaction of ROBERT FLETCHER ROGERS, an alumnus, two prizes were recently established at Harvard University for the best papers presented before the Mathematical Club during the year. Thus far, the winners have been: 1921–22, H. W. BRINKMANN (first prize), M. H. STONE (second prize); 1922–23, B. O. KOOPMAN, J. L. HOLLEY; 1923–24, D. V. WIDDER, F. W. PERKINS; 1924–25, MORRIS MARDEN, M. S. DEMOS.

Mr. E. T. FRANKEL, formerly statistician and secretary to the chairman, The Celluloid Company, has opened an office in New York City as a consulting business mathematician.

Assistant Professor A. D. CAMPBELL, of the University of Arkansas, has been promoted to an associate professorship of mathematics.

At the University of Wyoming, Associate Professor H. C. GOSSARD has been promoted to a full professorship, Assistant Professor O. H. RECHARD has been promoted to an associate professorship, and Miss GRETA NEUBAUR has been appointed instructor.

Miss OLIVE C. HAZLETT, of Mount Holyoke College, has been appointed an assistant professor of mathematics at the University of Illinois.

At Oberlin College Associate Professor MARY EMILY SINCLAIR has been promoted to a full professorship, and Assistant Professors F. E. CARR and C. H. YEATON to associate professorships. Professor Sinclair has also been voted a leave of absence for the year 1925–26 and will study at the University of Rome.

At the University of Minnesota, Associate Professor W. L. HART has been promoted to the rank of professor, and Miss GLADYS GIBBENS from the rank of instructor to that of assistant professor.

At Harvard University, Professor J. L. WALSH has been granted leave of absence for the academic year 1925–26, and Professor G. D. BIRKHOFF for the second semester. Professor Birkhoff recently delivered a popular lecture on relativity at Indiana University under the auspices of the local chapter of Sigma Xi.

Professor A. A. BENNETT, of the University of Texas, has been appointed professor and head of the department of mathematics at Lehigh University.

At Teachers College, Columbia University, Professor D. E. SMITH will retire at the end of the first semester, 1925–26. Dr. W. D. REEVE has been appointed associate professor.

Dr. HENRY BLUMBERG, of the University of Illinois, has been appointed professor of mathematics at Ohio State University.

Professor J. V. DePorte, of the New York State Teachers College, Albany, N. Y., has been appointed director of the division of vital statistics of the New York State Department of Health.

Professor S. Lefschetz, of the University of Kansas, has been appointed associate professor of mathematics at Princeton University.

Professor J. N. Michie, of the University of Texas, has been appointed head of the department of mathematics at Texas Technology College.

Associate Professor W. L. Miser, of the Armour Institute of Technology, has been appointed professor of mathematics at Vanderbilt University.

Professor F. B. Wiley, of Denison University, served as acting head of the department of mathematics at Roberts College, Constantinople, for the academic year 1924–25.

Assistant Professor I. T. Wilson, of the United States Naval Academy, has been promoted to an associate professorship.

The following appointments to instructorships are announced: Columbia University, Mr. H. C. Lieber; Case School of Applied Science, Mr. P. D. Wilkins.

Dr. D. J. McAdam, professor emeritus of mathematics at Washington and Jefferson College, died February 15, 1925, at the age of eighty-two.

S. E. Roberts, formerly professor of mathematics at the University of the Philippines, died March 9, 1925, at the age of forty-seven.

Professor C. E. Stromquist, of the University of Wyoming, a charter member of this Association, died on April 3, 1925, after three years of ill health.

Number 1–2 of Volume I of the Japanese *Journal of Mathematics* has been received by the library of the Association. This quarterly journal is published by the National Research Council of Japan, Department of Education at Tokyo, Baron K. Furuichi, President. The initial issues contain sixteen shorter papers ("transactions") and twenty-five abstracts, chiefly by Japanese mathematicians. This is to be recorded as a notable advance in the development of Japanese mathematics.

The Mathematical Association of America wishes again to call the attention of all its members to the working arrangement between the Association and the *Annals of Mathematics* by which, in return for a certain subsidy contribution from the Association, the *Annals* has extended the size of its volume to include approximately one hundred pages of expositional articles and at the same time has made the special subscription rate to individual members of the Association of one half the regular price. A goodly number of Association members have already taken advantage of this reduced rate, but it is felt that a much larger number would probably do so if their attention were sufficiently arrested.

In this connection it is pertinent to mention the fact that one expository article of nearly one hundred pages on "Differential equations from the group standpoint" by Professor L. E. DICKSON, which appeared in the last issue of the *Annals*, occupying the entire number, is on sale in the form of reprints at a price equal to two thirds the special price of a whole year's subscription to the *Annals*. This fact will emphasize the desirability of having the *Annals* regularly at hand. Some of the many other expository articles which have appeared in the *Annals* under this agreement, reprints of which are obtainable, are:

"The Gamma function in the integral calculus," by T. H. GRONWALL. 89 pages. Price 90 cents.

"An elementary exposition of the Gamma function," by J. L. W. V. JENSEN, translated by T. H. GRONWALL. 43 pages. Price 50 cents.

"Fermat's last theorem and the origin and nature of algebraic numbers," by L. E. DICKSON. 27 pages. Price 35 cents.

"An introduction to the theory of elliptic functions," by G. MITTAG-LEFFLER. 81 pages. Price 90 cents.

This is sufficient to emphasize the rare opportunity open to Association members by becoming subscribers to the *Annals* at $1.50 per year.

Reprints of the article on BENJAMIN PEIRCE are now ready for distribution. They comprise a biographical sketch and bibliography by R. C. ARCHIBALD and reminiscences by CHARLES W. ELIOT, A. LAWRENCE LOWELL, W. E. BYERLY, and ARNOLD B. CHACE, together with five portrait half-tones of Peirce representing him at various dates from 1845 to 1879. This brochure covers forty pages and may be secured in paper covers from Secretary CAIRNS at 75 cents (50 cents to members), or in board covers suited to library purposes at $1.00 per copy from the Open Court Publishing Company, 122 South Michigan Avenue, Chicago, Ill.

The THIRD PAN-AMERICAN SCIENTIFIC CONGRESS was held in Lima, Peru, from December 20, 1924, to January 6, 1925, immediately following the elaborate celebration of the hundredth anniversary of the Battle of Ayacucho (by which the independence of the Latin-American republics was finally secured from Spain).

All the republics in the western hemisphere were represented (except Chile), 138 members coming from Peru and 131 from foreign countries, including 30 from the United States, 25 from Argentina, 15 from Cuba, 9 from Venezuela, 8 from Colombia, 7 from Uruguay, 6 from Brazil, 5 from Ecuador, 5 from Mexico, etc. Of the thirty delegates from the United States, ten were official representatives of the United States Government, under the chairmanship of Dr. L. S. ROWE, Director-General of the Pan-American Union in Washington, and Honorary President of the Congress.

At the impressive opening ceremony, the Congress was formally welcomed by President LEGUIA of the Peruvian Republic and Dr. SALAMÓN, Minister of Foreign Affairs, and throughout the two and a half weeks of its sessions it was enter-

tained with the most lavish hospitality by the government of Peru, the American Embassy, the University of San Marcos, and the people of Lima.

On the scientific side, the Congress was divided into nine sections, as follows: (1) Anthropology and History. (2) Physical and Mathematical Sciences (including Geology and Geography). (3) Mining and Metallurgy. (4) Engineering. (5) Medicine and Sanitation. (6) Biology and Agriculture. (7) Law. (8) Economics and Sociology. (9) Education.

The subject of Pure Mathematics was assigned to Subsection 1 under Section 2. This subsection held one meeting for the reading of papers (December 28), at which about fifteen members were present, and the following three papers were read:

"Conjugate ordinates and their geometrical applications," by Dr. FLORENCIO D. JAIME, President of the recently organized Argentinian Mathematical Society, and delegate from the Ministry of Public Instruction of Argentina.

"Elementary types of order," by Dr. E. V. HUNTINGTON, vice-president of the American Mathematical Society, and delegate from the American Mathematical Society, the Mathematical Association of America, and the American Academy of Arts and Sciences.

"On the descriptive geometry of the sphere," by Dr. ALEJANDRO GUEVARA, Honorary Professor of the School of Engineers of Lima.

At the closing general session of the Congress, 147 resolutions were read and adopted, including a decision to hold the next Pan-American Scientific Congress in the city of San José in Costa Rica in 1929.

The University of Manchester, England, announces its second session of summer courses in post-graduate mathematics to be held at University College, Bangor, Monday, August 24, to Saturday, September 5, 1925. The following courses are offered:

"Atomic structure and the quantum theory," by Professor SYDNEY CHAPMAN (Imperial College of Science, London).

"Theory of functions," by Professor L. J. MORDELL (Manchester University).

"Higher geometry," by Mr. H. W. RICHMOND (King's College, Cambridge).

The annual spring meeting of the Middle States and Maryland Association of Teachers of Mathematics was held at Teachers College, New York City, May 9, 1925. The program was as follows:

1. General Mathematics in the Junior High School, Professor D. E. SMITH, of Columbia University.
2. General Mathematics in the Senior High School, J. A. SIVENSON, of the Wadleigh High School, N. Y. City.
3. General Mathematics in the Junior College, Professor R. W. BURGESS, of Brown University.
4. Conference on a one-year course in plane and solid geometry, directed by Professor TOMLINSON FORT of Hunter College; G. R. MIRICK of the Lincoln School led the discussion.

The annual meeting of the ASSOCIATION FRANÇAISE POUR L'AVANCEMENT DES SCIENCES will be held at Grenoble, July 27 to August 1, 1925.

Attention of American mathematicians is called to the fact that a year's subscription to *Sphinx-Œdipe* costs only \$1.20. By sending that amount (or more) to the editor, A. GÉRARDIN, 32 Quai Claude le Lorrain, Nancy, France, you will be helping a worthy mathematical journal which has already reached its twentieth year of usefulness.

A questionnaire was recently addressed by Professor H. S. EVERETT of Bucknell University to 48 colleges and universities in the state of Pennsylvania concerning existing courses in mathematics as applied in the fields of finance and statistics. Replies were received from 41 institutions and indicate that at least 19 are now offering (and several others are planning to offer) such courses. Amounts range from 2 to 21 semester hours, the average being slightly less than 6.

The following 32 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1924; the university and title of dissertation are given in each case.

R. W. BABCOCK, Wisconsin, On thermal convection; HERMAN BETZ, Yale, Surface transformations applied to dynamical systems with two degrees of freedom; A. D. CAMPBELL, Cornell, Linear systems of conics in the Galois field; ELIZABETH CARLSON, Minnesota, On the convergence of certain methods of closest approximation; G. H. COLLIGNON, Johns Hopkins, Problems of flow in connection with mapping of spherical polygons; JULIA T. COLPITTS, Cornell, On a certain class of entire functions; JULIA DALE, Cornell, Some properties of the exponential mean; MARGUERITE D. DARKOW, Chicago, Arithmetics of certain algebras of generalized quaternions; W. W. ELLIOTT, Cornell, Generalized Green's functions for compatible differential systems; F. J. GERST, Johns Hopkins, Image points and Riemann's theorem; CORNELIUS GOUWENS, Chicago, Invariants of the linear group modulo $p_1^{\lambda_1} \cdots p_n^{\lambda_n}$; L. M. GRAVES, Chicago, The derivatives as independent function in the calculus of variations; C. F. GUMMER, Chicago, The relative distribution of the real roots of a system of polynomials; J. W. HEDLEY, Chicago, Ruled surfaces whose flecnode curves belong to linear complexes; P. E. HEMKE, Johns Hopkins, A transformation involving ζ-functions with an aerodynamical application; J. L. HOLLEY, Harvard, Linear spaces and their fixed points; HAROLD HOTELLING, Princeton, Three dimensional manifolds of states of motion; J. C. HUGHES, Chicago, A problem in the calculus of variations in which one end-point is variable on a one-parameter family of curves; MILDRED HUNT, Chicago, The arithmetics of certain linear algebras; M. H. INGRAHAM, Chicago, A general theory of linear sets; C. M. JENSEN, Minnesota, Some problems in the approximate representation of a function by a Sturm-Liouville interpolation formula; C. G. Latimer, Chicago, Arithmetic of generalized quaternions; HARRY LEVY, Princeton, Tensors determined by a hypersurface in a Riemann space; J. B. LINKER, Johns Hopkins, Equations of motion; L. H. MCFARLAN, Missouri, Transformation of the Euler equations in

the calculus of variations; ARISTOTLE MICHAL, Rice Institute, (*a*) Integro-differential expressions invariant under Volterra group of transformations, (*b*) Functions of curves invariant under point transformations of the plane; J. A. NYSWANDER, Chicago, A direct method of obtaining the solutions of systems of linear differential equations having constant coefficients; R. G. PUTNAM, Chicago, On solutions of special, linear, third-order differential systems; B. P. REINSCH, Illinois, Expansion problems in connection with the hypergeometric differential equation; J. H. TAYLOR, Chicago, A generalization of Levi-Civita's parallelism and the Frenet formulas; MARIAN M. TORREY, Cornell, On monoidal space transformations in which the monoids have a fixed tangent cone; D. V. WIDDER, Harvard, Theorems of mean value and trigonometric interpolation.

The following degree, conferred in 1923, should have been included in the list for that year: FREDERICK WOOD, Wisconsin, Group-velocity and the propagation of disturbances in dispersive media.

THE SUMMER MEETING OF THE ASSOCIATION.

The summer meeting of the Mathematical Association will be held at Cornell University, Ithaca, N. Y., on Tuesday and Wednesday, September 8–9, in connection with the summer meeting and colloquium of the Society. Addresses will be given by Professor G. D. BIRKHOFF on "The mathematical basis of art" (illustrated), by Mr. H. E. WEBB on "The foundations of geometry from an elementary standpoint," by Professor IRVING FISHER on "The mathematics of economics" and by Professor H. L. RIETZ on "Certain applications of differential and integral calculus in actuarial science" (retiring presidential address), with probably one other paper. The full program with information as to room and board will be sent to the members of the Association about the first of August and reservations can be made at that time through Professor W. A. HURWITZ of Cornell University.

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 8–9, 1925;
Tenth Annual Meeting, Kansas City, Mo., December, 1925.
The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Peoria, May 9–10, 1925
INDIANA, Bloomington, May 8–9, 1925
IOWA, Coe College, Cedar Rapids, April 30–May 1, 1926.
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA–MISSISSIPPI, Jackson, Miss., March 20–21
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, George Washington Univ., Washington, Dec. 5, 1925.
MICHIGAN, Ann Arbor, April 1, 1926
MINNESOTA, St. Johns Univ., Collegeville, May 16
MISSOURI, Kansas City, December, 1925
NEBRASKA, Creighton Univ., Omaha, May 2
OHIO, Ohio State Univ., Columbus, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 28
TEXAS, Dallas, November 27–28, 1925

The Carus Mathematical Monographs

The expositions of mathematical subjects which these Monographs will contain are to be set forth in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields, and especially to the wide circle of thoughtful readers who, having a moderate acquaintance with elementary mathematics, wish to extend this knowledge without prolonged and critical study of the mathematical journals and treatises.

THE FIRST MONOGRAPH

Now being distributed

Calculus of Variations

By Gilbert Ames Bliss, of the University of Chicago

The author of this monograph assumes that the reader has an acquaintance with the elementary principles of the differential and integral Calculus, but even without such knowledge the geometrical or mechanical statements of the problems, the introductions to the various chapters, and the italicized theorems throughout the book, should be intelligible to any reader interested in mathematics.

These Monographs are published under the auspices of

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The First Monograph, 187 + xiii pages, will be sold at cost ($1.25 postpaid) to individual and institutional members of the Association (one copy to each member). Such orders should be sent with remittance to the Secretary-Treasurer, Oberlin, Ohio.

The sale price to non-members of the Association will be $2.00 postpaid and all such applications should be made directly to

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122 South Wabash Ave. Chicago, Illinois

THE MAY MEETING OF THE ILLINOIS SECTION.

The sixth annual meeting of the Illinois Section of the Mathematical Association of America was held at Bradley Polytechnic Institute, Peoria, on May 8–9, 1925. Chairman E. J. Moulton presided at both sessions. A short address of welcome was given by Acting President Wyckoff of Bradley College.

The attendance was thirty-four, including the following twenty-three members of the Association:

H. A. Bender, G. A. Bliss, C. E. Comstock, A. R. Crathorne, D. R. Curtiss, C. A. Garabedian, A. E. Gault, M. Gertrude Haseman, Mabel M. Heren, Mildred Hunt, E. C. Kiefer, Mayme I. Logsdon, E. J. Moulton, Mary W. Newson, H. P. Pettit, Theresa M. Renner, G. T. Sellew, H. E. Slaught, C. J. Stowell, Mildred E. Taylor, C. A. Van Velzer, Alice Winbigler, F. E. Wood.

The following officers for the next year were elected: Chairman, E. B. LYTLE of the University of Illinois; Vice-Chairman, E. C. KIEFER, James Millikin University; Secretary-Treasurer, BESSIE I. MILLER, Rockford College. The committee also recommended that the next meeting be held at James Millikin University at Decatur, May 7–8, 1926. The final decision was left to the committee with power to act. The members of the committee are Professors Comstock, Sellew and Slaught.

The following program was presented:

(1) "The next step in a unified mathematics course for freshmen" by Professor BESSIE I. MILLER, Rockford College.

(2) "The first Carus monograph" by Professor G. A. BLISS, University of Chicago.

(3) "Teaching mathematics to girls" by Professor J. B. SHAW, University of Illinois (by invitation).

(4) "Music and mathematics" by Professor C. A. GARABEDIAN, Northwestern University.

(5) "Graphing by ruler and compasses" by Dr. H. A. BENDER, University of Illinois.

(6) "A projective method of curve tracing" by Professor H. P. PETTIT, Illinois Wesleyan University.

(7) "Cross ratios in the complex plane" by Dr. MAYME I. LOGSDON, University of Chicago.

(8) "The Pearson method of curve fitting for statistical data" by Professor G. T. SELLEW, Knox College.

(9) "The line of best fit for certain statistical data" by Professor E. J. MOULTON, Northwestern University.

(10) "The individual in statistics" by Professor A. R. CRATHORNE, University of Illinois.

1. In the absence of Professor Miller, Professor Comstock read and discussed her paper. This showed that the next step in a unified mathematics course for

freshmen might well be the introduction of supplementary reading, class discussion, or lectures, of such a character as to secure (*a*) the introduction of material which will bring out other definitions of mathematics than the one now emphasized, namely, mathematics is the science of measurement; (*b*) the relation of mathematics to philosophy, religion, conduct; (*c*) the relation of mathematics to art and to music; (*d*) the introduction of mathematics in a literary form instead of in the form of a textbook.

2. Professor Bliss summarized the material in the first Carus mathematical monograph which is an elementary introduction to the theory of the calculus of variations. The problems of the calculus of variations are problems of maxima and minima some of which were formulated soon after Newton and Leibniz had discussed the simpler maximum and minimum problems of the differential calculus. The problem of finding the shortest distance from a point to a curve involves a simple illustration of one of the most important criteria of the theory. The brachistochrone problem is that of finding the curve down which a particle will fall from one point to another in the shortest time. It is historically interesting because with it the systematic study of the calculus of variations began. The most satisfactory elementary illustration of the theory of the calculus of variations is the problem of finding the form of a curve joining two given points and generating a surface of revolution of minimum area when rotated about the x-axis. The solution of this problem is equivalent to the determination of the form of a soap film stretched between two wire circles having a common axis. All of these problems are special cases of a more general theory. In conclusion Professor Bliss mentioned the wide variety of applications of the calculus of variations in other mechanical and physical domains as indicating the desirability of a presentation of the subject, like that attempted in the monograph, which should be elementary and yet descriptive of as many as possible of the more important characteristics of the theory.

3. It is regrettable that the vast majority of girls that come to freshman classes dread, and even loathe, mathematics. This is partly due to overstressing the applications of the subject to engineering and physics to the omission of the artistic and philosophic aspects. Also many instructors do not teach but merely assign tasks and record grades.

Girls are highly impressionable and affected by the atmosphere of the classroom. They are interested in persons and not things or abstractions. They feel the weight of public opinion that mathematics is not for women. They resent being treated impersonally, being left to survive or perish with indifference. They are not very analytic but do most of their thinking by imaginative and synthetic methods. Their "intuition" is a fact and must be utilized in the method of teaching. They excel in observation and can learn to read diagrams and schemes. They make concrete and particular cases serve as types for general cases, and the natural method of teaching should follow this line. They are interested in persons and this is utilized by connecting the course with the interesting lives of great mathematicians. Sylvester in particular is fascinating.

Properly taught, mathematics produces a sense of security in one branch at least of human thought that is not subject to the vicissitudes of time. All the philosophy connected with mathematics should be mentioned at the proper time and made to serve as a vitalizing element in the course. The beauty in mathematics should be dwelt upon, and exhibited frequently. This kind of a course minimizes the tactical or manipulative work as far as possible in order to get the student acquainted with the deep principles of the subject. The girl's imaginative, inventive, and intuitive ability should be cultivated to the utmost, as being the best thing for her development.

4. The paper by Professor Garabedian was in two parts. In the first section, devoted to a brief survey of the literature, it was pointed out that contributions to the subject "Music and Mathematics" have been made chiefly by the professional musician or the professional mathematician, and frequently give evidence of the bias of the specialist. Next to nothing has been written from that comprehensive point of view in which the writer, disparaging neither art in favor of the other, inquires, with love for both, if there be not perhaps certain bonds of kinship between the two. In the second section of the paper, music and mathematics were considered as two arts eminently deserving of comparative study; attention was called to certain similarities and dissimilarities, and some suggestive questions were raised—and answers attempted—with reference to the human significance and philosophical meaning of music and mathematics.

5. Most of our Analytic Geometry textbooks give the equation of the ellipse $(x^2/a^2) + (y^2/a^2) = 1$ in the parametric form $x = a \cos \theta$, $y = b \sin \theta$, and state that the coördinates of any point of the ellipse may be obtained by the intersection of the lines drawn parallel to the x- and y-axis through the points where the same radius cuts the circles of radius a and b respectively. The finding of the line values of different trigonometric expressions may be applied to the construction of the remaining conics as well as many of the standard curves in both rectangular and polar coördinates.

6. Professor Pettit considered a generalization of the ordinary projective construction of a conic, obtained by relating two pencils of lines by means of two perspective curves of order m and n respectively. The intersections of corresponding rays are points of a curve of order $2mn$ having two mn-fold points, m n-fold points, n m-fold points and $mn(mn - m - n + 1)$ nodes as well as a node corresponding to each node of the base curves. If one base curve is a straight line and the other a rational curve of order n, the resulting curve is rational, of order $2n$ and has 3 n-fold points. By proper choice of the base curves and pencils certain reductions can be introduced, among which are, notably, the ordinary construction of the ellipse, the rational cubic, and the rational sextic with ten distinct nodes. The application of the principle of duality leads directly to the construction of corresponding class curves.

7. Let A, B, C, D be four points in the complex plane associated with the complex numbers a, b, c, d. Put $(abcd) = \lambda = re^{i\varphi}$. The angle φ is the angle which the tangent to the circle ABD makes with the tangent to the circle ABC

at B. With A, B, C and λ given, D may be constructed with ruler and compasses. It will be one of the two intersections of two circles; the second intersection, D', satisfies $(ABCD') = re^{i(\pi-\varphi)}$.

Assuming two points, A and B, as fixed and calling two variable points Z and W, Dr. Logsdon discussed some interesting properties of the conformal representation established by $(abzw) = \lambda$ for a fixed complex λ.

8. The paper of Professor Sellew originated in a desire to answer some questions raised in the Chicago meeting of the Society a year ago, with regard to the number of curves used by Professor Karl Pearson in fitting statistical data.

Earlier work suggested the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{\psi(x)}{\varphi(x)},$$

where the functions $\psi(x)$ and $\varphi(x)$ are to be determined. For unimodal curves,

$$\psi(x) = x + a.$$

Professor Pearson gives his reasons for using only four moments and the simple development

$$\varphi(x) = c_0 + c_1x + c_2x^2.$$

The character of the roots of this quadratic determines twelve types, but of these three types are of unusual importance. The other types are limiting or transitional types as the character of the statistics changes from one of these main types to the others.

The particular type of curve to be used in any case is easily obtained by calculating the fundamental constants β_1 and β_2, functions of the moments.

9. An important problem in statistical theory is that of determining a straight line giving "the best fit" to a set of points not collinear. Professor Moulton discussed the solution when "the best fit" is determined by making the sum of the squares of the perpendicular distances from the points to a line a minimum. He showed how to apply the well-known theory of the ellipsoid of inertia to this problem and discussed advantages and disadvantages of this definition of "the best fit" as compared with the one more commonly used.

10. Starting with the notion of measuring individuality by the position of an individual on the Gaussian scale corresponding to the frequency distribution of the population to which the individual belongs, Professor Crathorne considered what might be called a weighted correlation problem. A specific example is that of grading the contestants in a stock-judging contest. Each contestant ranks the animals to be judged. Placing the best animal in first place should count more than placing the second best animal in second place. Placing the second correctly should count more than placing the third best correctly, and so on. A theory was outlined for grading contestants based on a rank correlation in which the ordinal numbers giving the ranks were replaced by the numbers on

the Gaussian scale corresponding to the individual animals considered as the extreme individuals in a frequency distribution of a large population.

BESSIE I. MILLER, *Secretary-Treasurer.*

THE MAY MEETING OF THE INDIANA SECTION.

The second meeting of the Indiana Section of the Mathematical Association of America was held May 8–9, 1925, at Indiana University, Bloomington, in connection with the visit of Professor J. HADAMARD of Paris to this institution.

There were forty-three present including the following thirty members of the Association: R. J. Aley, Gladys L. Banes, C. F. Barr, J. C. Bennett, E. M. Berry, H. T. Davis, S. C. Davisson, C. S. Doan, J. E. Dotterer, W. E. Edington, E. D. Grant, G. H. Graves, J. Hadamard, L. Hadley, U. S. Hanna, C. T. Hazard, Cora B. Hennel, F. H. Hodge, E. N. Johnson, J. J. Knox, Florence Long, Juna M. Lutz, Wm. Marshall, T. E. Mason, G. E. Moore, C. K. Robbins, D. A. Rothrock, K. P. Williams, H. E. Wolfe, H. N. Wright.

On Friday evening the visiting members were present at a joint dinner of the Indiana and Purdue sections of the American Association of University Professors.

At eight o'clock Professor Hadamard gave an address under the auspices of the Indiana chapter of Sigma Xi on the subject: "The equilibrium of the solar system, past and future." The introduction was made by Professor S. C. Davisson of Indiana University.

Professor Hadamard first discussed the nature of the motions of the planets of the solar system and pointed out that the greatest cause for perturbations in the orbits of the planets is the mass of Jupiter. Although these disturbances are largely compensating, there still remains a small residual disturbance which may ultimately lead to the disintegration of the solar system. The speaker showed how the study of this problem can be based upon geometrical analogy.

The program Saturday morning consisted of an address by Professor Hadamard on the subject: "The modern notion of a function." The speaker first showed how the notion of functional relationship soon outgrew Euler's definition of "functio continua." He then pointed out that analytic functions, as a class, possess three fundamental properties. First: they can be represented by means of formal expressions. Second: they possess the group property, *i.e.*, that an analytic function of an analytic function is again an analytic function. Third: they possess the property of continuation. The speaker then inquired whether other classes of functions might not exist which would fail to have one or more of these properties, but would still possess the others. Such a class is furnished by those functions which admit the limitation $|f^{(n)}| < k^n(an)!\quad a > 1$. Functions thus characterized arise in the parabolic case of the boundary-value problem of partial differential equations and are called *quasi-analytic*. As a class, they fail to possess the third property of analytic functions. The work of Gevrey, Denjoy and Carleman was quoted.

It should be included in the training of all prospective teachers and majors in mathematics.

6. The paper of Professor Davis, based upon a bibliography of 500 titles, traced the development of integral equations during the twenty-five-year period since the appearance of Fredholm's first paper in 1900.

The time and place of the next meeting were left for the decision of the executive committee.

H. T. DAVIS, *Secretary.*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The sixteenth regular meeting of the Maryland-Virginia-District of Columbia Section was held at Johns Hopkins University, Baltimore, Md., on December 6, 1924. The members were the guests of the University at luncheon. There were two sessions; Dr. F. D. Murnaghan presided at each session.

The attendance was forty-five including the following thirty-two members of the Association: O. S. Adams, R. N. Ashmun, H. G. Avers, Clara L. Bacon, W. W. Bigelow, G. A. Bingley, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, J. B. Eppes, H. Gwinner, W. M. Hamilton, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, E. S. Mayer, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, C. H. Rawlins, Jr., J. N. Rice, H. M. Robert, Jr., A. C. Robinson, H. A. Robinson, R. E. Root, G. A. Ross, J. B. Scarborough, Elizabeth W. Wilson, E. W. Woolard.

The following nine papers were read:

(1) "The slighted pyramid" by Professor HARRY GWINNER, Vice Dean, University of Maryland.

(2) "Needs of symbols for partial and total differential coefficients" by Professor A. COHEN, Johns Hopkins University.

(3) "On the Jonquières curve" by Dr. C. A. NELSON, Johns Hopkins University.

(4) "The distance between two points on an ellipsoidal earth" by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey.

(5) "The summation method for the determination of the Pearsonian coefficient of correlation" by Miss ELIZABETH W. WILSON of the Central High School, Washington, D. C.

(6) "Parametric equations of the perimeter of regular polygons" by Dr. O. S. ADAMS, U. S. Coast and Geodetic Survey.

(7) "A geometrical discussion of right-angled triangles with integral sides" by Mr. R. L. CARY of Baltimore, Md.

(8) "A laboratory course in mathematics" by Professor R. E. ROOT, U. S. Naval Academy.

(9) "A mathematical instruments laboratory" by Professor C. C. BRAMBLE, U. S. Naval Academy.

The papers led to considerable discussion. Abstracts follow below, the numbers corresponding to the numbers in the list of titles.

1. Professor Gwinner called attention to pyramids other than those known in elementary solid geometry as regular pyramids, reference being made especially to pyramids whose bases are not regular polygons and whose vertices do not lie in a line erected to the center of the base. For want of a better name, these were designated as oblique irregular pyramids.

The properties of such pyramids are studied very elegantly by the methods of spherical trigonometry. Attention was called also to articles in the *Engineering News* by C. A. P. Turner and Hart Vance as well as in the text of Carlton T. Bishop on *Structural Details of Hip and Valley Rafters.*

The solution of the oblique irregular pyramid is made by spherical trigonometry but the details of the primary plane are somewhat more tedious than in the case of the regular pyramid. Relative to the regular pyramid, it is an instructive exercise to present to a class in spherical trigonometry the problem of finding the dihedral angles between adjacent lateral faces having given the sides of the base and the altitude. Let the student decide as to whether he shall place the vertex of the pyramid at the center of the sphere; or place one corner of the base at the center. If the proper conception is made, the usual spherical triangle is made in either case and the solution follows. Wentworth's *Trigonometry,* Crockett's *Trigonometry,* and Fletcher Durell's *Trigonometry* have given this matter some attention. This problem of the dihedral angle of the pyramid is handled very nicely in the solid portion of analytic geometry.

2. In this communication, Dr. Cohen suggests a convenient notation for derivatives of a function of several variables, intermediate between its partial derivative with respect to one of its variables when all the remaining variables are considered constant, and its total derivative with respect to that variable, when all the other variables are considered functions of it.

3. Dr. Nelson discussed a suggestion of Fubini's for an extension of the concept of the flex-line of a rational plane cubic to any rational plane curve of order n with a multiple point of order $n - 1$ (Jonquières curve). He found the conditions under which this line may be found uniquely and the limiting position assumed by the line in the alternate case.

4. Mr. Lambert's paper dealt with the problem of finding the distance between two points on an ellipsoidal earth, the two points being given by their latitudes and longitudes. Many solutions of this problem have been given to meet various conditions of distance, accuracy or convenience in computation. When the flattening of the earth is allowed for, there are various connecting lines which might be used, but the most natural one is the shortest or geodesic line. The complete solution for the geodesic was given by Bessel in 1825. It involves elliptic integrals and is decidedly laborious. Associated with the spheroidal triangle, the vertices of which are a pole of the earth and the two points, there is a spherical triangle somewhat resembling the spheroidal one and related to it in certain rather simple ways. The spherical triangle should be

conceived as a convenient geometrical representation of certain quantities occurring in the computation rather than as a deformation or projection of the spheroidal triangle. The rigorous solution involves successive approximations each requiring the solution of spherical triangles. If, however, too great accuracy is not demanded, the calculation may be made by applying to the first approximation certain small corrections deduced from the differential variations of the parts of a triangle. This is done in the proposed approximate solution. If five-place logarithms are used, this approximate solution is, in general, as accurate numerically as the theoretically more exact one; sometimes this would be true even for six-place logarithms.

5. The summation method is an efficient means of computing r for N, M, and σ may be easily derived from the first, second, and third cumulative summations of a twofold frequency table. ΣXY equals the sum of the second cumulative summation. This method, suggested by Elderton, is more easily applied than the Yule method because it lends itself more readily to computation on an adding machine. A check is determined easily.

6. In the integral

$$w = \int_0^z \frac{dz}{(1 - z^n)^{2/n}},$$

let n be a positive integer greater than 2. Now expand this integral into a series and we get

$$w = z + \frac{2}{n} \cdot \frac{1}{n+1} z^{n+1} + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} z^{2n+1} + \cdots,$$

which series is convergent for $|z| \leqq 1$. In this series let z become the complex variable of the circumference of the unit circle with center the origin; that is, let $z = e^{i\theta}$ and let $w = u + iv$. Then

$$u + iv = e^{i\theta} + \frac{2}{n} \cdot \frac{1}{n+1} e^{(n+1)i\theta} + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} e^{(2n+1)i\theta} + \cdots$$

or

$$u = \cos\theta + \frac{2}{n} \cdot \frac{1}{n+1} \cos(n+1)\theta + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} \cos(2n+1)\theta + \cdots$$

and

$$v = \sin\theta + \frac{2}{n} \cdot \frac{1}{n+1} \sin(n+1)\theta + \frac{2}{n} \cdot \frac{n+2}{2n} \cdot \frac{1}{2n+1} \sin(2n+1)\theta + \cdots.$$

For any assumed value of n as θ varies from 0 to 2π, u and v are the running coördinates of a regular polygon of n sides with the center of the polygon at the origin and with one vertex on the positive axis of u. For n equals 3, we have the equilateral triangle; for n equals 4, the square; and so on for other values of n.

7. Mr. Cary's treatment of right-angled triangles with integral sides was developed from the relation $x + iy = (\xi + i\eta)^2$. The network formed by $\eta = k$, $\xi = k$, $\xi - \eta = k$, $\xi + \eta = k$, ($k = 1, 2, 3$, etc.), corresponds to a network of parabolas in the x, y plane whose equations are

$$y^2 = \pm 2(2k^2)x + (2k^2)^2$$

and

$$x^2 = \pm 2(k^2)y + (k^2)^2.$$

Integral values of x and y lie at the intersections of these parabolas which meet at angles $\theta = n\pi/4$. ($n = 1, 2, 3$.) Solutions obtained by repeated applications of a set of recursion formulæ of the type: $A = a + 2b + 2c$, $B = 2a + b + 2c$, $C = 2a + 2b + 3c$, lie on a parabola through a, b or on $x - y = \pm (a - b)$.

8. Professor Root mentioned briefly the development of the idea of the mathematical laboratory, with particular reference to the courses at the University of Edinburgh and at Massachusetts Institute of Technology. He then indicated the character of the laboratory course as given in the postgraduate school at the Naval Academy, describing briefly some of the exercises given. The laboratory course was shown to differ both in method and subject matter from other courses in mathematics given concurrently. Holding that the laboratory method of instruction is most effective for those topics which require a minimum of explanation and discussion by the teacher and a maximum of performance by the student, he indicated the construction of alignment charts as peculiarly suitable for laboratory exercises.

9. Professor Bramble discussed the possibilities of mathematical laboratory work in which instruments such as planimeters, integrators, integraphs and ellipsographs are used. The exercises as given in this work at the Naval post-graduate school were described. Especial attention was called to an exercise in which the principal axes of inertia of a given area (such as a beam section) are located by means of the Amsler three-wheel integrator.

HARRY ENGLISH, *Secretary-Treasurer.*

TENTH ANNUAL MEETING OF THE OHIO SECTION.

The tenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on April 3, 1925, in connection with the meetings of the Ohio College Association and allied societies. Chairman Harris Hancock presided, being relieved by Professor R. B. Wildermuth for an interval.

Forty-seven persons were registered, the following thirty-five being members of the Association:

W. E. Anderson, G. N. Armstrong, C. L. Arnold, Grace Bareis, I. A. Barnett, J. B. Brandeberry, V. B. Caris, E. H. Clarke, Rufus Crane, W. Dancer, O. L.

Dustheimer, T. M. Focke, B. C. Glover, H. Hancock, F. C. Hartwick, E. M. Justin, H. W. Kuhn, C. C. MacDuffee, W. C. McCoy, J. E. Merrill, E. S. Manson, C. N. Moore, Hortense Rickard, W. G. Simon, S. A. Singer, G. W. Spenceley, M. O. Tripp, J. H. Weaver, R. L. Wilder, R. B. Wildermuth, C. O. Williamson, K. D. Swartzel.

At the business session the secretary reported a membership of ninety-seven and eleven institutional members as against eighty-five and nine, respectively, last year. Officers elected for this year are: Chairman, Professor R. D. Bohannan, Ohio State University; Secretary-Treasurer, Professor G. N. Armstrong, Ohio Wesleyan University; Third Member of the Executive Committee, Professor C. C. MacDuffee, Ohio State University. Professor R. B. Allen, having resigned from the program committee, Professor I. A. Barnett was elected to take his place, and Professor E. H. Clarke was selected to fill the vacancy due to the expiration of Professor C. N. Moore's term. The financial situation of the Section is satisfactory. The committee circulating letters among the high schools of the state on "Why Elect Mathematics" was instructed to continue its work. Professor R. B. Wildermuth was named chairman of the committee to prepare revised letters to be discussed next year.

The Section dinner, with about forty present, served in one of the dining rooms of Ohio Union, was very successful. The evening session, which began with informal after dinner speeches, was held in the same room and was devoted to the last two numbers on the program.

The following nine numbers were presented at the two sessions:

(1) Chairman's Address: "A notion which includes that of divisibility" by Professor Harris Hancock, University of Cincinnati.

(2) "Some criteria for the rejection of doubtful data" by Professor W. E. Anderson, Miami University.

(3) "College algebra for freshmen" by Professor E. H. Clarke, Hiram College.

(4) "On the definition of a continuous curve" by Professor R. L. Wilder, Ohio State University.

(5) "Some new examples of modular non-euclidean geometries and a tactical method for their discovery" by Professor C. L. Arnold, Ohio State University.

(6) "Some new methods for finding the roots of algebraic equations" by Professor C. W. Spenceley, Miami University.

(7) "Mathematics as related to the American college" by Professor O. L. Dustheimer, Baldwin-Wallace College.

(8) "Side-lights on the international congress at Toronto" by various Ohio mathematicians who attended the congress.

(9) "Some statistical data relative to the lack of preparation of college freshmen in mathematics"; discussion opened by Professor Harris Hancock, University of Cincinnati.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In the beginning Professor Hancock gave definitions of *realms of rationality, algebraic numbers, integers* and *units*. Then it was shown by the well-known example that 21 in the realm $R(\sqrt{-5})$ may be factored in several different ways. Kummer numbers were introduced and their connection with the Dedekind ideals was indicated. By the introduction of the prime ideals a unique factorization in the extended realms may be effected. A number λ is divisible by the complex $[\alpha, \beta]$ if λ is contained in this complex, that is, if integers ξ and η may be found such that $\lambda = \alpha\xi + \beta\eta$. This notion, superfluous in the usual realm of natural (usual) numbers, is necessary in the more extended realms.

2. The criteria of Peirce, Stone, Chauvenet, and the U. S. Coast and Geodetic Survey were presented by Professor Anderson, especial attention being given to the development and application of Peirce's criterion. Criticisms by Airy, Stone, Winlock, and Glaisher were given. The present practice of the U. S. Naval Observatory regarding rejection of doubtful observations was given. The object of the paper was to try to show the best and most approved methods of handling this problem which has engaged the attention particularly of those who have been concerned with the adjustment of observations, including data of more or less doubtful value.

3. Professor Clarke pointed out the lack of well-defined objectives in our present courses in algebra for college freshmen. He suggested as essential aims: (1) the development of the idea of functional relationship, (2) clear-headed thinking on the equation, (3) accuracy in numerical work.

4. In Professor Wilder's paper there was presented a resumé of some of the criteria developed in recent years in the field of analysis situs for determining whether given point-sets fall in the class of curves which can be defined by continuous functions $x = f_1(t)$, $y = f_2(t)$, $[0 \leqq t \leqq 1]$. The following simple test, similar to that given by Schoenflies for plane continuous curves, was introduced: M being a plane continuum, and K any circle, denote by N the set of points of M interior to K and by T the set $K + N$. Then in order that M should be a continuous curve, it is necessary and sufficient that for every such set T and every positive number E the number of domains complementary to T and of diameter greater than E should be finite.

5. Professor Arnold noted that the finite modular geometries (without parallels) have $n(n - 1) + 1$ elements, where "n" is the number of points on a line. It was shown that the set of *differences* between the numbers of any line may be used to assure the discovery of all the possible geometries for any given value of n. Geometries exist for $n = 2, 3, 4, 5, 6, 8$, and 9, also probably for many greater values of n. The number of known geometries is: For $n = 2, 3, 5$, one each; for $n = 4$, two; for $n = 6$, five; for $n = 7$, none; for $n = 8$, six; for $n = 9$, at least one. Those for n greater than 5 are believed to be new.

6. In a polynomial $p_0x^n + p_1x^{n-1} + p_2x^{n-2} \cdots + p_n = 0$, Professor Spenceley pointed out that a positive real root x, which is large in comparison with S, the sum of all the other roots, is approximately given by $-p_1/p_0$. Any polynomial whatever may be transformed into one whose roots are of this peculiar

type by several repetitions of the transformation

$$x = \frac{r^2}{y - b} + a,$$

which is geometrically equivalent to reflecting the roots of the original equation in a circle in such a way as to make the desired root go over into a large root while the sum of the remaining roots is made small.

7. Professor Dustheimer had 300 catalogues examined from every state in the U. S. and every province of Canada. 50% require 2 units of secondary mathematics for entrance. 50% require from 18 to 24 hours for a mathematics major. 50% require 6 hours or less of mathematics for graduation. 30% offer from 20 to 30 hours of mathematics. 80% have no connection between mathematics and physics. 46% offer from 2 to 6 hours of astronomy; 30% have some connection between mathematics and astronomy; 30% teach no astronomy.

Answers to the questionnaire from 55 colleges (33 Ohio colleges) show the following: Only 16% require students majoring in mathematics to take history of mathematics. Sixty-one per cent. give courses in the teaching of mathematics while very few give courses combining the history and teaching of mathematics. Many of these courses are lecture courses, but nearly all of them use a textbook. The courses average 2 hours per semester and are generally open to juniors and seniors.

8. These reports were informal and humorous rather than scientific, dealing mainly with personal experiences.

9. Professor Hancock had Professor Barnett report the result of identical tests given to college freshmen who had had some college mathematics in the University of Cincinnati, Ohio State University and Adelbert College. The results in the three institutions were in fair agreement but decidedly unsatisfactory although the tests were extremely easy. Professor Swartzel explained a plan in use at the University of Pittsburgh whereby they demote students until they find their true level of preparation. Few failures need be recorded by this device, which works beneficially. Professor Holl reported the method used this year in Ohio Wesleyan University in sectioning freshmen entering with only one year of algebra. The discussion extended itself to nearly all phases of present-day education.

Professor Barnett presented the matter of the Carus monographs in an effective talk.

G. N. ARMSTRONG, *Secretary-Treasurer.*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION.

The ninth annual meeting of the Rocky Mountain Section was held at the University of Wyoming, Laramie, on April 10 and 11. There were thirty-six present, including the following ten members of the Association: I. M. DeLong,

Philip Fitch, J. C. Fitterer, H. C. Gossard, Claribel Kendall, G. H. Light, S. L. Macdonald, Letitia Odell, O. H. Rechard and H. E. Russell.

The section voted to hold the next meeting at the Colorado Agricultural College. The following officers were elected: S. L. MACDONALD, chairman; W. V. LOVITT, vice-chairman; PHILIP FITCH, secretary; G. H. LIGHT, treasurer.

Three committees were appointed, one to draft resolutions on the death of Dr. Carl E. Stromquist, formerly head of the department of mathematics at the University of Wyoming and a member of this section; one to formulate plans for the use of standard tests in connection with the teaching of mathematics in the colleges; and one to consider the advisability of procuring a speaker from outside the section for the next meeting.

On Friday evening, the members of the section were guests at a dinner given by the home economics department of the University. President A. C. Crane delivered an address of welcome on this occasion and expressed his pleasure at having the section meeting held at his institution. Professor S. L. Macdonald responded in a fitting manner in behalf of the guests.

Later in the evening the women members of the section were entertained at a production given by the Coffer-Miller players.

The following nine papers were read:

(1) "An endowment for the publication of the results of mathematical research" by Professor I. M. DELONG.

(2) "Mathematics as an aid in agricultural experimentation" by Professor A. G. CLARK (by invitation).

(3) "The relation of standard tests to the teaching of collegiate mathematics" by Professor H. C. GOSSARD.

(4) "Baade's asteroid" by Dean H. A. HOWE.

(5) "Points of view on the multiplicative axiom" by Professor C. H. RECHARD.

(6) "On the quinquenary cubic expressible as the sum of seven cubes" by Professor C. H. SISAM.

(7) "Integration in series" by Professor G. H. LIGHT.

(8) "Analysis of certain types of composite curves" by Mr. PHILIP FITCH.

(9) "On a tetrahedron" by Professor H. C. GOSSARD.

In the absence of the authors, the paper by Dean Howe was read by Professor Russell and the one by Professor Sisam by title only.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor DeLong pointed out the urgent need of funds for publishing results of mathematical research and made an appeal to all interested in mathematics to assist the American Mathematical Society in establishing an endowment large enough to insure the publication of the fine results that are being achieved by mathematicians in this country.

2. Professor Clark mentioned, in a brief way, various typical problems arising in agricultural experimental work where the need for mathematical treatment

was obvious. An actual problem, dealing with the peculiar results of the cross of two varieties of barley, was solved.

3. This paper was a report of experiments with speed, accuracy, and power tests in freshman and sophomore college mathematics. As a result of this report, the colleges and universities of the Rocky Mountain Section of the Association voted to coöperate in a continuation of this experiment.

4. The discovery of Baade's asteroid last October was made by photography. The computation of a preliminary orbit at the Student's Observatory of the University of California was beset with special difficulties, but when a reasonably correct orbit was finally obtained, a request from Denver brought an ephemeris of the planet, furnished in advance of publication, so that measures of its position with the 20-inch Denver telescope might begin at once. The orbit has been found to be the most eccentric planetary orbit known, with one exception. The inclination of the orbit is among the dozen highest. Only three planetoids come nearer to the earth than it does. Furthermore it is unusual in that it varies regularly in brightness in a period of a few hours. This is probably an indication of axial rotation, perhaps the first shown by any planetoid. Despite the faintness of this planet, a very large number of observations has been made at Denver, but the approaching opposition of the asteroid will cause a termination of the series.

5. Professor Rechard discussed the multiplicative axiom as brought to the fore by Zermelo's Theorem. The discussion centered around the points of view of various mathematicians, especially Baire, Borel, Hadamard, and Lebesgue. A summary of the principal problems and special fields affected by one's point of view on the axiom was included.

7. This paper gives methods for finding the solution of the differential equation

$$(x - a)^2 p_0 \frac{d^2y}{dx^2} + (x - a)p_1 \frac{dy}{dx} + p_2 y = X.$$

It is shown when this equation will have a solution in ascending and descending powers of $(x - a)$ as well as when a logarithmic solution occurs. The particular integral is found in a very simple manner.

8. Mr. Fitch discussed the analysis of composite curves obtained from experimental data arising from observations made on thermoluminescent and similar effects. He pointed out the necessity of being guided by the scientific facts involved while attempting such an analysis.

9. Following a suggestion by Dr. Morley of Johns Hopkins the author of this paper presented twenty-one relations between the edges and faces of a tetrahedron. The equation of the absolute is expressed first so that its coefficients are in terms of the edges of the tetrahedron and second in terms of its faces. The twenty-one resulting coefficients are then equated giving the desired relations.

PHILIP FITCH, *Secretary.*

OUTLINES OF RESEARCH: GENERAL ANALYSIS.[1]

By T. H. HILDEBRANDT, University of Michigan.

I feel that there is some need of apology that I should be the one to speak on the subject indicated. In thinking about the matter, it seemed to me that there were others who were far better suited to the task, others who had had considerable experience in directing research in general analysis or who had made rather significant contributions to the subject. This feeling was quite strong when the chairman of your program committee wrote to me about the matter of giving this paper at this time, and I tried to sidestep the issue. But he was insistent, and the general manager of the Association contributed his share towards spurring me on, so that I reluctantly acquiesced. If the results are not a credit to the Association or to the subject, the responsibility is theirs.

In a subject developing so rapidly as general analysis, it is to be expected that some of the things which I suggest as possibilities have been already proved, or disproved, and it may be that some have been published and have escaped my attention. I offer proper apologies for this contingency at this point.

There is no need of offering apologies for the subject of general analysis. By this term I mean to cover that body of mathematical science which in recent years has devoted itself to the task of replacing particular instances by general inclusive ones, especially in the field of analysis. The habit of generalization is one that is inculcated into the budding mathematician from the time when he begins the study of elementary algebra. The training continues, and for most of us the habit thus formed never ceases. Any one who has attempted generalization in any field can easily subscribe to the fact that such generalization has resulted, on the one hand, in a sharper analysis of the essentials of the theory, *i.e.*, in separating out the items peculiar to the special situation from those which are of general import; secondly, to a simplification; and lastly, to a widening of the sphere of influence and inclusion of greater results. General analysis has done this very thing for the realms into which it has entered up to the present time, *viz.*, the theory of abstract sets, theory of functions on abstract sets, and theory of functionals or transformations. It has abundantly verified the truth of the tenet of faith of general analysis, so ably phrased by one of the leaders in this field, E. H. Moore:[2]

The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.

In this paper, it is obviously impossible to give what would be a complete outline of research in a field which has extended its scope and developed so rapidly in the two decades of its existence. As a matter of fact the word "outline" in the title is rather a misnomer. A more appropriate title would be

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[1] Read before the Mathematical Association of America at Washington, D. C., on December 31, 1924.

[2] *New Haven Colloquium Lectures* (1910), p. 1.

"Some Suggestions for Research in General Analysis." Naturally a selection is rather a difficult matter. I have confined myself to a few indications in two fields: I. Theory of Abstract Sets, and II. Functionals, particularly Linear Functionals. Most of the suggestions made are in the nature of questions which have arisen in the course of reading. We might well say that "the beginning of research is the question and the question mark," and that the person who has formed the habit of being profuse with questions in his reading is on the high road to research.

I. Theory of Abstract Sets. Before treating questions which relate to the theory of abstract sets, it may be desirable to give a bird's-eye view of the theory in its historical development.

We postulate in such a theory a set of general elements, the class $\mathfrak{P}$ of elements p.[1] We can assume that the class $\mathfrak{P}$ is left entirely unconditioned, *i.e.*, the elements of the class are general. In that case, in order to be able to say anything, we must shift the responsibility to other quarters. The usual method is to assume the existence of functions or rather a class of functions on $\mathfrak{P}$ with real or complex values, and study the interrelations between the functions, as being properties of the class of functions. This was the point of view of E. H. Moore in his general analysis, whose primary interest lay in the building of a foundation for the generalization of the theory of linear integral equations, and later, Hellinger integrals. We shall not discuss this point of view. The other method of approach is to study interrelations as existing between the elements of the class $\mathfrak{P}$, *i.e.*, condition the class. This is the point of view of Fréchet as exposed first in his thesis, "Sur quelques points du Calcul Fonctionnel,"[2] whose purpose was to generalize the existing theory of point-sets.

Now, in the theory of point-sets a central rôle is played by the notion of *limiting element.* Fréchet approached this notion *via* the notion of *limit of a sequence,* and hence postulated in the first place that in the general class $\mathfrak{P}$ there was defined the limit of a sequence, this notion being subject to the following three conditions: (a) the limit of a sequence is unique, (b) a sequence which consists of one element repeated has this element as limit, (c) any subsequence of a sequence having a limit has the same element as limit. Such a class he calls a class $\mathfrak{L}$. A limiting element of a subclass of a class $\mathfrak{L}$ is defined as the limit of a sequence of distinct elements chosen from the subclass. A class $\mathfrak{L}$ suffices as a basis for the discussion of some continuity properties of functions,[3] but does not carry very far in the theory of abstract sets. For the latter he assumes that between pairs of elements of the class $\mathfrak{P}$[4] there is defined a function

[1] We follow E. H. Moore in notation, for the simple reason that he has made a good advance towards the question of uniformising notation in this field, *viz.:* small letters for elements, capital letters for transformations, Greek letters for functions, and capital German letters for classes. The use of the letter p is due to the fact that it does not have a preconceived connotation.

[2] *Rendiconti di Palermo*, vol. XXII (1906), pp. 1–74.

[3] Continuity properties of functions seem to demand very little, as indicated for instance by Fréchet: *Ann. de l'Éc. Norm. Sup.*, ser. 3, vol. 37 (1920–1), pp. 356 ff.

[4] Fréchet assumes that this is true for *all* pairs of elements, but this is not necessary. *Cf.* Gross: *Sitzungsberichte der Wiener Akademie*, vol. 123, Abt. 2a, pp. 802 ff.

$\delta(p_1p_2)$, subject to the conditions: (1) $\delta(p_1p_2) \geqq 0$; (2) If $\delta(p_1p_2)$ exists, then $\delta(p_2p_1)$ exists and they are equal; (3) $\delta(p_1p_2) = 0$ if and only if $p_1 = p_2$; (4) If $\delta(p_1p_2)$ and $\delta(p_2p_3)$ exist, then $\delta(p_1p_3)$ exists and

$$\delta(p_1p_3) \leqq \delta(p_1p_2) + \delta(p_2p_3).$$

Limit in such a space is defined as being equivalent to

$$\lim_{n\to\infty} \delta(p_np) = 0.$$

Such a space is generally called a metric space, and is denoted by the symbol $\mathfrak{D}$. It forms a natural and elegant basis for a general theory of point-sets in the abstract field, as has been illustrated for instance by Hahn in his *Theorie der reellen Funktionen.*

The principal later advance in the foundations of the theory has come through the realization that *limiting element* of a set of elements is the central figure in a theory of abstract sets, and need not be approached through the limit of a sequence, but can be postulated directly. This was pointed out by Riesz in his paper "Stetigskeitbegriff und abstrakte Mengenlehre" before the Mathematical Congress in Rome in 1908. Limiting element is subjected by him to the following postulates: (1) if p is a limiting element of a subclass $\mathfrak{P}_0$ of $\mathfrak{P}$, then it is also a limiting element of every subclass of $\mathfrak{P}$ containing $\mathfrak{P}_0$, (2) if p is a limiting element of a subclass $\mathfrak{P}_0$ of $\mathfrak{P}$, and $\mathfrak{P}_0$ is divided into two classes, then it is also a limiting element of one of these two classes, (3) a class consisting of a single element has no limiting element, and (4) limiting element of a subclass of $\mathfrak{P}$ is determined uniquely by the totality of classes of which it is a limiting element.

Starting from these ideas on limiting elements, R. E. Root [1] was led to connect the idea of limiting element with a special group of subclasses of the fundamental class usually called vicinities. The same idea was obtained independently by Hausdorff [2] in 1914 and again by Fréchet [3] in 1918. The latter considers what properties of a limiting element are necessary and sufficient for the existence of a vicinity notion. He finds that the notion of vicinity with limiting element defined, namely, p is a limiting element of a subclass $\mathfrak{P}_0$ of $\mathfrak{P}$, if every vicinity of p contains an element of $\mathfrak{P}_0$ different from p, is equivalent to the condition (1) of Riesz, together with the fact that whether an element p of $\mathfrak{P}$ is or is not a limiting element of a subclass depends on the elements of the subclass other than p. These conditions are very slight, and it seems that the future will

[1] "Limits in Terms of Order," *Trans. Amer. Math. Soc.*, vol. 15 (1914), pp. 51–71. Although Root bases his discussion fundamentally on a notion of betweenness, *i.e.*, of order, he uses this notion only to define the set of special subclasses in terms of which all later reasoning is done. It might be interesting to study limiting element directly in terms of order without the intervention of the vicinity notion, and determine what properties of order produce for instance the Riesz properties of limiting element.

[2] *Grundzuege der Mengenlehre*, pp. 212 ff.

[3] *Bull. des Sci. Math.*, 53 (1918), pp. 138–156.

probably link up the theory of abstract sets with classes in which a vicinity is defined (denoted by $\mathfrak{V}$), properly conditioned.[1]

We note finally some work on a general limit due to E. H. Moore and H. L. Smith[2] the prime object of which is to obtain a general limit to include the kind of limits which occur in connection with the definitions of integration of the Riemann type. They postulate an order defined by a relation R. This order is subject to two conditions: (1) transitivity: if p_1Rp_2 and p_2Rp_3, then p_1Rp_3, and (2) for every p_1 and p_2 there exists a p_3 such that p_3Rp_1 and p_3Rp_2. Viewed from the point of view of abstract sets[3] this naturally is a special instance in the theory given by Root which shows that any order based on a betweenness relation (obviously resultant from the transitivity postulate in this case) gives rise to a vicinity notion.

We come now to the questions which might be suggested with respect to the existing theory. In the theory of linear point sets, the difference between the notion of limit and that of limiting element is first that limit is connected with a sequence, *i.e.*, an ordered set, while limiting element is associated with a set of elements, order not prerequisite, and secondly that if a sequence has a limit, it has only one, while limiting elements of a set may be many. So far we have abstracted the notion of limit by preserving the sequence idea. It might be interesting to study what kind of a theory would result if we assumed as fundamental the notion of limit as attached to a class (ordered or not) emphasizing the uniqueness feature rather than the sequential idea.[4] From such a limit suitably conditioned we might get then notions of limiting elements. If the notion of order is to reign supreme in the notion of limit, it might be interesting to study what happens if limit is attached to sets which are ordered, the sets being no longer denumerable, necessarily, the order being for instance of the monotonic type where of two elements one precedes another, or of the type suggested by Moore and Smith, where every two elements have a common successor. One of the points of separation between limit and limiting element is that limiting element need not attach itself to a denumerable set.[5] Perhaps by starting with denumerability thrown into the discard, but with unique element and order present, we may be able to get a limit which will give rise to as general a limiting element as can be obtained by direct postulation.

Another thing which might well be done is to clear up the relationships which exist between different definitions, in the theory of abstract sets. For instance take the definition of limiting elements of different power or order. In Fréchet's thesis we find, besides the concept of limiting element, the notion of element of condensation—a limiting element being an element of condensation

[1] This sentence is the burden of the remarks made by E. W. Chittenden in discussing this paper.

[2] *Cf. Amer. Jour. of Math.*, 44 (1922), pp. 102–121.

[3] Moore and Smith are interested mainly in limit as applied to functions, of the type: limit as n increases indefinitely.

[4] *Cf.* Hausdorff, *loc. cit.*, pp. 232 ff.

[5] *Cf.* Root, *loc. cit.*, pp. 67 ff.

of a class, if it remains a limiting element of every subclass obtainable by removing a denumerable set of elements. Then in Hausdorff [1] we have a limiting element of order α in a class of type $\mathfrak{B}$, if every vicinity of p contains one element of the class (not necessarily different from p); a limiting element of order β if every vicinity of p contains a denumerable infinity of elements, and of order γ if every vicinity of p contains a nondenumerable infinity. Along the same line one can define a limiting element of *order* $\mathfrak{m}$ if every vicinity of p contains a set of elements of power $\mathfrak{m}$, but not every vicinity contains a set of power greater than $\mathfrak{m}$. Kuratowski and Sierpinski [2] have defined a limiting element which involves the notion of interiority (p is interior to $\mathfrak{P}_0$ if it belongs to $\mathfrak{P}_0$ and is not a limiting element of any set consisting entirely of elements not belonging to $\mathfrak{P}_0$) which might perhaps be called limiting elements of *power* $\mathfrak{m}$: p is a limiting element of $\mathfrak{P}_0$ of power $\mathfrak{m}$ if every vicinity of p has interior to it a set of elements of $\mathfrak{P}_0$ of power $\mathfrak{m}$, but not every vicinity contains in the interior sense a set of power greater than $\mathfrak{m}$.[3] It would obviously be desirable to find out to what extent these various concepts overlap, to determine existing relationships and to point out which had best be discarded and which used.

Closely connected with these different kinds of limiting elements one finds properties of classes which are for instance generalizations of the Weierstrass-Bolzano theorem. In Fréchet we find compactness, in which every infinite subclass has a limiting element, and condensed, in which every nondenumerably infinite subclass has an element of condensation. Then in Gross [4] we find compactness of different orders: a class being $\mathfrak{m}$-compact if every subclass of order greater than $\mathfrak{m}$ has a limiting element. Obviously compactness of various kinds could be combined with limiting elements of different orders, thus arriving at varied definitions. The interrelations and applicability of these ideas is desired. Among these things it might perhaps be possible to obtain theorems similar to the one due to Gross and Fréchet [5] that in a class $\mathfrak{D}$ the notions condensed and separable (a class is separable when it belongs to the derived class of a denumerable set) are equivalent. In this connection too one ought not to overlook the notion of perfectly compact due to R. L. Moore,[6] and defined in terms of a monotonic set of classes, *viz.*, that every monotonic set of subclasses of $\mathfrak{P}_0$ has a common element or common limiting element.[7]

Of the theorems which have engaged the attention of workers in the theory of abstract sets, the Borel-Lebesgue theorem has no doubt claimed the maximum

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[1] *Loc. cit.*, pp. 219 ff.

[2] *Fundamenta Mathematicæ*, vol. II (1921), p. 173.

[3] In an article by E. W. Chittenden, *Bull. of Amer. Math. Soc.*, vol. 30 (1924), pp. 512, 3, which has appeared since this paper was read, limiting elements of order and power $\mathfrak{m}$ are used and called nuclear and hypernuclear points respectively.

[4] *Loc. cit.*, p. 805.

[5] Gross, *loc. cit.*, p. 806. Fréchet, *Ann. de l'Ec. Norm. Sup.*, ser. 3, vol. 37 (1920–1), pp. 349 ff.

[6] *Proceedings of the Nat. Acad. of Sci.*, 5 (1919), pp. 206–10.

[7] In the article by Chittenden referred to above, it is shown that the notion of perfect compactness is equivalent to a kind of complete compactness based on limiting elements of various orders, thus justifying the name perfectly compact.

interest. The theorem of Borel-Lebesgue states that under suitable conditions a set of subclasses, such that each member of a given class is interior to some one of the subclasses of this set, can be replaced by a finite number of classes chosen from this set, and having the same property. Fréchet's proof of this theorem involved the class $\mathfrak{D}$; later it was extended to classes of the type $\mathfrak{L}$ and $\mathfrak{V}$. In the proofs of the latter [1] there intervene properties of well-ordered sets of the transfinite variety, *i.e.*, the Zermelo axiom and theorem is used. In consideration of other cases in which the theory of transfinites has been used in analysis, and replaced by other reasoning not involving them, it might naturally be asked whether such is possible with respect to the proofs of the Borel-Lebesgue theorem in a class $\mathfrak{V}$ for instance.

Another direction does not seem to have been pursued. The theorem of Borel-Lebesgue abstractly considered is concerned with the possibility of replacing a set of subclasses having a certain property by a finite (or denumerable?) set having the same property. In the theorems so far considered, the property in question is that every element of a given set shall be interior to a given subclass. The question may be suggested whether it is possible to derive theorems of the Borel-Lebesgue type, where this property is replaced by some other. In the theory of measure one finds somewhat similar theorems, and there may be some of similar character in general point-set theory.

II. Functionals. In discussing functionals, *i.e.*, functions of functions, it is possible to proceed in many different ways. We may follow E. H. Moore in his first general analysis,[2] starting with a general class $\mathfrak{P}$, a class of complex-valued functions on $\mathfrak{P}$, subject to definite properties, and a linear operator with postulated properties; or follow him in his later work in which the linear operator is obtained constructively and the properties of the general class of functions derived therefrom. We shall instead follow a trend which seems to be indicated in the recent developments of analysis, Lebesgue integration, Stieltjes integration and kindred operators. We proceed by postulating a class $\mathfrak{P}$ of elements p, general and unconditioned; a class $\mathfrak{M}$ of functions μ, complex-valued, on the range $\mathfrak{P}$. This class $\mathfrak{M}$ is assumed to be linear, *i.e.*, if μ_1 and μ_2 are two functions of $\mathfrak{M}$ and a_1 and a_2 any complex numbers, then $a_1\mu_1 + a_2\mu_2$ also belongs to $\mathfrak{M}$. On $\mathfrak{M}$ we assume the existence of a norm or modulus,[3] $N\mu$, with positive or zero values and with the properties:

$$(1)\quad N(\mu_1 + \mu_2) \leqq N\mu_1 + N\mu_2,$$
$$(2)\quad N(a\mu) = |a| N\mu.$$

We assume that any function μ such that $N\mu = 0$ is equivalent to the function which is identically zero on $\mathfrak{P}$, and as a consequence two functions, the norm of

[1] *Cf.* R. L. Moore, *loc. cit.;* Fréchet, *Ann. de l'Ec.*, article pp. 348–9; Kuratowski and Sierpinski, *loc. cit.*, and Chittenden, *loc. cit.*

[2] *New Haven Colloquium Lectures*, 1910.

[3] This norm $N\mu$ usually appears symbolized $|| \mu ||$.

whose difference is zero, are considered as being identical. With respect to the norm N we assume that the class $\mathfrak{M}$ is complete or satisfies a Cauchy condition, *i.e.*, if the sequence μ_n is such that

$$\lim_{n,\,m} N(\mu_n - \mu_m) = 0,$$

then there exists a μ of class $\mathfrak{M}$ such that

$$\lim_n N(\mu_n - \mu) = 0.$$

Examples of classes $\mathfrak{M}$ and norms are plentiful. *E.g.*, if the class $\mathfrak{P}$ is the class of integers 1, 2, 3, $\cdots$, which we shall call $\mathfrak{P}^{\mathrm{III}}$ in the sequel, then $\mathfrak{M}$ is some class of sequences. If $\mathfrak{M}$ is the class of sequences such that $\Sigma_p |\mu_p|^2$ is convergent, then $N\mu = \sqrt{\Sigma_p|\mu_p|^2}$; or if $\mathfrak{M}$ is the class such that $\Sigma|\mu_p - \mu_{p+1}|$ is convergent, then $N\mu = \Sigma|\mu_p - \mu_{p+1}| + |\mu_1|$; and so on. Or if $\mathfrak{P}$ is the linear interval $0 \leqq p \leqq 1$, which we shall call $\mathfrak{P}^{\mathrm{IV}}$, then $\mathfrak{M}$ is a class of functions on the real interval; *e.g.*, the class of continuous functions with $N\mu$ = maximum of the $|\mu(p)|$ for p on $\mathfrak{P}^{\mathrm{IV}}$; or if $\mathfrak{M}$ is the class of functions of bounded variation, then $N\mu$ is the total variation of μ; and so on.[1]

A class $\mathfrak{M}$ of the type discussed here is evidently of the character of the class $\mathfrak{D}$ of Fréchet with $\delta(\mu_1, \mu_2) = N(\mu_1 - \mu_2)$, provided $N\mu = 0$ is equivalent to μ identically zero.

This basis can be replaced by what seems to be a more general basis, called a *vectorial space*.[2] In this type of space, the function μ on $\mathfrak{P}$, complex-valued, is replaced by a *vector*, a suitably conditioned element. With respect to the class of these vectors, it is assumed that there is defined an addition process which is commutative and associative; and multiplication by real (or complex) numbers, which is commutative, associative, and distributive with respect to addition. Finally there is assumed a norm or modulus (usually denoted $\|\ \|$) which transforms the elements into real non-negative numbers, with properties similar to those given above.

This suggests a problem, which might be investigated, *viz.*, whether a vectorial space of this kind is in fact more general than a function space on a general range. Most of the examples of vectorial spaces so far considered can be thought of as complex-valued functions on a certain range. If it could be shown that every vectorial space can so be expressed (the converse is obvious), there would be a considerable gain in simplicity.

A question might be asked with respect to the norm as postulated above, especially as regards the condition (2):

$$Na\mu = |a|N\mu.$$

[1] A large number of interesting examples of this kind are discussed by Hahn, *Monatshefte f. Math. u. Physik*, vol. 32 (1922), pp. 1 ff.

[2] *Cf.* Banach, *Fund. Math.*, 2 (1921), pp. 134 ff.; Wiener, *Bull. de la Soc. Math. de France*, 50 (1922), pp. 123 ff.

Fréchet[1] has shown that in the class of all sequences, with limit defined

$$\lim_n \mu_n = \mu \text{ if and only if } \lim_n \mu_n(p) = \mu(p) \text{ for every } p,$$

it is possible to find a distance function giving an equivalent definition of limit. However this distance function considered as a norm does not satisfy this condition (2), and the question is whether it is possible to find such a distance function in this case. A similar question might be asked with respect to functions on the infinite interval.[2] On the other hand, it may be possible to obtain certain results in this theory, especially those relative to linear functionals which seem to require this postulate on norm, without this postulate, or by replacing it with some weaker postulate.[3]

Along the same lines comes the question: Is there for each class of functions a "most natural norm"?[4] This question is of course indefinite because of the indefiniteness of the term "most natural." Without attempting to make it precise, perhaps a few examples will make my meaning clear. In the class of continuous functions on $\mathfrak{P}^{IV}$ the most natural norm seems to be the maximum of the absolute value of the function, *i.e.*, connected with uniform convergence; in the class of Lebesgue integrable functions, the most natural norm seems to be the Lebesgue integral of the absolute value of the function; in the class of functions which are measurable and bounded, if a set of measure zero be neglected, the most natural norm seems to be the least upper bound of the absolute value of the function, obtainable by neglecting sets of measure zero; for functions of Lebesgue integrable square it seems to be the square root of the integral of the square; for functions defined on $\mathfrak{P}^{III}$, for the class of absolutely convergent series, $N\mu = \Sigma|\mu|$, for the class of convergent series $N\mu = \max |\Sigma_1{}^n\mu|$ and so on.[5]

For the sequel we shall assume that we are working in a general function space with a norm satisfying the conditions postulated at the beginning of this section.

We take up first some problems with respect to linear modular operations. We shall say that U is an *operation* on $\mathfrak{M}$ if it transforms every function μ of $\mathfrak{M}$ into a real or complex number; *linear* if for μ_1 and μ_2 of $\mathfrak{M}$, and a_1 and a_2 complex numbers,

$$U(a_1\mu_1 + a_2\mu_2) = a_1U(\mu_1) + a_2U(\mu_2);$$

modular if there exists a number M such that for every μ,

$$|U(\mu)| \leqq M \cdot N\mu.$$

In most of the current theories, this type of linear modular operator has a

[1] Thesis (*Pal. Rend.*, vol. 22), p. 40.

[2] *Cf.* Gateaux, *Atti di Lincei, Rend.*, ser. 5, v. 23^2 (1914), p. 310.

[3] *Cf.*, *e.g.*, Wiener, *loc. cit.*, p. 124.

[4] *Cf.* in this connection Levy, *Analyse Fonctionelle*, pp. 17 ff.

[5] Further illustrations may be found in the article by Hahn referred to above.

kind of bilinear character. For instance if $\mathfrak{M}$ is class of continuous functions on $\mathfrak{P}^{\mathrm{IV}}$, then there exists a function of bounded variation α such that

$$U(\mu) = \int \mu d\alpha,$$

the integral being a Stieltjes integral; if $\mathfrak{M}$ is the class of functions of integrable square on $\mathfrak{P}^{\mathrm{IV}}$, then there exists a function of integrable square α such that

$$U(\mu) = \int \alpha\mu,$$

the integral being taken in the Lebesgue sense. If $\mathfrak{M}$ is the class of absolutely convergent series on $\mathfrak{P}^{\mathrm{III}}$, then there exists a bounded sequence α such that

$$U(\mu) = \Sigma\alpha\mu$$

and so on. This naturally suggests the question under what conditions on N and the class $\mathfrak{M}$ does there exist a bilinear operator and a function μ' of a class $\mathfrak{M}'$ preferably of the same type as $\mathfrak{M}$ such that a linear modular operator on $\mathfrak{M}$ has the form

$$U(\mu) = J\mu'\mu?$$

If it were possible to derive such a theorem, it would not be difficult to consider such questions as biorthogonality (orthogonality of functions seems to be a property confined to functions where $\mathfrak{M}'$ and $\mathfrak{M}$ are the same class, which in ordinary analysis is true on $\mathfrak{P}^{\mathrm{IV}}$ only (?) for functions of Lebesgue integrable square, and on $\mathfrak{P}^{\mathrm{III}}$ for sequences of convergent square) as between functions of two associated classes.[1]

We come next to the linear modular functional transformations or functionals. T is said to be a functional transformation on $\mathfrak{M}'$ to $\mathfrak{M}''$ if it transforms every function μ' of $\mathfrak{M}'$ into a function of the class $\mathfrak{M}''$, it being assumed that the two classes in question are of the type of the class $\mathfrak{M}$ which we are considering. Linearity and modular properties are similar to the case of operations, which are special functional transformations in which the class $\mathfrak{M}''$ is a class of complex numbers, $\mathfrak{P}$ consisting of a single element. Probably the most interesting subject in functional transformations is the question of finding inverses, *i.e.*, solving functional equations of the form $T\mu' = \mu''$. Linear integral equations are a special case of this situation with the class $\mathfrak{M}'$ the same as $\mathfrak{M}''$. Some work in connection with the case in which $\mathfrak{M}'$ and $\mathfrak{M}''$ are the same and with respect to so-called completely continuous transformations has been done by Riesz.[2] It is a question whether his methods cannot perhaps be extended to other kinds of transformations. Perhaps the methods discussed in the valuable manual by Riesz on infinitely many variables[3] can be carried over or adapted to the func-

[1] This bilinearity feature has attracted attention previously. *Cf.* the work of E. H. Moore on Integral Equations in General Analysis: *Bull. Amer. Soc.*, vol. 17 (1912), pp. 348 ff. See also the article by Hahn previously mentioned.

[2] *Cf. Acta Mathematica*, vol. 41 (1918), pp. 71–98.

[3] *Equations Linéaires à une Infinité d'Inconnues*, Paris, 1913.

tional field. It might be desirable to do so. Possibly too this problem can be discussed by reducing it to a case of solution of equations in infinitely many variables, and then applying existent results in this latter theory.[1] The usual mode of procedure, especially when dealing with functions on the linear interval, is to perform a transformation *via* a complete system of normed orthogonal functions, which establishes a one-to-one correspondence between a sequence and a function on $\mathfrak{P}^{IV}$. In the general theory under discussion, however, we are not equipped with orthogonal, or even with biorthogonal, systems of functions. A method of attacking this latter problem would be to utilize some of the properties of such a system of functions. Some which might be possible of generalization are the following: If the sequence μ_n is a system of Lebesgue integrable square functions on $\mathfrak{P}^{IV}$, which is normed and orthogonal, then the coefficients in the expansion of any function whose square is Lebesgue integrable in a series of the μ_n can be determined by finding for each n the constants $a_1, \cdots a_n$ such that

$$\int_0^1 (\mu - \Sigma_1^n a_m \mu_m)^2$$

is a minimum. If we extract the square root of this integral, then we have here a case of a norm in this class of functions. These a_m are unique for a given set of linearly independent functions. The other property is that the a_m do not change with n. If the set μ_n is complete, then the minimum value of the integral approaches zero with n.

The first idea is easily generalized. Assume a sequence of linearly independent functions μ_n of the class $\mathfrak{M}$. Then for every function μ there exist constants a_{mn} such that

$$N(\mu - \Sigma_1^n a_{mn} \mu_m)$$

is a minimum. These constants are not necessarily unique. They will be unique if for instance the norm has the property that the equality

$$N(\mu_1 + \mu_2) = N\mu_1 + N\mu_2$$

holds if and only if there exists a real positive constant a such that $\mu_1 = a\mu_2$. Whether the second property that the a_{mn} are independent of n can be generalized is a question. Perhaps it may be possible to show that there exist linear combinations of the given functions, which are linearly equivalent to the given set of functions, and which have this property. Just how that could be done might be found out. Perhaps it is not possible. The notion of completeness of a set of functions is obviously easily transferable to the more general case. If it were possible to get for each complete sequence of functions a linearly equivalent set, in which the constants for the minimizing combination were independent of the number of functions (in order) of the sequence used, and a generalization of

[1] This method has been variously applied. *Cf.*, for instance, A. J. Pell, *Trans. Amer. Math. Soc.*, vol. 20 (1919), pp. 343–355; W. L. Hart, *Annals of Math.*, vol. 24 (1922), pp. 23–38. Levy, *loc. cit.*, pp. 117–129, 292–313.

the Riesz-Fischer theorem were obtained, then perhaps this matter of reducing the functional equation to the corresponding case in infinitely many variables might be a possibility. In any case, the minimal property of orthogonal functions is no doubt worth while.

A question which is of another kind is whether there is any relation between functional transformations and bilinear operations. In n-variables a system of algebraic equations and a bilinear form go together, similarly in infinitely many variables and in integral equations. Is there any relationship in this more general basis, and if so what?

We might continue further in this subject of functional transformations, but the above indications are no doubt sufficient to indicate that there are many unanswered questions, and many things to be done. To any one interested in other questions relating to this field, I am sure that there will be found much that is suggestive in the book of Levy on *Analyse Fonctionelle,* already referred to. This book, taken together with existing theories for instance of partial differential equations in n-space, is an admirable instance of the first part of Moore's tenet of faith for general analysis. Logically then there should be opportunity as stated in the second part of this statement for a general theory, and no doubt the near future will produce much of this. Among other things, perhaps there will be shown the possibility of generalizing the mean value integration of Gateaux as expounded by Levy. Perhaps the above suggestion of introducing an n-dimensional field into the reasoning *via* a sequence of functions which is complete for the class of functions, and the minimal process above referred to, may be a line of attack on this problem.

In closing, I wish to express to you my appreciation of the opportunity of presenting these ideas to you, and the hope that perhaps they may contain something which will be of value for further research in general analysis.

SOLUTIONS OF A PROBABILITY DIFFERENCE EQUATION.

By OTTO DUNKEL, Washington University.

1. Introduction. The writer has recently given solutions of two types of probability problems in coin tossing which were obtained by counting the number of favorable cases.[1] The first type will be considered here; the three cases under this type may be stated as follows:

I. *The probability,* $p_1(n)$, *of a sequence of a tail followed by r or more consecutive heads turning up in n tosses of a coin.*

II. *The probability,* $p_2(n)$, *of tossing at first r or more consecutive heads and in the subsequent tosses no set of a tail followed by as many as r consecutive heads.*

III. *The probability,* $p_3(n)$, *of r or more consecutive heads turning up in n tosses of a coin.*

[1] *Washington University Studies, Scientific Series,* vol. XII, Jan. 1925, no. 2, pp. 119–136. See also in the MONTHLY the solution of **3046** [*1924*, 403].

It is clear that

$$p_3(n) = p_1(n) + p_2(n), \tag{1}$$

and it was shown that

$$p_1(n) = 1 - 2^r p_2(n + r). \tag{2}$$

It suffices then to determine $p_2(n)$ in order to have a determination of $p_1(n)$ and $p_3(n)$. It was also shown that $p_2(n)$ satisfies the difference equation

$$p_2(n + 1) - p_2(n) = -\frac{p_2(n - r)}{2^{r+1}}, \tag{3'}$$

$$p_2(n) = 0, \qquad 0 \leqq n < r, \qquad p_2(r) = 1/2^r.$$

The object of the present paper is to discuss the solutions of a simple transformation of the equation (3′), and it will be foreseen from the simplicity of that equation that the discussion will be elementary. But it is believed that it is of sufficient interest to develop since it affords a simple illustration of important general theorems on difference equations, of a method of determining the real roots of algebraic equations similar to that of Bernoulli, and of a method of obtaining bounds for the absolute values of the imaginary roots. It will also be shown that the probability, $p_2(n)$, is related in a simple way to a generalization of the Fibonacci sequence of numbers. A determination of equation (3′) will be given by a method different from that previously used since it shows that $p_2(n)$ satisfies also an equation of lower order but of non-homogeneous form. There will also be given a different determination of $p_2(n)$ which leads to additional information. The well-known fact that the ratio $p_2(n + 1)/p_2(n)$ approaches the largest real root of the characteristic algebraic equation will be shown in two ways.

If the finite sum which defines $p_2(n)$ is made an infinite series, this infinite series will be shown to be a solution; and it will be further shown that the ratio of the two series for the values $n + 1$ and n is precisely a root of the algebraic equation.[1] Most of the theorems and proofs given here may be generalized.

2. Derivation of the Difference Equation. Let $P_2(n)$ be the number of favorable cases for probability II when there are n tosses of a coin, and consider the number of favorable cases when an additional toss, the $(n + 1)$th, is made. If the $(n + 1)$th toss is a tail, then the number of favorable cases is $P_2(n)$. Suppose now that it is a head. There is just one case in which all the n tosses are heads. If they are not all heads, there will be favorable cases when the first $n - k - 1$ of the n tosses are favorable while the last $k + 1$ tosses consist of a tail followed by k heads, $0 \leqq k < r - 1$. For each value of k there are $P_2(n - k - 1)$ favorable cases. Summing up these results we have the theorem.

[1] This result bears a certain relation to the theorem in Problem **3071** [*1924*, 206], E. L. Post, of this MONTHLY. It seems probable that this theorem may be derived from the results given here by a limiting process.

THEOREM. *The number of favorable cases in* II, $P_2(n)$, *satisfies the non-homogeneous equation of the rth order*

$$P_2(n+1) = P_2(n) + P_2(n-1) + \cdots + P_2(n-r+1) + 1, \tag{4}$$
$$P_2(n) = 0, \quad 0 \leqq n < r; \qquad P_2(r) = 1.$$

COROLLARIES. Replace in (4) n by $n-1$ and subtract this result from (4). There results the

THEOREM. $P_2(n)$ *must satisfy the homogeneous equation*

$$P_2(n+1) = 2P_2(n) - P_2(n-r), \tag{3}$$

with the same initial conditions.

In order to pass to the corresponding equations for $p_2(n)$ we have merely to divide both sides of (3) and (4) by 2^{n+1}. In the case of (3) we find the equation and conditions given in (3′), and from (4) we have

$$p_2(n+1) = \frac{p_2(n)}{2} + \frac{p_2(n-1)}{2^2} + \cdots + \frac{p_2(n-r+1)}{2^r} + \frac{1}{2^{n+1}}. \tag{5}$$

3. The Homogeneous Linear Difference Equation. We shall here state a few facts, which will be referred to later, regarding the difference equation

$$P(n+r) = A_{r-1}(n)P(n+r-1) + A_{r-2}(n)P(n+r-2) + \cdots + A_0(n)P(n), \tag{6}$$

where r is a positive integer, $A_i(n)$ and $P(n)$ are functions of the integer n, and $A_0(n) \neq 0$. All of the statements are easily proved.

If we have r solutions of (6), $P^{(1)}(n), P^{(2)}(n), \cdots, P^{(r)}(n)$, the determinant

$$\Delta(n) = \begin{vmatrix} P^{(1)}(n) & P^{(2)}(n) & \cdots & P^{(r)}(n) \\ P^{(1)}(n+1) & P^{(2)}(n+1) & \cdots & P^{(r)}(n+1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P^{(1)}(n+r-1) & P^{(2)}(n+r-1) & \cdots & P^{(r)}(n+r-1) \end{vmatrix}, \tag{7}$$

will be called the *determinant* of these solutions. This determinant satisfies the relation

$$\Delta(n+m+1) = (-1)^{(m+1)(r-1)}A_0(n+m)A_0(n+m-1) \cdots A_0(n)\Delta(n), \tag{8}$$

and, therefore, if it is not zero for one value of n, it never vanishes. In this case the r solutions are linearly independent; and if $P(n)$ is any solution, then

$$P(n) = C_1P^{(1)}(n) + C_2P^{(2)}(n) + \cdots + C_rP^{(r)}(n), \tag{9}$$

where the C's are constants.

If the A's in (6) are constants, solutions may be obtained by solving the *characteristic algebraic equation*

$$x^r = A_{r-1}x^{r-1} + A_{r-2}x^{r-2} + \cdots + A_1 x + A_0, \qquad A_0 \neq 0, \tag{10}$$

and then setting $P(n) = a^n$, where a is a root. If the r roots of (10) are distinct, the determinant (7) is not zero for the r solutions thus obtained and they are linearly independent solutions. If a is a p-fold root, then there are corresponding to it the p solutions

$$a^n, \quad na^n, \quad n^2a^n, \quad \cdots, \quad n^{p-1}a^n.$$

The r solutions obtained in this way are again linearly independent. If the A's are real, then to each pair of conjugate imaginary roots there corresponds a pair of real solutions.

If $P(n)$ is a solution such that

$$P(n) = 0, \qquad 1 \leqq n \leqq r-1, \qquad P(r) = 1, \tag{11}$$

then $P(n+r-1)$, $P(n+r-2)$, $\cdots$, $P(n)$ is a system of linearly independent solutions for which

$$\Delta(n) = (-1)^{(n-1)(r-1)}A_0{}^{n-1}. \tag{11'}$$

4. The Probability Algebraic Equation. The characteristic algebraic equation for (3′) is

$$x^{r+1} - x^r + \frac{1}{2^{r+1}} = 0. \tag{12'}$$

If the roots of this equation are multiplied by 2, there results the equation

$$f(x) \equiv x^{r+1} - 2x^r + 1 = 0, \tag{12}$$

which corresponds to the difference equation (3) for $P_2(n)$. For simplicity we consider this second equation. If $r = 1$, this equation has the double root 1, and we shall exclude this case in what follows.

THEOREM. *There are only two positive roots, unity and a root between one and two. These are the only real roots if r is odd; if r is even, there is only one more real root and it lies between -1 and 0.*

There are no multiple roots and no two roots have equal absolute values unless they constitute a pair of conjugate imaginaries. The absolute values of the imaginary roots are less than unity.

PROOF. The first statement follows from the facts that $f(0) = 1$, $f(1) = 0$, $f(2) = 1$, and that the derivative

$$f'(x) = x^{r-1}[(r+1)x - 2r] \tag{13}$$

vanishes only at $x = 0$ and at $x = 2r/(r+1)$. Since $f'(1) = 1 - r < 0$ and

$f'(2) = 2^r$, it is obvious that first $f'(x)$ and then $f(x)$ vanishes between 1 and 2. The facts regarding the negative root and the non-existence of multiple roots are easily verified. It has been shown above that $f[2r/(r+1)]$ is actually negative, and we shall set down here a result which will be used later. By a simple transformation of this inequality it becomes

$$\frac{r+1}{2^{r+1}}\left(\frac{r+1}{r}\right)^r < 1, \qquad r \geqq 2. \tag{14}$$

In order to examine the imaginary roots we set $x = \rho(\cos\theta + i\sin\theta)$, where $0 < |\theta| < \pi$. This gives the pair of equations

$$\rho^{r+1}\cos(r+1)\theta - 2\rho^r\cos r\theta = -1, \qquad \rho^{r+1}\sin(r+1)\theta - 2\rho^r\sin r\theta = 0.$$

Squaring both sides and adding the results, we find

$$\rho^{2(r+1)} - 4\rho^{2r+1}\cos\theta + 4\rho^{2r} = 1.$$

For a given value of ρ, $\cos\theta$ can have only one value, and hence θ can have only the two values corresponding to a single conjugate pair of roots with the absolute value ρ. This shows also that no imaginary root can have an absolute value equal to that of a real root.

In order to prove the last statement consider the equation whose roots are the reciprocals of the roots of (12),

$$y^{r+1} - 2y + 1 = 0. \tag{15}$$

This equation has the root 1 and a second positive root, say b, which is less than 1. Then, since $y^{r+1} - b^{r+1} = 2(y-b)$, we have

$$y^r + y^{r-1}b + \cdots + b^{r-1}y + b^r = 2, \qquad y \neq b.$$

Suppose that y is an imaginary root such that $|y| = \rho \leqq 1$, then

$$2 = |y^r + y^{r-1}b + \cdots + b^{r-1}y + b^r| < \rho^r + b\rho^{r-1} + \cdots + b^{r-1}\rho + b^r$$
$$< 1 + b + \cdots + b^{r-1} + b^r = 2.$$

Since y is imaginary, the equality sign is excluded above. The last equality on the right follows since b is a root of the equation obtained from (15) by removing the root unity. From the contradiction above it follows that we must have $\rho > 1$; and this means that the modulus of an imaginary root of (12) must be less than unity.

Remarks. It will be shown later how some of these results may be obtained in a different way.

It may be shown that the absolute values of the imaginary roots of (12) are greater than $1/\sqrt[r]{3} > 1/2$.

If r is even, the absolute values of the imaginary roots lie in an interval whose bounds are unity and the absolute value of the negative root. The length of this interval is less than .23 and it decreases as r increases.

An approximation, in excess, to the larger positive root of (12) is given by

$$2\left[1-\frac{1}{2^{r+1}-r}\right], \tag{16}$$

which is obtained by applying Newton's method to (15) rather than to (12). It will now be shown how to approximate this root in a different manner.

5. Approximations to the Larger Positive Root.

THEOREM. *The ratio*

$$\frac{P_2(n+1)}{P_2(n)} \tag{17}$$

decreases as n increases from $2r-1$ and approaches as a limit the larger positive root a_1 of equation (12).

PROOF. For convenience set the ratio (17) equal to $R(n)$, then we may write (3) in the form

$$R(n) = 2 - \frac{1}{R(n-1)R(n-2)\cdots R(n-r)}\cdot \tag{18}$$

From the conditions imposed upon $P_2(n)$ we have

$$R(n) = 2, \qquad r \leqq n \leqq 2r-1; \qquad R(2r) = 2\left[1-\frac{1}{2^{r+1}}\right].$$ [1]

If in (18) the denominator decreases, the whole expression, *i.e.*, $R(n)$, decreases. Now from the values of $R(n)$ above this happens when n in (18) increases from $2r$ to $2r+1$, and hence $R(n)$ will continue to decrease as n increases from this point on. It now follows easily that $R(n)$ approaches a limit $\bar{a}_1$ which is a root of (12).

It remains to show that $\bar{a}_1$ is not 1. If each factor in the denominator of (18) is greater than a_1, the larger positive root of (12), then the whole expression on the right, *i.e.*, $R(n)$, is such that $R(n) > 2 - 1/a_1^r = a_1$. Now when n in (18) is equal to $2r$, this is true, for then each factor is 2. Hence $R(2r) > a_1$, and it then follows that every $R(n)$ is greater than a_1. Therefore $\bar{a}_1 = a_1$.

REMARKS. From this it follows at once that

$$\operatorname*{Limit}_{n=\infty} \frac{p_2(n+1)}{p_2(n)} = \frac{a_1}{2} = c_1, \tag{19}$$

where c_1 is the larger positive root of (12′). This would furnish an approxima-

[1] This expression is also given as an approximation to a_1 when Newton's method is applied to (12).

tion formula for $p_2(n)$ for large values of n. Such a formula will be furnished in another way later. The above results will be shown in a second way which applies more generally but does not prove so much.

6. Two Expressions for $P_2(n)$ and $p_2(n)$. The evaluation of probability II is furnished by the following theorem.

THEOREM. *If* $[(n+1)/(r+1)]=i$ *means the greatest integer in* $(n+1)/(r+1)$,

$$P_2(n) = \sum_{k=1}^{k=i} (-1)^{k+1}{}_{n-kr}C_{k-1}2^{n-k(r+1)+1},$$

$$p_2(n) = \frac{1}{2^r}\sum_{k=1}^{k=i}(-1)^{k+1}{}_{n-kr}C_{k-1}\left(\frac{1}{2^{r+1}}\right)^{k-1}, \qquad {}_mC_p = \frac{m!}{p!(m-p)!}\cdot \tag{20}$$

If $a_1, a_2, \cdots, a_{r+1}$ *are the roots of* (12), *then*

$$P_2(n) = \sum_{k=1}^{k=r+1}\frac{a_k^{\,n}}{f'(a_k)}\cdot \tag{21}$$

PROOF. Since $a_1^n, a_2^n, \cdots, a_{r+1}^n$ are linearly independent solutions of (3), we must have

$$P_2(n) = \sum_{k=1}^{k=r+1} C_k a_k^{\,n},$$

where the constants C_k are determined uniquely by the equations

$$1 = \sum_{k=1}^{k=r+1} C_k a_k^{\,r}, \qquad 0 = \sum_{k=1}^{k=r+1} C_k a_k^{\,n}, \qquad 0 \leqq n \leqq r-1.$$

It is obvious that no C_k can be zero. One of the roots of (12) is 1 and the corresponding C_k is $1/(1-r)$. For subtract from each term of the first equation the sum of the corresponding terms of the last r equations and note that $a_k^r = a_k^{r-1} + a_k^{r-2} + \cdots + a_k + 1$, if $a_k \neq 1$. This is what we should expect, for $1/(1-r)$ is a solution of the non-homogeneous equation (4), and since $P_2(n)$ is also a solution, it must necessarily contain this term plus a solution of the homogeneous equation obtained by dropping the constant term 1 in (4).

The coefficients C_k are the quotients of certain determinants formed from the roots a_k which are easily written. But to avoid the labor of the reduction of these determinants and at the same time to obtain other results, we shall employ a method which is often used for this purpose. We set

$$\frac{\varphi(x)}{f(x)} = \sum_{n=1}^{n=\infty}\frac{P_2(n)}{x^n},$$

where $\varphi(x)$ is a polynomial in x of degree not higher than r, and we determine it so that $P_2(n) = 0$, $1 \leqq n \leqq r-1$, and $P_2(r) = 1$, $P_2(r+1) = 2$. Multiplying

both sides by $f(x)$, we have

$$\begin{aligned}\varphi(x) &= \sum[P_2(n+r+1) - 2P_2(n+r) + P_2(n)]x^{-n}\\ &= A_r x^r + A_{r-1}x^{r-1} + \cdots + A_1 x + A_0.\end{aligned}$$

This gives

$$A_0 = P_2(r+1) - 2P_2(r) = 0, \qquad A_1 = P_2(r) = 1,$$
$$A_2 = A_3 = \cdots = A_r = 0,$$
$$P_2(n+r+1) - 2P_2(n+r) + P_2(n) = 0, \qquad n \geqq 1.$$

Thus $\varphi(x) = x$, and, if we suppose that $x \geqq 3$, we may write

$$\frac{x}{f(x)} = \frac{1}{x^r\left[1 - \left(\frac{2}{x} - \frac{1}{x^{r+1}}\right)\right]} = \sum_{n=0}^{n=\infty} \frac{1}{x^r}\left(\frac{2}{x} - \frac{1}{x^{r+1}}\right)^n.$$

We may develop each term of the series by the binomial theorem and rearrange the terms of the result. This gives after a slight change in the notation

$$\sum_{n=r}^{n=\infty} \frac{1}{x^n} \sum_{k=1}^{k=i} (-1)^{k+1}{}_{n-kr}C_{k-1} 2^{n-k(r+1)+1}, \qquad i = \left[\frac{n+1}{r+1}\right];$$

and, since $P_2(n)$ is the coefficient of x^{-n}, we have the first part of the result in (20). The expression for $p_2(n)$ is now obtained by dividing $P_2(n)$ by 2^n.

Returning to the rational function of x above, we may write it by aid of the theory of partial fractions in the form

$$\frac{x}{f(x)} = \sum_{k=1}^{k=r+1} \frac{a_k}{f'(a_k)} \frac{1}{x - a_k} = \sum_{k=1}^{k=r+1} \frac{a_k}{f'(a_k)x} \frac{1}{\left(1 - \frac{a_k}{x}\right)}.$$

Since $|a_k/x| < 2/3$, we may develop each term of the finite sum in an infinite series and collect the terms thus:

$$\frac{x}{f(x)} = \sum_{n=1}^{n=\infty} \frac{1}{x^n} \sum_{k=1}^{k=r+1} \frac{a_k{}^n}{f'(a_k)}.$$

From this the expression (21) now follows.

REMARKS. The expressions for $P_2(n)$ and $p_2(n)$ in (20) define these functions only for positive values of n. However, the difference equations define them for negative values also. Expressions corresponding to (20) when n is negative may be obtained without difficulty by similar developments. On the other hand, the solution of the difference equation given by (21) is valid for both positive and negative values of n. The expression for $p_2(n)$ corresponding to (21) is easily written.

7. The Rapidity of Convergence of $P_2(n+1)/P_2(n)$. The proof of formula (21) applies to the general linear homogeneous difference equation with constant coefficients provided that the roots of the algebraic equation are distinct. In case there are equal roots there is no difficulty in suitably modifying the discussion. Thus a second and more general proof that the limit of the ratio $P_2(n+1)/P_2(n)$ is a_1 may be obtained from (21). This shows also that for sufficiently large values of n we may write

$$P_2(n) = \frac{a_1{}^n}{f'(a_1)} \text{ approximately,} \tag{22}$$

if the absolute value of a_1 is greater than that of any other root.

We shall now prove the following theorem.

THEOREM. *The difference*

$$\frac{P_2(n+2)}{P_2(n+1)} - \frac{P_2(n+1)}{P_2(n)} = \frac{P_2(n+2)P_2(n) - P_2{}^2(n+1)}{P_2(n+1)P_2(n)} \tag{23}$$

is negative after a sufficiently large value of n and it approaches zero like $1/a_1{}^n$.

PROOF. The numerator of (23) is

$$\begin{vmatrix} P_2(n+2) & P_2(n+1) \\ P_2(n+1) & P_2(n) \end{vmatrix} = \sum\sum C_iC_j(a_ia_j)^n(a_i - a_j)^2, \qquad j > i = 1, 2, \cdots, r,$$

while the denominator is

$$P_2(n+1)P_2(n) = \sum_{i=1}^{r+1} C_i^2 a_i^{2n+1} + \sum\sum C_iC_j(a_ia_j)^n(a_i + a_j), \qquad j > i,$$

where the values of the C's are given in (21). Hence for large values of n (23) is approximately

$$\frac{C_2a_2{}^n(a_1 - a_2)^2}{C_1a_1{}^{n+1}} = \frac{f'(a_1)(a_1 - 1)^2}{f'(1)a_1{}^{n+1}} = \frac{[(r+1)a_1 - 2r](a_1 - 1)^2a_1{}^{r-2}}{(1-r)a_1{}^n}, \tag{24}$$

since $a_1 > 1$ ($r \geqq 2$) and the absolute values of the remaining $r-1$ roots are less than the root $a_2 = 1$. Since here a_1 and 1 are consecutive simple roots, $f'(a_1)/f'(1)$ is negative; hence for sufficiently large values of n the ratio $P_2(n+1)/P_2(n)$ decreases. The previous proof determined this value of n.

REMARKS. In the case of the general equation, if $P_2(n+1)/P_2(n)$ approaches a limit a_1, and no two roots of the algebraic equation have equal absolute values unless they form a conjugate pair of imaginaries, it may be shown by the use of the formula for (23) that a_1 is a root with a greater absolute value than any other root. If $P_2(n+1)/P_2(n)$ ultimately always decreases, it may be shown by (23) that the root whose absolute value is next in magnitude to that of a_1 is the real root a_2. For simplicity of statement we have assumed that a_1 and a_2 are both positive and that the coefficients in the difference equation are real. A

second proof of the first part of this theorem which does not require so much is given later.

8. The Solution of Numerical Equations. If it is known that a given numerical equation with real coefficients has a real root a_1 whose absolute value is greater than that of any other root, this root may be approximated by the ratio $P(n+1)/P(n)$, where $P(n)$ is a solution of the corresponding difference equation similar to $P_2(n)$. The initial values of $P(n)$ may be chosen arbitrarily, but in this case, if the algebraic equation is reducible, the approximation may lead to the real root of the reduced equation whose absolute value is greater than that of its other roots. The method also applies when a_1 is a multiple root. Also if a_1 and $-a_1$ are two real roots of greater absolute value than those of the remaining roots, a slight change in the equation permits the method to be applied to find a_1. If the root a_2 whose absolute value is next in magnitude to that of a_1 is real, the ratio a_2/a_1 may be approximated by the ratio of two of the expressions (23) for the values $n+1$ and n. Then a_2 may be found after a_1 has been determined.

Theoretically, any real root may be approximated in this way after making the transformation of the equation by $y = 1/(x-c)$, where c is chosen near enough to the desired root to make the corresponding root of the y-equation have the greatest absolute value. However, this would not in general give a good practical process. If the equation has a real root of absolute value smaller than that of any other root, then $y = 1/x$ transforms the equation and the root so that the method may be applied to the resulting equation.

The equation (12) becomes after the transformation $y = 1/x$

$$y^{r+1} - 2y + 1 = 0, \tag{25}$$

and the corresponding ratio is

$$\frac{P_2(-n-1)}{P_2(-n)} = \frac{Q(n+1)}{Q(n)}, \tag{26}$$

where $Q(n)$ satisfies the equation

$$Q(n+r+1) - 2Q(n+1) + Q(n) = 0. \tag{27}$$

If r is odd, the ratio (26) cannot have a limit as we see from the nature of the roots. If r is even, (25) has a third real root of absolute value greater than 1. From the facts stated above, page 359, regarding the roots it follows that (26) must approach this root as a limit. But this process is useless as a practical method since the convergence is extremely slow due to the very slight difference between the absolute value of this root and that of an imaginary root.

9. Relations between Powers of a Root.

THEOREM. *If x is any root of* (12),

$$x^n = P_2(n)x^r - \sum_{k=0}^{k=r-1} P_2(n-k-1)x^k. \tag{28}$$

PROOF. Writing the equation (12) in the form $x^{r+1} = 2x^r - 1$, we multiply both sides by $x, x^2, x^3, \cdots$, and after each multiplication we reduce the right side to the degree r by means of the original equation above. The result may be written

$$x^n = \sum_{k=0}^{k=r} F_k(n)x^k, \tag{29}$$

$$F_k(n) = 0, \qquad n \neq k, \qquad F_k(k) = 1, \qquad 0 \leqq n \leqq r.$$

Multiply both sides of (29) by x and reduce the right side to the degree r as before; we obtain

$$x^{n+1} = [2F_r(n) + F_{r-1}(n)]x^r + \sum_{k=0}^{r-2} F_k(n)x^{k+1} - F_r(n).$$

Comparing this with (29), we find

$$F_r(n+1) = 2F_r(n) + F_{r-1}(n), \qquad F_{k+1}(n+1) = F_k(n),$$
$$F_0(n+1) = -F_r(n), \qquad 0 \leqq k \leqq r-2.$$

From these equations we obtain in turn

$$F_k(n) = -F_r(n-k-1), \qquad F_r(n+1) = 2F_r(n) - F_r(n-r), \qquad 0 \leqq k \leqq r-1.$$

The last equation shows that $F_r(n)$ is a solution of (3) and from the initial conditions in (29) we see that $F_r(n) = P_2(n)$. Inserting the values of $F_k(n)$ in (29), we obtain (28).

REMARKS. Setting $x = 1$ in (28), it follows that if $P_2(n)$ is a solution of (3) with the given initial conditions then it must also satisfy (4).

The equation (28) furnishes a proof of the

THEOREM. *Given that the roots of* (12) *are distinct, that $P_2(n)$ is a solution of* (3) *with the given initial conditions, that $P_2(n+1)/P_2(n)$ approaches a limit a_1 as n becomes infinite, then a_1 is a root of* (12) *whose absolute value is greater than that of any other root.*

This theorem and its proof may be generalized.

PROOF. It follows that a_1 is a root of (12) by the same reasoning indicated before. Divide both sides of (28) by $P_2(n-r)$ and take the limit of each side for $n = \infty$; we have as the result

$$\operatorname*{Limit}_{n=\infty} \frac{x^n}{P_2(n-r)} = a_1^r x^r - \sum_{k=0}^{k=r-1} a_1^{r-k-1}x^k = a_1^r x^r - \frac{a_1^r - x^r}{a_1 - x}, \tag{30}$$

where in the last form to the right $x \neq a_1$. If $x \neq a_1$, the right side of (30) is zero; for it is equal to

$$\frac{x^r f(a_1) - a_1^r f(x)}{a_1 - x}.$$

If $x = a_1$, the right side reduces to $a_1^r f'(a_1)$, and then (30) gives after a slight change

$$\operatorname*{Limit}_{n=\infty} \frac{a_1^n}{P_2(n)} = f'(a_1).$$

If $x \neq a_1$, we have now

$$\operatorname*{Limit}_{n=\infty} \left(\frac{x}{a_1}\right)^n = \frac{1}{a_1^r f'(a_1)} \operatorname*{Limit}_{n=\infty} \frac{x^n}{P_2(n-r)} = 0.$$

This shows that $|x| < |a_1|$.

If both sides of (28) are multiplied by $1/f'(x)$ and the sum of each side is taken for $x = a_1, a_2, \cdots, a_{r+1}$, we obtain again the result (21) making use of the known relations

$$\sum_{i=1}^{i=r+1} \frac{a_i^r}{f'(a_i)} = 1, \qquad \sum_{i=1}^{i=r+1} \frac{a_i^m}{f'(a_i)} = 0, \qquad 0 \leqq m < r.$$

From the equation (28) relations between the functions $P_2(n)$ for different values of n may be obtained. One procedure is to multiply both sides by x^m, then reduce the right side to the degree r and compare the result with the expression for x^{n+m} given directly by (28).

10. The Fibonacci Sequence. Since $P_2(n)$ is a solution of the non-homogeneous equation (4), it is at once evident that

$$P_2(n+1) - P_2(n) = F(n) \tag{30A}$$

must be a solution of the homogeneous equation

$$\begin{gathered} F(n+r) = F(n+r-1) + F(n+r-2) + \cdots + F(n), \\ F(n) = 0, \qquad 0 \leqq n \leqq r-2, \qquad F(r-1) = 1. \end{gathered} \tag{31}$$

When $r = 2$, this gives the series of numbers usually called the Fibonacci series. Other relations are given in the following theorem.

THEOREM. *If $F(n)$ is defined as in* (30),

$$P_2(n) = \frac{1}{r-1}\left[F(n+r-1) - \sum_{k=3}^{k=r} (k-2)F(n+r-k) - 1\right],$$

$$= \frac{1}{r-1}\left[F(n+r) - \sum_{k=2}^{k=r} (k-1)F(n+r-k) - 1\right], \tag{32}$$

$$= \sum_{n=r-1}^{n-1} F(n). \tag{33}$$

PROOF. Write (4) in the equivalent form

$$P_2(n+r) - P_2(n+r-1)$$
$$= \sum_{k=3}^{k=r} (k-2)[P_2(n+r-k+1) - P_2(n+r-k)] + (r-1)P_2(n) + 1.$$

Then from (30A) we have at once the first equation in (32). The second form follows by use of (31).

If we multiply both sides of

$$P_2(n-k) = P_2(n-k-1) + \cdots + P_2(n-k-r) + 1$$

by $F(r+k-1)$, and take the sum of the corresponding sides from $k = 0$ to $k = n - r$, there results after certain cancellations by use of (31) the equation given in (33).

REMARKS. From (11′) we may obtain a theorem regarding the determinant of the $F(n)$'s which is well known in the case of $r = 2$. Other relations involving this function may be obtained by the method indicated at the end of the previous section.

11. An Infinite Series as a Solution. It will be more convenient now to consider the equation (3′). It will be readily seen that the second expression of (20) is formally a solution of this equation when the upper limit i in the finite sum is made infinite. It remains to show that

$$p(n) = \frac{1}{2^r}\left[1 + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{(n-kr)(n-kr-1)\cdots(n-kr-k+2)}{(k-1)!}\left(\frac{1}{2^{r+1}}\right)^{k-1}\right], \quad (34)$$

is convergent. Here we shall have a solution for both positive and negative values of n. To show the convergence of (34) it suffices to examine the infinite series

$$q(n) = 1 + \sum_{k=2}^{\infty} \frac{(n+kr)(n+kr+1)\cdots(n+kr+k-2)}{(k-1)!}\left(\frac{1}{2^{r+1}}\right)^{k-1}, \quad (34')$$

where n is positive. The ratio of two consecutive terms of this series for $k \geqq r+2$ may be written after certain cancellations

$$\left(1+\frac{k-1}{n+kr}\right)\left(1+\frac{k-1}{n+kr+1}\right)\cdots\left(1+\frac{k-1}{n+kr+r-1}\right)\left(r+1+\frac{n+r-1}{k}\right)\frac{1}{2^{r+1}}.$$

This ratio is less than

$$\left(1+\frac{1}{r}\right)^r\left(r+1+\frac{n+r-1}{k}\right)\frac{1}{2^{r+1}}.$$

Hence the series (34′) converges if

$$\frac{r+1}{2^{r+1}}\left(1+\frac{1}{r}\right)^r < 1,$$

and this has already been shown to be true if $r \geqq 2$ in (14). Thus the series (34) converges for all values of n.

THEOREM. *If c_1 is the larger positive root of* (12′), *then*

$$\frac{p(n+1)}{p(n)} = c_1, \qquad p(n) = Ac_1{}^n, \tag{35}$$

where A is a constant and n is an integer.

PROOF. The function $q(n) = 2^r p(-n)$ defined in (34′) satisfies the equation

$$q(n+1) = q(n) + \frac{q(n+r+1)}{2^{r+1}}, \tag{36}$$

for which the algebraic equation is

$$y = 1 + \frac{y^{r+1}}{2^{r+1}}\cdot \tag{36′}$$

The roots of this equation are the reciprocals of $c_1, c_2, \cdots, c_{r+1}$, the roots of (12′).

Consider the two terms in the ratio

$$\frac{q(n+1)}{q(n)}, \qquad n \geqq 0, \tag{37}$$

one in the numerator and the other in the denominator, which correspond to the same value of k. The ratio of these two terms is

$$\frac{(n+1+kr)(n+2+kr)\cdots(n+k-1+kr)}{(n+kr)(n+1+kr)\cdots(n+k-2+kr)} = 1 + \frac{k-1}{n+kr} < 1 + \frac{1}{r}\cdot$$

Hence the ratio (37) lies between 1 and $1 + 1/r$. If we set this ratio equal to $R(n)$, we may write (36) in the form

$$R(n) = 1 + \frac{R(n+r)R(n+r-1)\cdots R(n)}{2^{r+1}},$$

and this enables us to determine a series of upper bounds l_k for $R(n)$ and also a series of lower bounds m_k, where $m_1 = 1$ and $l_1 = 1 + 1/r$. Thus we have at once

$$m_2 = 1 + \frac{1}{2^{r+1}} < R(n) < 1 + \frac{1}{2^{r+1}}\left(\frac{r+1}{r}\right)^{r+1} = l_2.$$

Obviously $m_2 > m_1$, and since (14) may be written in the form

$$1 + \frac{1}{2^{r+1}}\left(\frac{r+1}{r}\right)^{r+1} < 1 + \frac{1}{r},$$

we see that $l_2 < l_1$. Each of the bounds m_k and l_k satisfies the relation

$$x_{k+1} = 1 + \frac{x_k{}^{r+1}}{2^{r+1}};$$

and it follows at once that as k increases l_k decreases and m_k increases. Hence each approaches a limit which must be a root of (36′) which lies between 1 and 2. Since there is only one such root, $1/c_1$, both l_k and m_k approach this root as a common limit. It now follows that the ratio (37) is equal to $1/c_1$. From this follows that $p(-n) = 2^{-r}q(n) = Ac_1^{-n}, n \geqq 0$. Since $p(n)$ is linearly dependent upon the $(r+1)$ solutions $c_1^n, c_2^n, \cdots, c_{r+1}^n$, we have for both positive and negative values of n

$$p(n) = Ac_1^n + A_2c_2^n + \cdots + A_{r+1}c_{r+1}^n.$$

But for negative values of n, $A_2 = A_3 = \cdots = A_{r+1} = 0$, and the proof of (35) is complete.

Remarks. A result similar to the above is given by the series[1]

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+nh)^n}{n!},$$

which satisfies the equation

$$f'(x) = f(x+h), \tag{40}$$

for which the corresponding equation is

$$e^{mh} = m. \tag{40'}$$

Here it may be shown that $f(x) = Ae^{mx}$, where A is a constant. A root of the equation (40′) is given by $f(x+h)/f(x)$. In this case the quotient may be evaluated in the form of a single series not containing x. It may also be possible to evaluate the quotient defining c_1 in (35) as a single series.

It will be seen that the methods used above may be applied to determine a solution in the form of an infinite series similar to (34) for the equation

$$p(x+d) - p(x) = dp(x+r+d);$$

and then by a limit process we may pass to the solution of (40) and (40′). This limit process, however, has been examined by the writer only in a formal way, but it seems likely that the formal steps can be justified so as to furnish the theorems above regarding (40).

12. Numerical Illustrations. A small table of values of $p_1(n)$, $p_2(n)$ and $p_3(n)$ for $r = 5$ was given in a previous paper.[2] These values were computed by first calculating $p_1(n)$ by a formula similar to that for $p_2(n)$ (20). These results could have been computed more easily and quickly by using the recurrent relation (4) to calculate a table of values of $P_2(n)$; from the values of $P_2(n)$ those for $p_2(n)$ may then be obtained by division by 2^n. Below is given a table of values of $P_2(n)$ for $r = 5$ calculated in this way. After writing down the first five values

[1] Post, Problem **3071** [*1924*, 206].

[2] Dunkel, *Washington University Studies*, *l.c.*, p. 126.

1, 2, 4, 8, 16, these are added together with 1 to give the next entry 32. A slip of paper may be used to cover all of the entries in a column except the last five, and a figure 1 should be marked on the edge of the slip in line with the unit column. After the addition of this 1 and the last five entries exposed the slip is advanced one space to leave exposed the next set of five, and so on. The preparation of the table below including the time for the verification of each entry takes about twenty minutes.

$r = 5$

n	$P_2(n)$	n	$P_2(n)$
5	1	23	210687
6	2	24	414200
7	4	25	814296
8	8	26	1600864
9	16	27	3147216
10	32	28	6187264
11	63	29	12163841
12	124	30	23913482
13	244	31	47012668
14	480	32	92424472
15	944	33	181701728
16	1856	34	357216192
17	3649	35	702268543
18	7174	36	1380623604
19	14104	37	2714234540
20	27728	38	5336044608
21	54512	39	10490387488
22	107168	40	20623558784

From the last two entries we have

$$\frac{P_2(40)}{P_2(39)} = 1.9659482366682240\cdots$$

as an approximation to a_1. The correct figures may be determined by calculating $P_2(41)/P_2(40)$. This does not require an extension of the table, for

$$\frac{P_2(41)}{P_2(40)} = 2 - \frac{P_2(35)}{P_2(40)} = 1.9659482366571560\cdots$$

and therefore

$$a_1 = 1.96594823665\cdots.$$

The following values were also computed from the table by division and the use of (1) and (2):

$$p_2(35) = .020438704611, \qquad p_3(35) = .420214146347,$$
$$p_1(35) = .399775441736, \qquad p_2(40) = .018757017446.$$

From (22) we have approximately for large values of n, using the value of a_1 above,

$$p_2(n) = \left(\frac{a_1}{2}\right)^{n-4} \frac{1}{2^5(3a_1 - 5)} = .0348055734513(.98297411833)^{n-4}.$$

This formula gives by the aid of (1) and (2)

$$p_2(20) = .0264437, \qquad p_2(100) = .0066942,$$
$$p_2(50) = .0157974, \qquad p_1(100) = .803415,$$
$$p_3(100) = .810109.$$

All of these are perhaps correct in the figures given except $p_2(20)$ which may be incorrect in the last figure 7. The computation of $p_2(100)$ by the formula (20) would be quite tedious.

For $r = 2$, we have the exact formula

$$p_2(n) = \frac{1}{2^n}\left[\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n - 1\right].$$

For $r = 3$,

$$p_2(n) = C_1c_1^n + C_2c_2^n + \rho^n[C_3 \cos n\theta + C_4 \sin n\theta],$$

where

$$\begin{array}{llll} c_1 = .91964338, & c_2 = 1/2, & \rho = -\,.368676, & \theta = 55^\circ\,18'\,39.6'', \\ C_1 = .21780819, & C_2 = -\,1/2, & C_3 = \,.282164, & C_4 = -\,.359184. \end{array}$$

AN ALGEBRA WITH SINGULAR ZERO.

By E. T. BELL, University of Washington.

1. To the many existing "algebras" which have been devised by modifying or abolishing certain of the restrictions imposed by the fundamental laws of common algebra, we propose to add another, distinguished by the singular properties of its zero. In common algebra, $b - b = 0$, and if this be taken as the definition of (additive) zero, then $c + 0 = 0 + c = c$. In the modification, if β, γ are any elements of the algebra, we do not have $\gamma + \beta - \beta = \gamma$, although we do have $\beta - \beta + \gamma = \gamma + \beta - \beta$. Our purpose in calling attention to this algebra is not to exhibit a barren freak, but to point out an extremely powerful method of dealing symbolically with certain classes of numbers and functions. Its use greatly simplifies, for example, the deduction of general relations (general in the sense that arbitrary functions are involved) between the Bernoulli numbers, or the Bernoulli polynomials of one or more variables, or the Bessel coefficients. Here merely the algebra itself is stated; the applications are developed in other papers. From the interpretation given it will be easy for any one who is interested to construct his own applications.

2. For brevity we shall refer to common algebra as A, and to the algebra which is to be constructed with singular zero as U. The elements of A are ordinary algebraic quantities $a, b, c, \cdots$, or *ordinaries;* the elements of U are *umbræ* $\alpha, \beta, \gamma, \cdots$ of ordinaries. The umbra α is a representative of the entire class of ordinaries α_i $(i = 0, 1, 2, \cdots)$, and similarly for $\beta, \gamma, \cdots$. We say that α is an umbra of the class α_i $(i = 0, 1, 2, \cdots)$. It is assumed unless otherwise stated that umbræ denoted by different letters are distinct; thus α_i, β_i $(i = 0, 1, 2, \cdots)$ are different classes of ordinaries, so that at least one of the classes contains at least one element not in the other.

3. It will be of assistance here to recall a few points of the "representative" or umbral calculus of Blissard.[1] The nth power, α^n, of the umbra α is purely symbolic, or umbral, and represents the ordinary, α_n. Operations upon umbral powers are to be performed as if the powers were ordinary until the last step,

[1] *Quarterly Journal of Mathematics*, vols. 4, 5; 3 papers; also Lucas, *Théorie des Nombres*, chap. XIII.

when exponents are degraded to suffixes. Thus, if m, n are integers > 0,

$$\alpha^m\alpha^n = \alpha^{m+n} = \alpha_{m+n}; \qquad \frac{\partial^r\alpha^m}{\partial\alpha^r} = \frac{m!}{(m-r)!}\alpha_{m-r};$$

$$(\alpha+\beta)^n = \sum_{r=0}^{n}\binom{n}{r}\alpha^{n-r}\beta^r = \sum_{r=0}^{n}\binom{n}{r}\alpha_{n-r}\beta_r.$$

These sufficiently illustrate the nature of the processes, which are fully discussed in the works cited.

4. If a is an ordinary, α an umbra, $a\alpha$ is by definition the umbra of the class $a\alpha_n$ $(n = 0, 1, 2, \cdots)$.

We define the equality $\alpha = \beta$ between umbræ α, β to mean either of the equivalent relations

$$\alpha^n = \beta^n, \qquad \alpha_n = \beta_n \quad (n = 0, 1, 2, \cdots).$$

The umbral multinomial theorem, which can be taken as the definition of $(a\alpha + b\beta + \cdots + c\gamma)^n$ for $a, b, \cdots, c$ ordinaries, $\alpha, \beta, \cdots, \gamma$ umbræ, and $n \geqq 0$ an integer, is

$$(a\alpha + b\beta + \cdots + c\gamma)^n = \sum \frac{n!}{f!\,g!\,\cdots\,h!}a^f b^g \cdots c^h \alpha_f\beta_g \cdots \gamma_h,$$

the Σ extending to all sets of integers $f, g, \cdots, h$ each $\geqq 0$ whose sum is n and whose number is equal to that of the umbræ. The right-hand side of this defines a polynomial δ_n in $a, b, \cdots, c$ and $\alpha_f, \beta_g, \cdots, \gamma_h$, for $f, g, \cdots, h$ as stated, which is uniquely determined when $a, b, \cdots, c, \alpha, \beta, \cdots, \gamma$ and n are given. Hence by the definition of equality we can write

$$\delta = a\alpha + b\beta + \cdots + c\gamma.$$

Suppose we have also $\delta = r\rho + s\sigma + \cdots + t\tau$. Then evidently the meaning of

$$a\alpha + b\beta + \cdots + c\gamma = r\rho + s\sigma + \cdots + t\tau$$

is uniquely defined.

From the foregoing definitions it follows immediately that

$$a\alpha + b\beta + c\gamma = a\alpha + c\gamma + b\beta = (a\alpha + b\beta) + c\gamma = a\alpha + (b\beta + c\gamma),$$

so that, *provided the umbræ are distinct*, the laws of addition and subtraction in U are precisely as in A.

The divergence between U, A is seen in the simplest instance. We do *not* have $\alpha + \alpha = 2\alpha$, for this would imply

$$2^n\alpha_n = \sum_{r=0}^{n}\binom{n}{r}\alpha_{n-r}\alpha_r,$$

which is in general false.

Suppose there are precisely n (> 0) umbræ in the set $\alpha, \beta, \cdots, \gamma$, and precisely n ordinaries in the set $a, b, \cdots, c$, and let us write

$$\delta = a\alpha + b\beta + \cdots + c\gamma.$$

The result of replacing each of $\alpha, \beta, \cdots, \gamma$ by α in δ^m *after* degradation of all exponents of umbræ is defined to be the value of $(a\alpha + b\beta + \cdots + c\gamma)^m$. Hence if in any relation between umbræ a particular umbra occurs precisely n times, the n like umbræ are replaced by n distinct umbræ until after the degradation of exponents, when these n distinct umbræ are replaced by the original. In the particular case $a = b = \cdots = c = 1$ of the above, we write

$$n \cdot \alpha = \alpha + \alpha + \cdots + \alpha,$$

the α on the right being repeated precisely n times. Combining this with previous definitions, we have uniquely defined

$$n \cdot a\alpha + m \cdot b\beta + \cdots + s \cdot c\gamma$$

for $n, m, \cdots, s$ positive non-zero integers, $a, b, \cdots, c$ ordinaries, and $\alpha, \beta, \cdots, \gamma$ umbræ.

The symbol for zero, 0, is the symbol of an ordinary. The product 0α, written without the dot, is the umbra of the class each of whose elements is the ordinary zero, $0\alpha_i$ ($i = 0, 1, 2, \cdots$).

From what precedes, *viz.*, from the special case of $(a\alpha + b\beta + \cdots)^m$ in which $a = 1$, $b = -1$, and each succeeding coefficient $= 0$, it follows that $\alpha - \alpha, \beta - \beta, \gamma - \gamma, \cdots$ are distinct, and we have shown how their values are to be calculated. With the distinction that the difference of two identical umbræ is never to be replaced by the symbol 0, subtraction in U is subject to all the laws of subtraction in A. For example,

$$\alpha - \alpha + \beta = \alpha + \beta - \alpha.$$

Combining results, we have now defined

$$n \cdot a\alpha + m \cdot b\beta + \cdots + s \cdot c\gamma$$

for $n, m, \cdots, s$ integers $\gtreqless 0$; $a, b, \cdots, c$ ordinaries; $\alpha, \beta, \cdots, \gamma$ umbræ. Note that each of $n \cdot a\alpha$, $m \cdot b\beta$, $\cdots$, $s \cdot c\gamma$ is a single umbra.

5. Multiplication in U, with the exception of multiplication by $\alpha - \alpha$, is as in A. For, if we represent the umbral product of the umbræ α, β by $\alpha\beta$, and define $\alpha\beta$ to be the umbra of the class $\alpha_n\beta_n$ ($n = 0, 1, 2, \cdots$), then $\alpha\beta = \beta\alpha$ and

$$(\alpha + \beta)(\gamma + \delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta = (\gamma + \delta)(\alpha + \beta),$$

as can be verified at once from the definition of equality. Again, if all the letters

are as at the end of § 4, and accented letters have similar meanings, it is evident that

$$(n\cdot a\alpha + m\cdot b\beta)(n'\cdot a'\alpha' + m'\cdot b'\beta')$$
$$= (n\cdot a\alpha)(n'\cdot a'\alpha') + (n\cdot a\alpha)(m'\cdot b'\beta') + (m\cdot b\beta)(n'\cdot a'\alpha') + (m\cdot b\beta)(m'\cdot b'\beta'),$$

where the separate terms $(n\cdot a\alpha)$, etc., are enclosed in ( ) to avoid a possible confusion, thus

$$(n\cdot a\alpha)(n'\cdot a'\alpha') \mp nn'\cdot aa'\alpha\alpha'.$$

That is, $n\cdot a\alpha$, or any similar term, is treated as the *single* umbra which it is.

The treatment of the excepted case, multiplication by $\alpha - \alpha$, is referred by means of

$$(\alpha - \alpha)\beta = \alpha\beta - \alpha\beta$$

to § 4, so that this case also follows the same laws as above. The distinction here is that $\alpha\beta - \alpha\beta$ is not zero, as it would be in A.

6. It remains to say a word about the way in which U is applied. Assume for a moment that between the umbræ $\alpha, \beta, \cdots, \gamma$ and δ we have found the relation

$$\delta = \alpha + \beta + \cdots + \gamma,$$

and let x be a parametric ordinary, *viz.*, a parameter whose range of values is the range of ordinaries, such that $x^n \mp \xi_n$ $(n = 0, 1, 2, \cdots)$ for $\xi = \alpha, \beta, \cdots, \gamma, \delta$. Let $f(x)$ be a polynomial in x. Then, from the assumed umbral equality, it follows that

$$f(x + \delta) = f(x + \alpha + \beta + \cdots + \gamma),$$

and there is an extension of this, which we need not discuss here, to the case in which $f(x)$ is any function possessing a power series expansion in x.

To prove the polynomial relation it suffices to observe that for all integers $n \geqq 0$ we have

$$(x + \delta)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}\delta^r,$$
$$= \sum_{r=0}^{n} \binom{n}{r} x^{n-r}(\alpha + \beta + \cdots + \gamma)^{n-r},$$

by means of the umbral equality, and hence

$$(x + \delta)^n = (x + \alpha + \beta + \cdots + \gamma)^n,$$

which might have been written down immediately from the umbral equality, as follows. We can consider x as an umbra, *viz.*, x_n is here $\equiv x^n$, since x itself is an ordinary, and x, quâ umbra, is the umbra of the class of ordinaries x^n $(n = 0, 1,$

2, $\cdots$). Since by hypothesis the umbræ $x, \alpha, \beta, \cdots, \gamma, \delta$ are distinct, it follows from $\delta = \alpha + \beta + \cdots + \gamma$ by the properties of addition of umbræ that $x + \delta = x + \alpha + \beta + \cdots + \gamma$, and this is equivalent to the relation stated.

Hence from any umbral equality may be inferred immediately an equality between polynomials (or functions, as mentioned) whose arguments contain, in addition to the umbræ concerned, a parameter x. By giving to x special values, and by choosing for the polynomial (or function) particular forms, we obtain an unlimited number of algebraic relations between the $\alpha_n, \beta_n, \cdots, \gamma_n, \delta_n$ for different n's.

To obtain the initial umbral equality we proceed from the *generators* of $\alpha_n, \beta_n, \cdots$. The generator of α_n is by definition $e^{\alpha y}$, where y is a parametric ordinary, *viz.*,

$$e^{\alpha y} \equiv \alpha_0 + \alpha_1 y + \alpha_2 \frac{y^2}{2!} + \cdots.$$

Suppose that a certain function $F(y)$ of y has a MacLaurin expansion, so that we can write

$$F(y) = e^{\alpha y},$$

where α_n is the coefficient of $y^n/n!$ in the MacLaurin series for $F(y)$. Suppose further that in the same way we have

$$G(y) = e^{\beta y}, \ \cdots, \qquad H(y) = e^{\gamma y}, \qquad K(y) = e^{\delta y},$$

and that there exists the (ordinary) identity in y,

$$K(y) = F(y)G(y) \cdots H(y).$$

Then it follows that

$$e^{\delta y} = e^{(\alpha+\beta+\cdots+\gamma)y},$$

whence

$$\delta = \alpha + \beta + \cdots + \gamma.$$

The actual, fruitful process is somewhat different from this. We start from any ordinary function of y, and resolve it in a given domain into its irreducible factors. The function and each of its factors are then expanded and the expansion written in exponential umbral form. By combining the factors in various ways different umbral equalities between members of a set of related functions are obtained as above outlined.

To get interesting results, we proceed as follows. Let

$$I(a, x, y, \cdots, z) = 0$$

be an identity between ordinaries $a, x, y, \cdots, z$. Replace $a, x, y, \cdots, z$ by θ, $e^{px\theta}, e^{qy\theta}, \cdots, e^{rz\theta}$ respectively, where $\theta, p, q, \cdots, r$ are ordinaries. Call the

transformed identity I'. It is equivalent to an umbral identity

$$\alpha = 0,$$

which is obtained by putting the left-hand member of I' into its generator form $e^{\alpha\theta}$. From then on the procedure is as previously sketched. In this way we use the simplest ordinary identities as the points of departure for reading off general umbral polynomial or functional relations between classes of numbers or functions. A powerful feature of this isomorphism between A, U is its great suggestiveness as to new numbers and functions worthy of investigation.

We have omitted umbral division α/β, because its definition and discussion appear to be of but slight utility.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. PI AND THE FACTORS OF $x^2 + 1$.

BY A. A. BENNETT, University of Texas.

The three term Diophantine arccotangent relation

$$\operatorname{arccot} x = \operatorname{arccot} y + \operatorname{arccot} z \tag{1}$$

is of importance[1] in the computation of π. On the other hand the problem of determining integers to satisfy (1) is equivalent to the problem of finding the numerical factors of $x^2 + 1$. Indeed (1) may be written in the form

$$\operatorname{arccot} x = \operatorname{arccot} (x + u) + \operatorname{arccot} (x + v), \qquad uv = x^2 + 1, \tag{2}$$

as may be seen from the equation $z = (xy + 1)/(y - x)$, to which (1) may be reduced, if y is replaced by $x + u$.

A table of the values of $x^2 + 1$ may readily be constructed, for successive integral values of x, and these values of $x^2 + 1$ may be tested for prime and composite factors in the usual ways. What is of particular interest is that, as often happens, such a table can be checked with great ease although at first sight it might be natural to suppose that a composite number with only large prime factors might be mistaken for a prime by some slip in computation. We have in the first place the following theorem.

A necessary and sufficient condition that a given integer, u, be a factor of

[1] *Cf.* Hobson, *Plane Trigonometry*, p. 310.

an integer of the form $x^2 + 1$ is that u contains no factor of the form $4k + 3$, and that u does not contain 4 as a factor. However u may be even.

The proof of this is merely an exercise in the first principles of quadratic residues. The hypothesis may be stated in familiar technical language in the form $x^2 \equiv -1$, mod u. By the usual rules, we see that since -1 is not congruent to 1, mod 4, u cannot contain 2 to a power higher than the first, and -1 must be a quadratic residue of each prime factor of u. For -1 to be a quadratic residue of an odd prime, p, it is necessary and sufficient that $(-1)^{(p-1)/2} \equiv 1$, mod p. Thus p cannot be of the form $4k + 3$ but must be of the form $4k + 1$. Incidentally, any product of such factors is again of this same form. The general theory shows that these conditions are sufficient as well as necessary.

In the second place we have the following as a consequence of the general propositions of elementary number theory, namely, the assurance that for any given odd prime of the form $4k + 1$ there will be two and only two distinct values of r, such that $0 < r < u$, while $r^2 + 1$ is divisible by u. The actual determination of these values of r for a given u may be effected without much difficulty for moderate values of u. It is perhaps best explained by reference to an example. Take for instance the prime, $853 = 4(213) + 1$, as u. Express u as the sum of two squares, which by the general theory is always uniquely possible for u of this form. This is not difficult for moderate values of u by the mere use of a table of squares, trying successively in decreasing order the squares which are less than u. Thus 853 is quickly found to be expressible as $23^2 + 18^2$. Use the smaller of these two numbers (not squared) as partial divisor, writing $23/18$ as $1 + (5/18)$. Then as a congruence we have $18^2(1 + [1 + (5/18)]^2) \equiv 0$, mod 853, from which the external factor, 18^2, may be dropped. The problem now is to express $5/18$ as congruent to an integer with respect to the given modulus. Now 853 is itself congruent to zero, that is, $18[47 + (7/18)]$ is congruent to zero, and hence $7/18 \equiv -47$, mod 853. Hence trying successive multiples of $7/18$, we find that $49/18 = 3 - (5/18) \equiv -7.47$, mod 853, or $1 + (5/18) \equiv 4 + 7.47 = 333$, mod 853. The two values of r may be thought of as differing only in sign, or if positive values are desired, they are such that their sum is the given u. Here the values are 333, and $853 - 333 = 520$. As a check we have $333^2 = 110889$, and $333^2 + 1 = 110890 = 853.130$. Similarly $520^2 + 1 = 270401 = 853.317$.

We have then the following facts to be used in checking a table of the factors of $x^2 + 1$,

(i) Each of the numbers 1, 2, 5, 10, 13, 17, 25, 26, 29, $\cdots$ is a factor of some $x^2 + 1$, and if one of these, say u, is a factor of a given $x^2 + 1$, it is a factor of each successive $(x + nu)^2 + 1$. The possible values of x smaller than u are also capable of fairly simple direct computation, as has been explained for the case in which u is prime.

(ii) Each factor of $x^2 + 1$ is in this sequence 1, 2, 5, 10, 13, $\cdots$, which consists of those integers which contain neither 4 nor any factor of the form $4k + 3$.

(iii) Whenever a u is a factor of an $x^2 + 1$, for an x less than u, another

factor, v, is obtained, where v is greater than x, which in turn serves as a u for later expressions.

(iv) Each u appears first as a factor of x^2+1, for an x less than u and indeed for at least two distinct such cases. It can be overlooked only if the other factor, say v, is ignored, and hence for v less than x, if for some previous cases, namely, $x-v$, $x-2v$, $\cdots$, this same factor v is ignored, and so on in infinite descent.

Thus the table when once started may be continued largely by its own momentum, with next to no computation, and may be rechecked in several ways at any stage.

The following table gives the values of x, u, v, for $|x| \leq 40$. In each case the smaller of the factors u, v is placed over the larger, and positive signs only are given.[1]

1.	$\frac{1}{2}$			
2.	$\frac{1}{5}$			
3.	$\frac{1}{10}$	$\frac{2}{5}$		
4.	$\frac{1}{17}$			
5.	$\frac{1}{26}$	$\frac{2}{13}$		
6.	$\frac{1}{37}$			
7.	$\frac{1}{50}$	$\frac{2}{25}$	$\frac{5}{10}$	
8.	$\frac{1}{65}$	$\frac{5}{13}$		
9.	$\frac{1}{82}$	$\frac{2}{41}$		
10.	$\frac{1}{101}$			
11.	$\frac{1}{122}$	$\frac{2}{61}$		
12.	$\frac{1}{145}$	$\frac{5}{29}$		
13.	$\frac{1}{170}$	$\frac{2}{85}$	$\frac{5}{34}$	$\frac{10}{17}$
14.	$\frac{1}{197}$			
15.	$\frac{1}{226}$	$\frac{2}{113}$		
16.	$\frac{1}{257}$			
17.	$\frac{1}{290}$	$\frac{2}{145}$	$\frac{5}{58}$	$\frac{10}{29}$
18.	$\frac{1}{325}$	$\frac{5}{65}$	$\frac{13}{25}$	
19.	$\frac{1}{362}$	$\frac{2}{181}$		
20.	$\frac{1}{401}$			
21.	$\frac{1}{442}$	$\frac{2}{221}$	$\frac{13}{34}$	$\frac{17}{26}$
22.	$\frac{1}{485}$	$\frac{5}{97}$		
23.	$\frac{1}{530}$	$\frac{2}{265}$	$\frac{5}{106}$	$\frac{10}{53}$
24.	$\frac{1}{577}$			
25.	$\frac{1}{626}$	$\frac{2}{313}$		
26.	$\frac{1}{677}$			
27.	$\frac{1}{730}$	$\frac{2}{365}$	$\frac{5}{146}$	$\frac{10}{73}$
28.	$\frac{1}{785}$	$\frac{5}{157}$		
29.	$\frac{1}{842}$	$\frac{2}{421}$		
30.	$\frac{1}{901}$	$\frac{17}{53}$		
31.	$\frac{1}{962}$	$\frac{2}{481}$	$\frac{13}{74}$	$\frac{26}{37}$
32.	$\frac{1}{1025}$	$\frac{5}{205}$	$\frac{25}{41}$	
33.	$\frac{1}{1090}$	$\frac{2}{545}$	$\frac{5}{218}$	$\frac{10}{109}$
34.	$\frac{1}{1157}$	$\frac{13}{89}$		
35.	$\frac{1}{1226}$	$\frac{2}{613}$		
36.	$\frac{1}{1297}$			
37.	$\frac{1}{1370}$	$\frac{2}{685}$	$\frac{5}{274}$	$\frac{10}{137}$
38.	$\frac{1}{1445}$	$\frac{5}{289}$	$\frac{17}{85}$	
39.	$\frac{1}{1522}$	$\frac{2}{761}$		
40.	$\frac{1}{1601}$			

II. On the Use of the Calculating Machine for Cube and Fifth Roots.

By Derrick Henry Lehmer, Berkeley, Calif.

The extraction of roots of higher degree than the second by means of the calculating machine is difficult for any but the most experienced computers and is at best a slow process. The subjoined table has been prepared to make this process both rapid and accurate for cube and fifth roots. It covers the range of numbers from *1* to *10,000,000,000* and presents a remarkable condensation when compared with other available tables of much smaller range.

It will be seen that the table has for arguments the first nine integers together with numbers of the form $1.0^n p$. Its use therefore depends on the decomposition

[1] To illustrate its application to (1), take 13, and the corresponding factors 5 and 34 of 13^2+1. Then we may infer that

$$\text{arccot } 13 = \text{arccot } (13+5) + \text{arccot } (13+34),$$

$$\text{arccot } 13 = \text{arccot } (13-5) + \text{arccot } (13-34).$$

of a number into factors of this particular type. This decomposition, which has been made use of for the computation of logarithms,[1] may be effected in the following manner which is well adapted to the computing machine:

If the given number lies between one and ten, divide first by the greatest integer in it. Divide the quotient by the greatest number of the form $1.0^n p$ that is less than the quotient. Proceed in the same way with this quotient and continue the process until a quotient is reached that is sensibly unity. If the given number does not lie between one and ten, it can be made to do so by division by an appropriate power of ten.

In practice one finds that only half the above process needs to be carried out, the last factors being obtained by inspection.

As an illustration let it be required to decompose the number 432.7699836. The number 4.327699836 is first divided by 4. The quotient, 1.081924959, is then divided by the number 1.08. The quotient, $1.0^2 1782369$, is in turn divided by $1.0^2 1$. The quotient, $1.0^3 781587$, is now divided by $1.0^3 7$. The quotient, $1.0^4 81530$, might now be divided by the number $1.0^4 8$ and the process continued, but the remaining factors are seen without division to be $1.0^5 1$, $1.0^6 5$, and $1.0^7 3$. The given number, 432.7699836, is thus the product of the following numbers: 10^2, 4, 1.08, $1.0^2 1$, $1.0^3 7$, $1.0^4 8$, $1.0^5 1$, $1.0^6 5$, $1.0^7 3$.

To apply this decomposition to the extraction of roots, we observe that the root of any given number will be the product of the corresponding roots of the factors. The subjoined table gives the cube and fifth roots of all numbers of the above form to ten significant figures. The following examples show the method of using the table.

Example 1. Required the cube root of 34625.00589. Decomposing this number according to the method indicated above, we have

$$34625.00589 = 10^4 \times 3 \times 1.1 \times 1.04 \times 1.0^2 8 \times 1.0^3 8 \times 1.0^4 8 \times 1.0^7 1 \times 1.0^8 3.$$

Multiplying together the values found opposite these factors in the cube root column, we get the required root, 32.59342142.

Example 2. Required the fifth root of .0005734267268. We have, decomposing the number 5.734267268,

$$5.734267268 = 5 \times 1.1 \times 1.04 \times 1.0^2 2 \times 1.0^3 4 \times 1.0^4 9 \times 1.0^5 3 \times 1.0^6 2 \times 1.0^7 5 \times 1.0^8 3.$$

The product of the numbers opposite these factors in the fifth root column is 1.418063200 which is the fifth root of 5.734267268. To get the root of the required number we multiply this result by $\sqrt[5]{10}/10$. The final result is .2247478713.

After so many successive multiplications it is quite possible that the last digit may be erroneous. If ten-figure accuracy is required, the following correction can easily be made: For a cube root, $c = e/(3r^2)$, where e is the difference between the given number and the cube of the tentative root r. The corresponding correction for the fifth root is $c = e/(5r^4)$. Thus in example 2 the error in the last digit is as much as 4. Applying the correction indicated, the true value is .2247478709.

[1] See last column of article on *Logarithms* by J. W. L. Glaisher in the *Encyc. Brit.*

TABLE OF CUBE AND FIFTH ROOTS.

n	$\sqrt[3]{n}$	$\sqrt[5]{n}$	n	$\sqrt[3]{n}$	$\sqrt[5]{n}$
1	1.0000 00000	1.0000 00000	$1.0^{4}1$	$1.0^{4}03333$	$1.0^{4}02000$
2	1.2599 21050	1.1486 98355	$1.0^{4}2$	$1.0^{4}06667$	$1.0^{4}04000$
3	1.4422 49570	1.2457 30940	$1.0^{4}3$	$1.0^{4}10000$	$1.0^{4}06000$
4	1.5874 01052	1.3195 07911	$1.0^{4}4$	$1.0^{4}13333$	$1.0^{4}08000$
5	1.7099 75947	1.3797 29662	$1.0^{4}5$	$1.0^{4}16666$	$1.0^{4}10000$
6	1.8171 20593	1.4309 69081	$1.0^{4}6$	$1.0^{4}20000$	$1.0^{4}12000$
7	1.9129 31183	1.4757 73162	$1.0^{4}7$	$1.0^{4}23333$	$1.0^{4}14000$
8	2.0000 00000	1.5157 16567	$1.0^{4}8$	$1.0^{4}26666$	$1.0^{4}15999$
9	2.0800 83823	1.5518 45574	$1.0^{4}9$	$1.0^{4}29999$	$1.0^{4}17999$
1.1	1.0322 80115	1.0192 44877	$1.0^{5}1$	$1.0^{5}0333$	$1.0^{5}0200$
1.2	1.0626 58569	1.0371 37289	$1.0^{5}2$	$1.0^{5}0667$	$1.0^{5}0400$
1.3	1.0913 92883	1.0538 73952	$1.0^{5}3$	$1.0^{5}1000$	$1.0^{5}0600$
1.4	1.1186 88942	1.0696 10376	$1.0^{5}4$	$1.0^{5}1333$	$1.0^{5}0800$
1.5	1.1447 14243	1.0844 71771	$1.0^{5}5$	$1.0^{5}1667$	$1.0^{5}1000$
1.6	1.1696 07095	1.0985 60543	$1.0^{5}6$	$1.0^{5}2000$	$1.0^{5}1200$
1.7	1.1934 83192	1.1119 61586	$1.0^{5}7$	$1.0^{5}2333$	$1.0^{5}1400$
1.8	1.2164 40399	1.1247 46113	$1.0^{5}8$	$1.0^{5}2667$	$1.0^{5}1600$
1.9	1.2385 62330	1.1369 74489	$1.0^{5}9$	$1.0^{5}3000$	$1.0^{5}1800$
1.01	1.0033 22284	1.0019 92048	$1.0^{6}1$	$1.0^{6}033$	$1.0^{6}020$
1.02	1.0066 27709	1.0039 68378	$1.0^{6}2$	$1.0^{6}067$	$1.0^{6}040$
1.03	1.0099 01634	1.0059 29269	$1.0^{6}3$	$1.0^{6}100$	$1.0^{6}060$
1.04	1.0131 59404	1.0078 74989	$1.0^{6}4$	$1.0^{6}133$	$1.0^{6}080$
1.05	1.0163 96357	1.0098 05798	$1.0^{6}5$	$1.0^{6}167$	$1.0^{6}100$
1.06	1.0196 12822	1.0117 21952	$1.0^{6}6$	$1.0^{6}200$	$1.0^{6}120$
1.07	1.0228 09122	1.0136 23698	$1.0^{6}7$	$1.0^{6}233$	$1.0^{6}140$
1.08	1.0259 85568	1.0155 11278	$1.0^{6}8$	$1.0^{6}267$	$1.0^{6}160$
1.09	1.0291 42467	1.0173 84928	$1.0^{6}9$	$1.0^{6}300$	$1.0^{6}180$
$1.0^{2}1$	1.0003 33222	1.0001 99920	$1.0^{7}1$	$1.0^{7}03$	$1.0^{7}02$
$1.0^{2}2$	1.0006 66223	1.0003 99680	$1.0^{7}2$	$1.0^{7}07$	$1.0^{7}04$
$1.0^{2}3$	1.0009 99002	1.0005 99281	$1.0^{7}3$	$1.0^{7}10$	$1.0^{7}06$
$1.0^{2}4$	1.0013 31559	1.0007 98723	$1.0^{7}4$	$1.0^{7}13$	$1.0^{7}08$
$1.0^{2}5$	1.0016 63897	1.0009 98006	$1.0^{7}5$	$1.0^{7}17$	$1.0^{7}10$
$1.0^{2}6$	1.0019 96013	1.0011 97130	$1.0^{7}6$	$1.0^{7}20$	$1.0^{7}12$
$1.0^{2}7$	1.0023 27910	1.0013 96097	$1.0^{7}7$	$1.0^{7}23$	$1.0^{7}14$
$1.0^{2}8$	1.0026 59587	1.0015 94905	$1.0^{7}8$	$1.0^{7}27$	$1.0^{7}16$
$1.0^{2}9$	1.0029 91045	1.0017 93555	$1.0^{7}9$	$1.0^{7}30$	$1.0^{7}18$
$1.0^{3}1$	1.0000 33332	1.0000 19999	$1.0^{8}1$	$1.0^{8}0$	$1.0^{8}0$
$1.0^{3}2$	1.0000 66662	1.0000 39997	$1.0^{8}2$	$1.0^{8}1$	$1.0^{8}0$
$1.0^{3}3$	1.0000 99990	1.0000 59993	$1.0^{8}3$	$1.0^{8}1$	$1.0^{8}1$
$1.0^{3}4$	1.0001 33316	1.0000 79987	$1.0^{8}4$	$1.0^{8}1$	$1.0^{8}1$
$1.0^{3}5$	1.0001 66639	1.0000 99980	$1.0^{8}5$	$1.0^{8}2$	$1.0^{8}1$
$1.0^{3}6$	1.0001 99960	1.0001 19971	$1.0^{8}6$	$1.0^{8}2$	$1.0^{8}1$
$1.0^{3}7$	1.0002 33279	1.0001 39961	$1.0^{8}7$	$1.0^{8}2$	$1.0^{8}1$
$1.0^{3}8$	1.0002 66596	1.0001 59949	$1.0^{8}8$	$1.0^{8}3$	$1.0^{8}2$
$1.0^{3}9$	1.0002 99910	1.0001 79935	$1.0^{8}9$	$1.0^{8}3$	$1.0^{8}2$

$\sqrt[3]{10} = 2.1544\ 34690$

$\sqrt[3]{100} = 4.6415\ 88834$

$\sqrt[5]{10} = 1.5848\ 93193$

$\sqrt[5]{100} = 2.5118\ 86432$

$\sqrt[5]{1000} = 3.9810\ 71705$

$\sqrt[5]{10000} = 6.3095\ 73445$

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

THE FIRST CARUS MONOGRAPH.

Calculus of Variations. By G. A. BLISS. Chicago, The Open Court Publishing Company, 1925. xiii + 189 pages. Price $2.00 post-paid to non-members of the Association.

This review might very properly begin with the quotation of the first paragraph of the author's preface which explains the origin and purpose of these monographs. "This book is the first of a series of monographs on mathematical subjects which are to be published under the auspices of the Mathematical Association of America and whose publication has been made possible by a very generous gift to the Association by Mrs. Mary Hegeler Carus as trustee of the Edward C. Hegeler Trust Fund. The purpose of the monographs is to make the essential features of various mathematical theories accessible and attractive to as many persons as possible who have an interest in mathematics but who may not be specialists in the particular theory presented, a purpose which Mrs. Carus has very appropriately described to be the 'diffusion of mathematical and formal thought as contributory to exact knowledge and clear thinking not only for mathematicians and teachers of mathematics but also for other scientists and the public at large.'"

The principal aim of this review is to give an appraisal of this work as to the extent to which it has fulfilled the purposes of these monographs, leaving to others to give a detailed description and criticism of the subjects discussed. I might say at the outset that the publication committee could have made no wiser choice for the subject of the first monograph than that of the calculus of variations. The beautiful geometrical and mechanical properties of curves and surfaces which unfold themselves so naturally in this subject are sure to arrest the attention of the reader who is even only mildly interested in mathematics. Coupled with this inherent beauty of the subject, we have here an exposition by a man whose ability both as a teacher and scholar is of such outstanding character as to insure the success of any mathematical work, let alone a subject with which he has such an intimate acquaintance and in which he has such a deep interest. Those of us who have always regarded mathematics merely as a science find in these pages an artistic structure and a beauty of technique that lead us to think that mathematics in this country is at last taking on the form of an art also.

The book opens with an introductory chapter giving a statement and historical sketch of some of the typical problems of the calculus of variations. This chapter is so clear and vivid that it must surely arouse the interest and curiosity of the reader. For those who have no acquaintance with this subject it may not

be amiss at this place to give a brief statement of one of the most interesting of these problems, viz., the brachistochrone problem which was first proposed by John Bernoulli in 1696. For some reason a rivalry had sprung up between him and his older brother James and this problem was proposed as a sort of a challenge. It is required to pick out of all the possible paths joining two points in a vertical plane that one down which a particle will fall in the shortest time. It is clear that this problem is one which is quite distinct from the maxima and minima problems solved in the differential calculus. In fact we have here to find not merely a value, or values, of a variable which will give an assigned function the largest or smallest value, but a whole curve or function which, when substituted in the integrand of a certain definite integral, will give the latter a value smaller than would be obtained by the substitution of any other function. The brothers both solved the problem in 1697; and they found much to their surprise that the curve down which a particle will fall in the least time is the same curve which will be generated by a particle on the rim of a wheel as the wheel rolls along a straight line, and more surprising still it is the same curve on which a particle starting at rest will fall to the lowest point in the same time no matter at what point of the curve the particle is started. In other words, a pendulum constrained to move along a cycloid, as this curve is called, would have a period independent of the arc of swing, a property not enjoyed by the so-called simple pendulum. We can feel with John Bernoulli when he says (see page 54), "With justice we admire Huygens because he first discovered that a heavy particle falls on a cycloid in the same time always, no matter what the starting point may be. But you will be petrified with astonishment when I say that exactly this same cycloid, the tautochrone of Huygens, is the brachistochrone which we are seeking."

In the next three chapters this and two other problems are taken up in great detail with a view to explaining and illustrating the modern methods of the calculus of variations. These modern methods of studying this subject are here presented for the first time in a form which it is possible for the intelligent layman with a thorough training in advanced calculus to understand. I do not mean to imply that the reader will always find it easy to follow every statement at a first glance. In fact I doubt if any one but a specialist could read everything in the book that way. But I am sure that it is possible for a reader with a good training in elementary mathematics to verify every statement made in this book, provided he has sufficient patience and concentration. If we recall that mathematicians had been working on this subject for over 300 years before it was finally put on a rigorous foundation by Weierstrass, we can see why a beginner should not expect to understand the calculus of variations in all its aspects at a first reading. In fact, I would suggest to the beginner to omit on a first reading all the difficult proofs in the book and come back to them only after he has seen the underlying general ideas of the subject. This, by the way, is I think the best and most efficient method of reading all mathematics, even technical articles. But, in spite of the inherent difficulty of the subject, Professor Bliss

has given us an exposition which I think will be considered a model for many years to come. Whenever he needs to use some result from an advanced portion of mathematics, as for example the theory of implicit functions, he either substitutes for it a clear geometric proof (see for example the proof that a cycloid can always be drawn through two points, page 56) or he gives a proof using nothing more than what one should know after a thorough course in calculus.

The book ends with a chapter on the general theory and we have to marvel that any one can in the space of fifty pages give such a clear and complete treatment of what is known as the *simplest problem* of the calculus of variations. Yet we actually find that the necessary conditions as well as the sufficient conditions for a minimum or maximum are stated and proved with all the rigor that one could desire. This chapter should be excellent reading for one who has already had some acquaintance with the calculus of variations, and even the specialist will find here a form which is not lacking in novelty. The chapter closes with an interesting historical account of the whole subject.

It may be seen from this brief description that this is an interesting and useful book for a large group of readers whose mathematical equipment varies all the way from a course in the differential and integral calculus to one in the calculus of variations. In fact, I should think it would make an excellent introduction for a course in this subject if not as a text, at least for collateral reading.

I. A. Barnett.

THREE BOOKS ON INFINITE SERIES.

1. *Theorie und Anwendungen der unendlichen Reihen.* By K. Knopp. Berlin, Julius Springer, 1922. 474 pages. Price 28 gold marks.

This is a text on fundamental theory. The subject matter treated and results obtained although generally well known are put forward here in a particularly clear and accessible manner. There is a treatment of the theory of aggregates and in particular of the number system. The treatment of convergence and divergence of sequences and series in general is good. A large number of theorems are clearly stated and proved. The comments of the author are scholarly. At times there seems to be quite needless repetition as many theorems are proved which appear as special cases of more general theorems proved later, and several statements of the same theorem will frequently be given where the change is merely a matter of wording. Special theories such as Dirichlet Series, summation of series, development of functions in series, etc., etc., are hurriedly and inadequately treated. Had they been treated with the thoroughness of the convergence and divergence theory, we should certainly have a very fine work on series. However, taken as a whole this is probably the best book on infinite series on the market at present.

2. *Elements of the Theory of Infinite Processes.* By L. L. Smail. New York, McGraw-Hill Book Co., 1923. 339 pages. Price $3.50.

This book is not unlike the preceding. It is intended for a text for advanced undergraduates or first year graduate students at an American university. As

the title indicates, the author does not limit himself to series but studies infinite products, continued fractions, integrals and determinants also. The whole is preceded by a study of the theory of aggregates. Theorems are well stated and proved. Infinite series naturally are the chief concern of the author. However, special types are not treated with sufficient fullness to serve as more than the merest introduction. The book is somewhat more elementary than that of Knopp and as a result, so far as the same topics are covered, less complete. But there is the advantage to the American of the English Language. As Bromwich's *Theory of Infinite Series* is now practically unobtainable, it fills a decided want and should be regarded as a worthwhile addition to American mathematical literature.

3. *An Introduction to the Operations with Series.* By I. J. SCHWATT. Philadelphia, Press of the University of Pennsylvania, 1924. 287 pages. $5.00 net.

This book is unlike any other thing known to the reviewer. It partakes as much of the nature of a table as of a text or treatise and whereas it is certainly a valuable book to have it is not a book that any one is likely to read in completeness. The obtaining of closed formulas for the sums of both finite and infinite series seems to be the principal end of the author although in many instances the reverse problem is handled. Formulas are obtained in large number and with but little comment. It is believed that many of his results are new and, although no effort at verification has been made, this seems extremely likely judging from their number and complexity. For reference the book is well worth having. It is a witness to the ingenuity and algebraic skill of its author. The book is handled by D. Appleton & Co., New York City.

TOMLINSON FORT.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

AMERICAN JOURNAL OF MATHEMATICS, volume 47, no. 2, April, 1925: "Some properties of the exponential mean" by J. Dale, 71–90; "The eliminant of a net of curves" by F. Morley, 91–97; "On an equation of planar motion" by F. Morley, 98–100; "An extension of the problem of the elastic bar" by H. T. Davis, 101–120; "On the Sylow subgroups of the symmetric and alternating groups" by L. Weisner, 121–124; "On certain theorems regarding summable series and their application to the double and triple Fourier's series" by G. M. Merriman, 125–139; "On the power characters of units in a cyclotomic field" by H. S. Vandiver, 140–147.

ANNALS OF MATHEMATICS, volume 26, nos. 1 and 2, September–December, 1924: "Contact transformations linear in x, y, z; applications to equilong transformations" by B. H. Brown, 1–7; "On the trigonometric representation of an ill-defined function" by D. Jackson, 8–20; "An irregular boundary value and expression problem" by L. E. Ward, 21–36; "Maximal cuspidal curves" by T. R. Hollcroft, 37–46; "Plane cubics associated with the quadrangle-quadrilateral configuration" by B. M. Turner, 47–58; "On Pellet's theorem concerning the roots of a polynomial" by J. L. Walsh, 59–64; "The setting of a proposition" by P. J. Daniell, 65–78; "Theta functions and arithmetic" by E. T. Bell, 79–87; "A new type of criteria for the first case of Fermat's last theorem" by H. S. Vandiver, 88–94; "Algebraic functions and their divisors"

by G. A. Bliss, 95–124; "An extension of the definition of the Green's function in one dimension" by W. M. Whyburn, 125–130; "Real representations of analytic complex curves" by W. C. Graustein, 131–143; "Note on Dirichlet series with complex exponents" by J. F. Ritt, 144; "On the order of an analytic function at a singular point" by M. H. Stone, 145–154; "Representations of integers in certain binary, ternary, quaternary, and quinary quadratic forms and allied class number relations" by E. T. Bell, 155–164.

Bulletin of the American Mathematical Society, volume 31, nos. 1–2, January–February, 1925: "A theorem on simple algebras" by J. H. M. Wedderburn, 11–13; "Note on a class of harmonic functions" by G. C. Evans, 14–16; "Three theorems on normal orthogonal sets" by M. H. Stone, 17–20; "On the complete independence of the functional equations of involution" by C. C. MacDuffee, 21–26; "The frequency law of a function of one variable" by E. L. Dodd, 27–31; "On the accessibility of an arc from its complement in space of three dimensions" by C. Kuratowski, 32; "On sets of three consecutive integers which are quadratic residues of primes" by H. S. Vandiver, 33–38; "Normal congruences of curves in Riemann space" by H. Levy, 39–41; "Limits for actual double points of space curves" by T. R. Hollcroft, 42–55; "Some mean-value theorems connected with Cotes' method of mechanical quadrature" by D. V. Widder, 56–62; "The geometry of frequency functions" by D. Jackson, 63–73. Nos. 3–4, March–April, 1925: "Remarks on the foundations of geometry" by O. Veblen, 121–141; "Ricci's coefficients of rotation" by H. Levy, 142–144; "Sur les valeurs asymptotiques des coefficients de Cotes" by J. Ouspensky, 145–156; "Functions with essential singularity" by P. Franklin, 157–162; "The calculus of variations" by L. Tonelli, 163–172.

Journal of Mathematics and Physics, M. I. T., volume 4, no. 3, May, 1925: "The Weierstrass approximation theorem" by P. Franklin, 148–152; "The solution of a difference equation by trigonometric integrals" by N. Wiener, 153–163; "Note on a method of evaluating the complex roots of sixth and higher order equations" by L. F. Woodruff, 164–166; "Minimal varieties of two and three dimensions whose element of arc is a perfect square" by C. L. E. Moore, 167–178; "The multiple complement of one or more polyadics" by F. L. Hitchcock and L. H. Rice, 179–187; "Note on De Sitter's universe" by G. Lemaitre, 188–192.

Journal für die reine und angewandte Mathematik, volume 154, no. 2, January, 1925: "The number e in $k(p)$" by G. E. Wahlin, 110–113.

L'Enseignement Mathematique, volume 24, nos. 1–2–3, March, 1925: "Histoire de cinq concepts fondamentaux des mathematiques" by G. A. Miller, 59–69.

Mathematische Annalen, volume 94, nos. 1–2, April, 1925: "Einstein spaces which are mapped conformally on each other" by H. W. Brinkmann, 119–145.

Mathematische Zeitschrift, volume 23, nos. 1–2, March, 1925: "Some problems in 'Partitio numerorum' (VI): Further researches in Waring's problem" by G. H. Hardy and J. E. Littlewood, 1–37.

Messenger of Mathematics, volume 54, nos. 9–10, January–February, 1925: "The stability of electrons and protons" by H. Bateman, 142–149.

Philosophical Magazine, volume 49, no. 293, May, 1925: "The motion of an electric charge" by A. Bramley, 912–923.

Proceedings of the National Academy of Sciences, volume 11, no. 4, April, 1925: "On the projective and equi-projective geometries of paths" by T. Y. Thomas, 199–203; "Projective normal co-ordinates for the geometry of paths" by O. Veblen and J. M. Thomas, 204–206; "Note on the projective geometry of paths" by J. M. Thomas, 207–208. No. 5, May, 1925: "Linear connections of a space which are determined by simply transitive continuous groups" by L. P. Eisenhart, 246–249; "Note on a theorem by H. Kneser" by J. W. Alexander, 250–251; "Associated types of linear connection" by L. Ingold, 252–256; "Conformal correspondence of Riemann spaces" by J. M. Thomas, 257–259. No. 6, June, 1925: "A note on the abundance of differential combinants in a fundamental system" by O. E. Glenn, 281–283; "On the application of Borel's method to the summation of Fourier's Series" by C. N. Moore, 284–286; "The intersection of complexes of manifolds" by S. Lefschetz, 287–289; "Continuous transformations of manifolds" by S. Lefschetz, 290–291; "Laws of Reciprocity and the first case of Fermat's last theorem" by H. S. Vandiver, 292–297.

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PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3144. Proposed by JOHN BIGGERSTAFF, University of Washington.

Find the point from which the sum of the distances to two given straight lines and the distance to a given point is a minimum.

[Joseph Bertrand.]

3145. Proposed by W. J. SIDIS, New York City.

Prove that if a number of n digits, expressed in the scale of r, is divisible by any factor of $r^n - 1$, that divisibility is not altered by a cyclical permutation of the digits of the original number.

3146. Proposed by L. H. BURNS, Student, Yale University.

Show, by elementary geometrical methods (preferably suitable for use in teaching a class in elementary solid geometry), that the volume of a regular icosahedron of edge a is

$$5a^3(3 + \sqrt{5})/12,$$

and obtain a similar formula for the volume of a regular dodecahedron of edge a.

3147. Proposed by N. MILLER, Queen's University, Canada.

If space of three dimensions be divided into cubes by means of three sets of equidistant parallel planes and a closed circuit be formed of their diagonals, what is the least number of diagonals necessary in order that this broken-line curve be knotted?

2672 [1918, 74]. Proposed by E. T. BELL, Seattle, Washington.

There is an identity in z, (1) $A(z) \equiv B(z)C(z)$; *e.g.*, $A(z) = 1/(1 - k^2z^2)$; $B(z) = 1/(1 - kz)$; $C(z) = 1/(1 + kz)$; and the formal expansions $A(z) = \Sigma a(n)z^n$, $B(z) = \Sigma b(n)z^n$, $C(z) = \Sigma c(n)z^n$,

($n = 0, 1, \cdots, \infty$), when substituted in (1), give, on equating coefficients:

$$a(n) = b(n)c(0) + b(n-1)c(1) + \cdots + b(0)c(n). \tag{2}$$

If (2) is an identity in n, justify such a use of non-convergent series to obtain it (*e.g.*, for $|k| \geqq 1$ in the above). This method of finding important identities (2) has been freely used by Hermite and many others without question of its validity, and without offering independent proofs of (2).

2694 [1918, 170]. Proposed by N. P. PANDYA, Sojitra, India.

Find the locus of the centroid of a triangle, whose vertex lies on a given parabola, whose base of given length is a segment of a given straight line of unlimited length, and one of whose base angles is known.

2723 [1918, 303]. Proposed by G. Y. SOSNOW, Newark, N. J.

The feet of the perpendiculars from the intersection of the diagonals on the sides of a cyclic quadrilateral M are joined to form a second quadrilateral N. Prove that N is a quadrilateral of minimum perimeter inscribed in M.

2724 [1918, 303]. Proposed by FRANK IRWIN, University of California.

Show that there is a unique set of real values, $x_1, x_2, x_3, \cdots, x_n$, that satisfy the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_1x_2 - x_2x_3 - x_3x_4 - \cdots - x_{n-1}x_n - x_n + \frac{n}{2(n+1)} = 0.$$

SOLUTIONS.

3051 [1924, 49]. Proposed by NORMAN ANNING, University of Michigan.

Given the sequence: $u_1 = 2$, $u_2 = 8$, $u_n = 4u_{n-1} - u_{n-2}$, ($n = 3, 4, 5, \cdots$), show that

$$(\pi/12) = \sum_{n=1}^{n=\infty} \operatorname{arccot} u_n^2.$$

SOLUTION BY THE PROPOSER.

It will first be shown that

$$u_n^2 - u_{n+1}u_{n-1} = 4. \tag{1}$$

For, when n is any integer greater than 2, we have

$$\begin{aligned} u_n(4u_{n-1}) &= u_{n-1}(4u_n) \\ u_n(u_n + u_{n-2}) &= u_{n-1}(u_{n+1} + u_{n-1}) \\ u_n^2 - u_{n+1}u_{n-1} &= u_{n-1}^2 - u_nu_{n-2} \\ &= u_{n-2}^2 - u_{n-1}u_{n-3} = \cdots \\ &= u_2^2 - u_3u_1 = 64 - 60 = 4. \end{aligned}$$

Now

$$\begin{aligned} \operatorname{arccot} u_n^2 &= \operatorname{arccot} [u_n(4u_n)/4] \\ &= \operatorname{arccot} [u_n(u_{n+1} + u_{n-1})/(u_n^2 - u_{n+1}u_{n-1})], \qquad \text{by (1)}, \\ &= \operatorname{arccot} (u_{n+1}/u_n) - \operatorname{arccot} (u_n/u_{n-1}). \end{aligned} \tag{2}$$

If, in (2), we allow n to range from 2 to r and take the sum, we have

$$\operatorname{arccot} u_1^2 + \sum_{n=2}^{n=r} \operatorname{arccot} u_n^2 = \operatorname{arccot} (u_{r+1}/u_r). \tag{3}$$

As r approaches ∞, the ratio (u_{r+1}/u_r) approaches the larger root of the quadratic equation: $x^2 = 4x - 1$, namely, $2 + \sqrt{3}$. So

$$\sum_{n=1}^{n=\infty} \operatorname{arccot} u_n^2 = \operatorname{arccot} (2 + \sqrt{3}) = \pi/12.$$

3090 [1924, 353]. Proposed by H. S. UHLER, Yale University.

Show that the volume of the (smallest) segment of a sphere (radius $= c$) cut out by two mutually perpendicular planes, the distances of which from the center are a and b respectively, may be expressed by the formula

$$\frac{2}{3}c^3\cos^{-1}\left[\left(\frac{a}{\sqrt{c^2-b^2}}\right)\left(\frac{b}{\sqrt{c^2-a^2}}\right)\right]-\frac{1}{3}b(3c^2-b^2)\cos^{-1}\left(\frac{a}{\sqrt{c^2-b^2}}\right)$$
$$-\frac{1}{3}a(3c^2-a^2)\cos^{-1}\left(\frac{b}{\sqrt{c^2-a^2}}\right)+\frac{2}{3}ab\sqrt{c^2-(a^2+b^2)}.$$

[*Note.* This formula may be used as major part of an alternative solution of problem **2947** [*1922*, 29] as given by J. B. Reynolds [*1923*, 209].]

Solution by the Proposer.

If the origin is taken at the center of the sphere, the required volume is

$$v=2\int_a^{\sqrt{c^2-b^2}}dy\int_b^{\sqrt{c^2-y^2}}dz\sqrt{c^2-y^2-z^2},\qquad (a^2+b^2\leqq c^2).$$

Performing the indicated integrations, we obtain the result given in the statement of the problem.

To apply this result to problem **2947**, we replace a, b, and c by $(a-h)$, $\sqrt{r^2-a^2}$, and r respectively, double the resulting volume, and add $l\left\{a^2\cos^{-1}\left(\frac{a-h}{a}\right)-(a-h)\sqrt{h(2a-h)}\right\}$, the volume of the cylindrical segment.

3095 [1924, 401]. Proposed by GEORGE RUTLEDGE, Massachusetts Institute of Technology.

Establish the following n identities involving the binomial coefficients of any even order:

$$1!\left\{\frac{1}{1}\binom{1}{1}\binom{2n}{n-1}-\frac{1}{2}\binom{2}{1}\binom{2n}{n-2}+\frac{1}{3}\binom{3}{1}\binom{2n}{n-3}-\cdots+(-1)^{n-1}\frac{1}{n}\binom{n}{1}\binom{2n}{0}\right\}=\frac{1}{2}\binom{2n}{n}\cdot$$

$$3!\left\{\frac{1}{2}\binom{3}{3}\binom{2n}{n-2}\overset{2}{\Sigma}_{\binom{n-1}{1}}-\frac{1}{3}\binom{4}{3}\binom{2n}{n-3}\overset{3}{\Sigma}_{\binom{n-1}{1}}+\cdots+(-1)^{n-2}\frac{1}{n}\binom{n+1}{3}\right.$$
$$\left.\times\binom{2n}{0}\overset{n}{\Sigma}_{\binom{n-1}{1}}\right\}=\frac{1}{2}\binom{2n}{n}\cdot$$

$$5!\left\{\frac{1}{3}\binom{5}{5}\binom{2n}{n-3}\overset{3}{\Sigma}_{\binom{n-1}{2}}+\cdots(-1)^{n-3}\frac{1}{n}\binom{n+2}{5}\binom{2n}{0}\overset{(n)}{\Sigma}_{\binom{n-1}{2}}\right\}=\frac{1}{2}\binom{2n}{n}\cdot$$

. . . .

$$(2n-1)!\left\{\frac{1}{n}\binom{2n-1}{2n-1}\binom{2n}{0}\overset{(n)}{\Sigma}_{\binom{n-1}{i-1}}\right\}=\frac{1}{2}\binom{2n}{n}\cdot$$

The notation $\overset{(j)}{\Sigma}_{\binom{n-1}{i-1}}$ is used to indicate the sum of the $\binom{n-1}{i-1}$ products of the squared reciprocals of the first n integers excepting j, taken $i-1$ at a time.

The first of these identities is well known in the form

$$\binom{2n}{0}-\binom{2n}{1}+\binom{2n}{2}-\cdots+\binom{2n}{2n}=0$$

and the last one is obvious.

Solution by the Proposer.

It will suffice to illustrate the general case if the third identity is established. Consider the determinant

$$D = \begin{vmatrix} 1 & 2^2 & \cdots & n^2 \\ 1 & 2^4 & \cdots & n^4 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & 2^{2n} & \cdots & n^{2n} \end{vmatrix}. \tag{1}$$

Multiply the first and second rows, respectively, by the coefficients of x^2 and x^4 in the polynomial $x^2(x^2 - 1)(x^2 - 2^2)$, and add the products to the third row. The first two elements of this row then reduce to zero and the kth ($k > 2$) element becomes

$$\frac{5!}{k}\binom{k+2}{5}k^2. \tag{2}$$

By an extension of this process all the elements below the principal diagonal may be reduced to zero. The value of the determinant is then found to be $n!3!5!\cdots(2n-1)!$. If A_{ij} is the cofactor of the element j^{2i} in the transformed determinant, then

$$\frac{1}{2}\binom{2n}{n}\frac{A_{3j}}{D} = (-1)^{3+j}\binom{2n}{n-j}\frac{1}{j^2}\sum^{(j)}\binom{n-1}{2}.$$

(See Pascal, *Die Determinanten*, p. 131. Also, *Journal of Mathematics and Physics of the Mass. Inst. of Tech.*, vol. 2, 1922, p. 47.) The development of the determinant in terms of the elements of the third row, using the reduced form in which the first two elements are zero and the values of the cofactors above, gives the desired identity. The proposer hopes that some reader of the Monthly may observe a simpler method for obtaining these identities, or an evaluation of the expressions $\sum^{(j)}\binom{n-1}{i-1}$.

3108 [1925, 46]. [Corrected.] Proposed by M. KURTZ, New York City.

Prove, or disprove, that

$$a^n = n! + n(a-1)^n - \frac{n(n-1)}{1\cdot 2}(a-2)^n + \cdots + (-1)^{n+1}(a-n)^n$$

holds for all values of n. Also prove or disprove the following corollary:

$$n(1^x) - \frac{n(n-1)}{1\cdot 2}(2^x) + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(3^x) + \cdots + (-1)^{n-1}(n^x) = y,$$

where $y = 1$ when $x = 0$, $y = 0$ when $x = 1, 2, 3, \cdots (n-1)$, and $y = (-1)^{n+1}n!$ when $x = n$, both x and n being integral.

I. Solution by Louis Weisner, University of Rochester.

The first equation may be written in the form

$$(1 - e^{-D})^n a^n = n! \tag{1}$$

where $D = d/da$ and $e^{hD}f(a) = f(a+h)$. From the generalisation of Leibniz's theorem:

$$F(D)uv = uF(D)v + Du\cdot F'(D)v + \frac{D^2u}{2!}\cdot F''(D)v + \cdots,$$

we have

$$\begin{aligned}(1 - e^{-D})^{n+1}a\cdot a^n &= a(1 - e^{-D})^{n+1}a^n + (n+1)e^{-D}(1 - e^{-D})^n a^n \\ &= [a(1 - e^{-D}) + (n+1)e^{-D}](1 - e^{-D})^n a^n \\ &= [a(1 - e^{-D}) + (n+1)e^{-D}]n! \\ &= (n+1)!,\end{aligned}$$

since $e^{hD}\cdot c = c$, where c is a constant. Hence if (1) is true for a given value of n, it is true when n is replaced by $n + 1$. But (1) is evidently true when $n = 1$. It follows by the usual argument employed in mathematical induction that (1) is true for all positive integral values of n.

From (1) it follows that

$$(1 - e^{-D})^{n+m}a^n = 0, \qquad m, n > 0.$$

Changing n to x and $n + m$ to n, we have

$$(1 - e^{-D})^n a^x = 0, \qquad n > x > 0.$$

Expanding and putting $a = 0$, we have the corollary as stated for $x = 1, 2, \cdots, (n - 1)$. From the expansion of $(1 - 1)^n$ by the binomial theorem it follows that when $x = 0$, $y = 1$; and from (1) it follows that when $x = n$, $y = (-1)^{n+1}n!$. Hence the theorem and corollary are correct as stated.

II. Solution by Harry Langman, New York City.

If u_x is a function of x, then

$$\Delta^n u_x = \sum_{r=0}^{n}(-1)^r C_r^n u_{x+n-r}.$$

If $u_x = x^n$, $\Delta^n u_x = n!$. Hence

$$\Delta^n u_{a-n} = n! = \sum_{r=0}^{n}(-1)^r C_r^n (a - r)^n,$$

which is the required identity. Since this is true for all values of a, successive differentiation with respect to a gives

$$0 = \sum_{r=0}^{n}(-1)^r C_r^n (a - r)^x, \qquad 0 \leqq x \leqq n - 1.$$

By setting $a = 0$ we obtain the results in the corollary.

Also solved by H. Betz, J. Ginsburg, C. A. Shook, and H. L. Smith.

3109 [1925, 46]. Proposed by the late J. W. NICHOLSON.

Find two rational numbers which separate the roots of the equation $x^3 - ax^2 + bx - c = 0$.

Solution by C. F. Gummer, Queen's University.

It will be understood that two rational functions of the coefficients $f(a, b, c)$ and $g(a, b, c)$ are to be found, such that, whenever the equation has three real unequal roots $x_1 < x_2 < x_3$, either $x_1 < f \leqq x_2 \leqq g < x_3$ or $x_1 < g \leqq x_2 \leqq f < x_3$.

If instead we attempt to make $x_1 < f < x_2 < g < x_3$ in every case, we find that the problem admits no solution. For, in the first place, it is readily seen, by means of a similarity transformation, that such an f, when expressed in terms of the roots, must be a rational homogeneous expression of degree unity (whether integral or fractional); and hence that f is replaced by $-f$ when x_1, x_2, and x_3 are replaced by $-x_1$, $-x_2$, and $-x_3$. If, therefore, x_1 changes continuously to $-x_3$, x_2 to $-x_2$, and x_3 to $-x_1$, which does not necessitate any two roots becoming equal at any stage, f will be found to have changed from a value in the interval between the two smaller roots to a value between the two greater. It is therefore impossible to find a suitable function f which will remain always in the left-hand interval.

It is, however, easy to select an f which remains always in the greater of the two intervals (x_1, x_2) and (x_2, x_3); except when these intervals are equal. For we may take

$$f = a/3 = (x_1 + x_2 + x_3)/3 = x_2 + (x_1 + x_3 - 2x_2)/3.$$

It remains to choose a g which shall be restricted to the smaller interval between the roots. This must be a symmetric function of the roots expressible in the form $x_2 - (x_1 + x_3 - 2x_2)U/V$, where V is a symmetric polynomial in the x's and U is a polynomial such that U/V is positive when x_2 is the middle root. If either x_1 or x_3 approaches x_2, g must approach x_2 also. Hence U is divisible by $x_2 - x_1$ and by $x_3 - x_2$.

In seeking the simplest possible form for g, we may try

$$U = (x_2 - x_1)(x_3 - x_2), \qquad V = p(x_1^2 + x_2^2 + x_3^2) + q(x_1x_2 + x_1x_3 + x_2x_3).$$

The conditions for the symmetry of g show that p must be 2 and q must be -2. This makes V a positive definite quadratic form in the roots, so that U/V has the required sign. Hence a suitable expression for g is

$$(\Sigma x_1^2x_2 - 6x_1x_2x_3)/(2\Sigma x_1^2 - 2\Sigma x_1x_2) = (ab - 9c)/(2a^2 - 6b).$$

The conclusion is that *if the roots of the cubic* $x^3 - ax^2 + bx - c = 0$ *are real and distinct and not in arithmetic progression, they are separated by* $a/3$ *and* $(ab - 9c)/(2a^2 - 6b)$; *and* $a/3$ *lies in the greater interval.*

There are no three rational functions of the coefficients of the general quartic which separate the roots in this manner. For if there were, one of the functions would approach x_1 as x_2 approached x_1. It would therefore take the form $x_1 + (x_2 - x_1)P/Q$, Q being a symmetric function not divisible by the differences of roots. By symmetry, the same quantity would be expressible in the form $x_3 + (x_4 - x_3)R/Q$, which leads to a contradiction when x_2 approaches x_1 and x_4 approaches x_3 at the same time.

It is another question, however, whether or not there exists a rational algebraic process to determine numbers separating the roots of the general numerical quartic.

REMARK BY OTTO DUNKEL, Washington University.

To show that g lies in the smaller interval, it suffices to show that

$$2[\Sigma x_1^2 - \Sigma x_1x_2]x_1 < \Sigma x_1^2x_2 - 6x_1x_2x_3 < 2[\Sigma x_1^2 - \Sigma x_1x_2]x_3.$$

The first inequality reduces to

$$(x_3 - x_1 + x_2 - x_1)(x_3 - x_1)(x_2 - x_1) > 0,$$

and the second to

$$(x_3 - x_1 + x_3 - x_2)(x_3 - x_1)(x_3 - x_2) > 0.$$

Both of these inequalities are true if $x_1 < x_2 < x_3$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

The Franklin Institute has conferred Franklin medals and certificates of honorary membership on Dr. P. ZEEMAN, professor of physics at the University of Amsterdam, and Dr. ELIHU THOMSON, of the General Electric Company.

J. R. FREEMAN, consulting engineer, of Providence, has made a gift of securities valued at $25,000 to the Boston Society of Civil Engineers for the establishment of a fund, the income of which is to be used for encouraging research by the younger engineers of the Society through the award of prizes for papers on hydraulics and allied subjects.

Professor NIELS BOHR and Professor A. S. EDDINGTON have been elected foreign members of the National Academy of Sciences. Professor S. LEFSCHETZ, of Princeton University, has been elected a member in the section of mathematics.

Professor G. G. CHAMBERS has received the honorary degree of doctor of science from Dickinson College.

Columbia University has conferred the honorary degree of doctor of science on Dr. IRVING LANGMUIR, of the General Electric Company.

Yale University has conferred the honorary degree of doctor of science on Professor R. A. MILLIKAN.

The Case School of Applied Science, and Union College have conferred honorary degrees on Professor M. I. PUPIN, of Columbia University.

Assistant Professor J. R. KLINE, of the University of Pennsylvania, has been awarded a Guggenheim Fellowship to study the analysis situs of three dimensions from a point-set standpoint, principally at the University of Göttingen.

Assistant Professor J. L. WALSH, of Harvard University, has been awarded a National Research Fellowship.

At West Virginia University, Assistant Professors MARGARET BUCHANAN, C. N. REYNOLDS, and B. M. TURNER have been promoted to associate professorships of mathematics.

Assistant Professor J. E. DAVIS, of the Drexel Institute, has been appointed associate professor of engineering extension at Pennsylvania State College.

Dr. PHILIP FRANKLIN has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

Dr. C. C. GROVE has been appointed assistant professor of mathematics at the Brooklyn Polytechnic Institute.

Associate Professor HILLEL HALPERIN has been promoted to a full professorship of mathematics at the Agricultural and Mechanical College of Texas.

Miss FRANCES HARSHBARGER, of West Virginia University, has been appointed head of the department of mathematics at Potomac State College, Keyser, W. Va.

Professor A. A. MICHELSON, of the University of Chicago, has been appointed to the first of the distinguished service fellowships recently established at that university.

Assistant Professor E. J. MILES, of Yale University, has been promoted to an associate professorship of mathematics.

Mr. MAX MORRIS has been promoted to an assistant professorship of mathematics at the Case School of Applied Science.

Miss ELIZABETH STAFFORD has been appointed adjunct professor of mathematics at the West Texas College of Technology, Lubbock.

Dr. J. H. TAYLOR has been appointed assistant professor of mathematics at Lehigh University.

Associate Professor H. S. UHLER, of Yale University, has been appointed head of the department of physics at Gettysburg College.

Professor H. A. WILSON, of the University of Glasgow, has accepted reappointment to the professorship of physics at Rice Institute which he held from 1912 to 1924.

Professor FREDRICK WOOD, of Lake Forest College, has been appointed professor of mathematics at Wesleyan College, Macon, Ga.

Mr. E. P. STARKE has been promoted to an assistant professorship of mathematics at Rutgers College.

Professor G. N. ARMSTRONG, of Ohio Wesleyan University, has been appointed head of the department of mathematics to fill the vacancy caused by the death of Professor C. B. AUSTIN.

At the University of Colorado, Professor I. M. DELONG has been made professor emeritus after forty-seven years of service. Professor A. J. KEMPNER, of the University of Illinois, has been appointed professor of mathematics and acting head of the department.

At Wesleyan College, Macon, Ga., Professor J. C. HINTON has ceased active work after thirty-five years of service. He has been dean of the college for the past twenty-five years.

Professor C. N. MILLS, of Aberdeen, South Dakota, has been appointed head of the mathematics department of the Illinois State Normal University.

Professor L. E. DICKSON, of the University of Chicago, would appreciate information as to where he could borrow or buy G. B. Jerrard's *Mathematical Researches, Part II*, 1834.

The following appointments to instructorships are announced: American College for Women, Constantinople, E. MARIE PLAPP; Connecticut College for Women, MILDRED E. CARLEN; University of Florida, CLAIRE HARKINS; West Virginia University, H. A. DAVIS; University of Michigan, B. DUSHNIK, N. C. FISKE, J. D. GRANT.

Dr. W. S. DENNETT, of New York City, died March 6, 1925. He was treasurer of the American Mathematical Society from 1900 to 1907.

Professor W. A. HAMILTON, of Antioch College, died June 25, 1925.

Professor S. J. LOCKNER, of the University of Pittsburgh, died May 10, 1925, at the age of fifty-five.

Dr. MANSFIELD MERRIMAN, professor of civil engineering at Lehigh University from 1878 to 1907, died June 7, 1925, at the age of seventy-seven.

Professor FELIX KLEIN, of the University of Göttingen, died June 22, 1925, at the age of seventy-six.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 8–9, 1925;
Tenth Annual Meeting, Kansas City, Mo., December, 1925.
The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Decatur, May 7–8, 1926
INDIANA, Bloomington, May 8–9, 1925
IOWA, Coe College, Cedar Rapids, April 30–May 1, 1926.
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 20–21
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, George Washington Univ., Washington, Dec. 5, 1925.
MICHIGAN, Ann Arbor, April 1, 1926
MINNESOTA, St. Johns Univ., Collegeville, May 16
MISSOURI, Kansas City, December, 1925
NEBRASKA, Creighton Univ., Omaha, May 2
OHIO, Ohio State Univ., Columbus, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 28
TEXAS, Dallas, November 27–28, 1925

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A DEVELOPMENT OF FIFTY YEARS IN DESIGN AND QUALITY

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THE SURNAMED CHOSEN CHEST.

By DAVID EUGENE SMITH, Columbia University.

II. Orientalia.

In the "Surnamed Chosen Chest" [1] there are many oriental manuscripts and early printed books,—some hundreds in all. It is proposed in this article to mention a few of the most interesting manuscripts of the mathematical classics of the East. There are not many copies of these works in this country, and it is probable that a brief list of some of the more important ones will be of service to scholars, not merely in the history of mathematics but in the field of oriental languages as well. The material is available for study, through photostat copies, by anyone who cares to have such copies made and who is prepared to undertake work of this nature.

Referring first to the Hindu classics, the anonymous *Sūrya Siddhānta* was the earliest noteworthy Indian work on astronomy. It is mathematically interesting because of its early trigonometric table,[2] and there is in the library a remarkably well-preserved copy on palm leaves. The text is Sańskrit, 78 leaves, $1\frac{1}{2}$ by 13 in.[3] The original work was written *c.* 400, and this copy was made *c.* 1450. Although slightly damaged, the text seems to be substantially complete.

The first of the prominent Hindu mathematicians whose names are known was Āryabhāta (*c.* 500), and there is in the library a manuscript of the entire *Āryabhatīya* (*Āryabhatīyam*). This is on palm leaves, 2 by 19 in., 76 leaves (152 pages). It is a modern copy of the Cadjan manuscript in Madras.

Of the native astronomers of India, those who joined mathematics with the science of the stars, the best-known is Varāhamihira (*c.* 505). There is in the library a complete manuscript on paper of his great *Pañca Siddhāntikā,* a work of much rarity in this complete form and consisting of several hundred leaves. The text is Sanskrit, and the earliest portions of the copy date from the sixteenth century.

The leading mathematician of India during the Middle Ages was Bhāskara (1114–*c.* 1185), a native of Biddur in the Deccan, but working at the great astronomical center Ujjain.[4] Of his various works the one which brought him the most fame is the *Lilāvati,* a treatise on arithmetic and mensuration. It was translated into Persian in 1587 by Fyzi, and the latter's story of Bhāskara's daughter is well known although not authenticated in Hindu sources. There are two translations in English.

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[1] The present article forms a continuation of that which appeared in this volume, pages 287–294.

[2] See the author's *History of Mathematics*, I, 34, 145; II, 625; hereafter referred to as *History*.

[3] These measurements are merely approximate, as is necessarily the case with all old manuscripts.

[4] *History*, I, 275.

The following manuscripts of this work are in the library:

1. Manuscript of the *Lilāvati* on palm leaves, $1\frac{1}{2}$ by $15\frac{1}{2}$ in., Sanskrit, 88 leaves (176 pages), *c*. 1400, with a commentary by Karmapradipika.[1]

2. Manuscript of the *Lilāvati* on paper, $4\frac{1}{2}$ by $9\frac{1}{2}$ in., Sanskrit, 45 leaves (90 pages), *c*. 1600.[2]

3. Manuscript of the *Lilāvati* on paper, $3\frac{3}{4}$ by $9\frac{1}{4}$ in., Sanskrit, 55 leaves (110 pages), *c*. 1600, incomplete.

4. Manuscript of the *Lilāvati* on paper, 7 by $10\frac{1}{8}$ in., Persian translation by Fyzi, 76 leaves (152 pages), dated 1071 A.H. (1661 A.D.). The *Lilāvati* seems incomplete, but this copy has not been carefully checked. The manuscript contains portions of other mathematical works.

5. Manuscript of the *Lilāvati* on paper, $5\frac{3}{4}$ by $8\frac{1}{4}$ in., Persian translation by Fyzi, 56 leaves (112 pages), dated 1143 A.H. (1731 A.D.).[3]

6. Manuscript of the *Lilāvati* on paper, $4\frac{1}{4}$ by 10 in., Sanskrit, 53 leaves (106 pages), dated 1785. Numerous glosses.

7. Manuscript of the *Lilāvati* on paper, $4\frac{1}{4}$ by $9\frac{1}{2}$ in., Sanskrit, 24 leaves (48 pages), *c*. 1840, incomplete.

8. Manuscript of the *Lilāvati* on paper, $5\frac{1}{4}$ by $13\frac{1}{2}$ in., Sanskrit, 48 leaves (96 pages), incomplete, *c*. 1850.

9. Manuscript of the *Lilāvati* on paper, $4\frac{1}{2}$ by 10 in., Sanskrit, 40 leaves (80 pages), dated 1851.

10. Manuscript of the *Lilāvati* on paper, $6\frac{1}{2}$ by $8\frac{1}{4}$ in., Sanskrit, 53 leaves (106 pages, 12 blank), 19th century, red ink.

11. Manuscript of a commentary on the *Lilāvati* on paper, $4\frac{1}{4}$ by 9 in., Sanskrit, 77 leaves (154 pages), *c*. 1800.

There is also in the library a copy of the first printed edition, entitled *Lílávatí; / A Treatise / on Algebra and Geometry. / By Srí Bháskara Áchárya*, Calcutta, 1832.[4]

It is a matter of dispute as to how much of Bhāskara's work was included in his *Siddhānta Siromani* (*Head jewel of accuracy*). He may have intended even the *Lilāvati* to be a part of this compendium, and the *Goladhia*, on the sphere, is commonly given as the fourth and last chapter. The following portions of this *Siddhānta*, in manuscript, are in the library:

1. Fragment of the *Siddhānta Siromani*, on paper, $3\frac{3}{4}$ by 9 in., Sanskrit, 24 leaves (48 pages, one blank), *c*. 1400, being the oldest paper manuscript in the Bhāskara collection.

2. Manuscript fragment of the *Siddhānta Siromani* on paper, 6 by 11 in., Sanskrit, 9 leaves (18 pages), 19th century.

3. Manuscript fragment of the *Goladhia* (being part IV of the *Siddhānta Siromani*) on paper, 5 by 13 in., Sanskrit, 11 leaves (22 pages), 19th century.[5]

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[1] Illustrated in *History*, I, 276.

[2] Fac-simile in *History*, I, 277.

[3] Fac-simile in *History*, I, 279.

[4] Fac-simile in *History*, I, 278.

[5] Fac-simile in *History*, I, 281.

Of Bhāskara's treatise on algebra there is one complete manuscript:

Manuscript of the *Bija Ganita* on paper, $4\frac{3}{4}$ by 13 in., Sanskrit, 42 leaves (84 pages), copied at Benares by Nundarama in 1825.

There is also in the library a copy of the first printed edition.[1]

Altogether there are fifteen manuscripts of the works of Bhāskara in the library, and the first Sanskrit printed editions of the above-mentioned treatises.

Of the best-known algebras of the Moslem civilization there are two rare specimens. The first is the *'Ilm al-jabr w'al muqabalah* (*Science of reduction and cancellation*) of Mohammed ibn Mûsâ al-Khowârizmî (*c.* 830), an Arabic manuscript on paper, $6\frac{1}{2}$ by $9\frac{1}{4}$ in., 132 numbered pages, without date, but probably *c.* 1800. Suter does not list any Arabic manuscripts in Europe, but the Rosen translation (London, 1831) was made from a copy in the Bodleian Library at Oxford, which was the only one that the translator had been able to find. There must, however, be others in the European collections of Orientalia.

The second noteworthy Oriental manuscript on algebra is one of the treatise of Omar Khayyam ('Omar ibn Ibrâhîm al-Khayyâmî, Giyât ed-dîn, Abû'l-Fath, *c.* 1044–1123/24). This is an Arabic copy on paper, $6\frac{3}{4}$ by $9\frac{1}{2}$ in., 49 numbered leaves (98 pages), without date, but probably written *c.* 1800. The text is identical, at least in its main features, with the Arabic version of Woepcke, Paris, 1851. Woepcke examined three manuscripts: (1) one in the Bibliothèque Nationale; (2) a fragment, also in the same library; and (3) one in Leyden. Suter mentions also one in the India Office, London. The text is written with unusual elegance and distinctness.

While not bearing upon the mathematical question, but upon the work of Omar Khayyam, it may be of interest to oriental scholars to know that there is in the library a beautifully written manuscript of his *Rubaiyat,* 78 numbered pages, $3\frac{1}{2}$ by $5\frac{3}{4}$ in., without date but probably the sixteenth or seventeenth century, in Persian stamped binding, gilded leather.

Of the oriental manuscripts on geometry in the library, two are of special interest. The first is the Arabic translation of Books I–VI of Euclid's *Elements,* made by Ishâq ibn Honein, *c.* 890, and revised by Tâbit ibn Qorra at about the same time. The manuscript is on paper, $6\frac{1}{2}$ by $11\frac{1}{4}$ in., 95 numbered leaves (190 pages). It is dated 751 A.H. (1350/51 A.D.).[2]

The second of the interesting manuscripts of Euclid is the Chinese version of Matteo Ricci (Li Ma-do, in Chinese), translated in 1603–1607 with the help of two learned mandarins. This copy is on paper, 6 by 11 in., and is bound in eight parts in two folders. It is a fine piece of Chinese writing and is undated, but probably was copied *c.* 1800. There is also in the library the first printed edition of this version.

The great Persian writer Nasîr ed-dîn al-Tûsî (1201–1274) left many works of importance, some of which are represented by manuscripts in this library, as follows:

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[1] Fac-simile in *History,* II, 426.

[2] Fac-simile in *History,* I, 173.

1. *Kitab-i sî fasl* (*Book of thirty sections*), manuscript on paper, $4\frac{1}{2}$ by 8 in., Persian, 32 leaves (64 pages, 2 blank), copied in 1354 A.D.

2. The *Centiloquium of Ptolemy*, manuscript on paper, 6 by 10 in., Persian, 27 leaves (54 pages). Other manuscript copies exist in Leyden, Florence, Constantinople, Cairo, and the British Museum.[1]

3. *Risâle-i; bist bâb*, on the astrolabe, manuscript on paper, $4\frac{1}{2}$ by $7\frac{1}{8}$ in., Persian, 23 leaves (46 pages). Other manuscript copies exist in Petrograd, Florence, Oxford, Constantinople, and the British Museum.

'Abderrahmân al-Khâzinî, Abû Mansûr, known as Abû'l Fath of Bagdad, wrote *c.* 1136 A.D. the Sinjaric Tables, dedicated to the sultan Sinjar, of which there is a copy in the Vatican. What is probably a complete copy of this work is in this library, dated 1011 A.H. (1602/3 A.D.), on paper, Persian manuscript, 86 leaves (172 pages), $6\frac{1}{8}$ by $9\frac{1}{4}$ in.

As a piece of writing, the most interesting work in the collection of Orientalia is a Persian manuscript of the tables of Ulugh Beg (1393–1449), the prince astronomer of Samarkand. It consists of 194 numbered leaves (388 pages, 4 blank + 1 with notes, 6 by $9\frac{5}{8}$ in. It bears the date 1214 A.H. (1836 A.D.)). While not fully checked, it contains the catalogue of stars and apparently most if not all of the other tables as given by Sédillot and quoted by Knobel.

Of the commentators on Ulugh Beg the most prominent was 'Abdel 'alî ibn Mohammed ibn al-Hosein, al Barjendî, who died *c.* 930 A.H. (*c.* 1524 A.D.). There is in the library a copy of this work, manuscript on paper, $5\frac{1}{2}$ by $9\frac{1}{4}$ in., Persian, 312 leaves (624 pages), written in 965 A.H. (1558 A.D.).

A second copy of this same manuscript by al Barjendî is in the library. Manuscript on paper, $5\frac{3}{4}$ by 9 in., 258 leaves (516 pages), not dated, but *c.* 1550.

There is also a manuscript of a later commentator, Molla Ali Qoshchy (?), paper, $4\frac{1}{2}$ by $8\frac{1}{2}$ in., Persian, 52 leaves (104 pages), apparently *c.* 1750.

The last of the notable scholars of Islam was Behâ ed-din al-'Âmilî (1547–1622), probably a Persian. In the library are two manuscripts relating to his works, as follows:

1. *Kholâsat al-Hisâb* (*Essence of Arithmetic*), manuscript on paper, $7\frac{3}{4}$ by $11\frac{7}{8}$ in., Persian, 320 leaves (640 pages), together with a commentary of 58 leaves (116 pages), in a different hand. The *Kholâsat* was written *c.* 1600. A date at the end of the commentary is 1024 A.H. (1615 A.D.), which may refer to the original treatise. The manuscript of the commentary bears no date. The manuscript of the text proper was written in 1135 A.H. (1723 A.D.). There are manuscripts in Berlin, Munich, British Museum, India Office, Cairo, Constantinople, and Paris.

2. Commentary on the above, manuscript on paper, $6\frac{1}{2}$ by $9\frac{1}{2}$ in., Persian, 30 leaves (60 pages), no date, but probably 17th century. The commentary is found in some of the manuscripts referred to above.

There are in the library a large number of manuscripts on astrology from

[1] On all such statements see Suter, *Abhandlungen*, Heft X, Leipzig, 1900.

India and adjacent regions. These are semi-mathematical in that they make more or less use of astronomical tables. Of these only two will be mentioned.

The first is the *Dyevagna-Kamadenuva*, the greatest of the Ceylon treatises on the subject. This copy is modern, on 190 palm leaves (380 pages), 1¾ by 12¼ in.

The second is a Madras copy of the *Vākyakaranaya*, on 117 palm leaves (234 pages), 2¼ by 6¾ in. The copy is modern.

Of the Chinese classics, represented in the library by a large number of printed works, the most interesting manuscript is one of the *I-king*.[1] This is a silk roll 16¾ in. by 6 ft. 4 in. The original is of uncertain date, but was probably written by Wön-wang (1182–1135 B.C.), and perhaps was based upon the works of earlier writers. This copy is modern and is a beautiful piece of Chinese writing in black and red.

The oldest manuscripts in the library are, naturally, the Babylonian cylinders. There are several bearing upon arithmetic, including receipts, tax lists, and bills. The most interesting one, however, is a cylinder 2¼ by 3¼ in., found at Warka, the Biblical Erech (Genesis, X, 10). It is a school boy's exercise tablet for practice in making numerals, and is filled with these forms. It comes from a mass of cylinders all of the Hammurabi period, *c.* 2100 B.C.

Concerning the Japanese manuscripts in the library it is impossible, in the space allowable, to speak in any great detail. There are more than a hundred specimens, including the complete *Kiku genpō chō ken* (on surveying) by Nobutomo Ogino (1718); the *Chō ken Tō monki* (also on surveying) by Murai Masahiro (*c.* 1732);[2] the *Honbun Hassen Hio* (on trigonometric tables) by Miju Rakŭsai (*c.* 1815); the *Saku yen riu kwai gi* (on the *yenri* theory applied to mensuration) by Toshihisa Isawaki (1831); the *Sampō Tengen jutsu* (algebra) by Moriyoshi Hanai; the *Yendan jutsu Kwaisho* (algebra) of Sataka Fusataka Iwasa and Shigeyuki Kato (*c.* 1865); the *Ken jutsu Denpō* (on surveying) of Hiroyoshi Ogawa (*c.* 1830); one of the well-known Hundred-Question collections copied by Nobujiro Hirata (*c.* 1818); the *Iyen Sampō* (on circles), written in 1825 by the celebrated Wada Yenzō Nei (1787–1840) and probably copied by one of his pupils at about the same time; a manuscript of the eighteenth century on Seki Kōwa's *Fukudai* process, in which Seki makes a step toward the theory of determinants; an extensive treatise in twelve volumes, the *Ichigen Kappo* by Hoshuku Iriye (1760); and a large number of treatises of the ancient *Wasan* (native mathematics) by minor or unnamed writers of the eighteenth and early nineteenth centuries. A complete list of this material, made by a Japanese scholar, would be of importance.

[1] *History*, I, 25.

[2] Fac-similes in *History*, II, 358, 359, 614.

THE PATH OF LIGHT IN A GRAVITATIONAL FIELD.

By C. C. WYLIE, University of Iowa.

1. Introduction. The earliest work on the deflection of light in the gravitational field of the sun, so far as is known, is that of Von Soldner of Munich.[1] On the assumptions that light has weight, and that it is deflected according to Newton's law of gravitation, he computed the bending of a ray passing the limb of the sun. A factor 2 was erroneously included, making his result double the correct.

In 1911, Einstein,[2] using a very different method, but the same basic assumptions, computed the gravitational deflection of light passing the limb of the sun, and arrived at the same result (Von Soldner's, with the mistake corrected). Using modern values for the astronomical constants, this result, ordinarily termed the Newtonian deflection, is 0″.87.

The relativity value, 1″.75, was published by Einstein [3] in 1916. This is the value computed on the assumption that light is deflected according to the general theory of relativity.

The published discussions of the deflection of light are nearly all in writings on relativity. Since the derivation of the Newtonian deflection is a problem of celestial mechanics, it is not given, and as a consequence persons unfamiliar with the subject occasionally have difficulty.[4]

In the formula for the relativity deflection the mass of the sun appears and is considered a "length" of 1.47 kilometers. This conception of mass as a length is puzzling to many. A little search brought to light a reference in a scientific journal to mass as "three dimensional." The evaluation of the mass m gives the viewpoint of the relativity writers. Further, the reader can convince himself that it is a perfectly definite length, whether the units be the day and the mean distance of the earth from the sun, or the centimeter-second system.[5]

In the discussion of photographic plates taken at the time of a total eclipse the statement occasionally appears that according to the relativity theory the deflection should vary inversely as the radial distance from the center of the sun's image. This form is used for computing the theoretical deflection of each star from that at the limb. The proof is not given, so a simple one is included in this paper.

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[1] Bode, *Berliner Astronomisches Jahrbuch*, 1804. A reprint of a portion of his work was published by Lenard in *Annalen der Physik*, **65**, 593, 1921.

[2] *Annalen der Physik*, **35**, 898, 1911.

[3] *Annalen der Physik*, **49**, 769, 1916. Eclipse measurements indicate a deflection of about the amount here predicted: Mitchell, *Eclipses of the Sun*, 395–417; *Lick Observatory Bulletin*, No. 346.

[4] *Literary Digest*, No. 1803, Page 20, November 8, 1924. Report of address of Professor T. J. J. See, before the California Academy of Sciences. He received national advertising on the claim that he had found a mistake in the computation of the deflection which "any high school student" could understand, the omission of a factor 2.

[5] *Science*, **60**, 221, 1924. Very different values of m are derived with different sets of units.

It may be added that at the distance of the earth from the sun the deflection is practically that for an infinite distance.[1]

2. The Newtonian Deflection at the Limb of the Sun. The equations for the path, acceleration, and areal velocity-constant of an infinitesimal planet moving about the sun in accordance with the Newtonian law of gravitation are:[2]

$$r = p/(1 + e \cos \theta), \qquad \alpha = -\, h^2/pr^2, \qquad h = r^2 d\theta/dt.$$

Consider light just grazing the limb of the sun. Let us denote the radius of the sun by R, the acceleration of gravity at the surface of the sun by G, and the velocity of the light at this point by V_0. At this instant we have:

$$r = R, \qquad \theta = 0, \qquad d\theta/dt = V_0/R, \qquad \alpha = -\, G.$$

Substituting:

$$h = R^2V_0/R = RV_0, \qquad G = R^2V_0^2/pR^2, \qquad R = V_0^2/G(1 + e),$$
$$e = (V_0^2/RG) - 1, \qquad p = V_0^2/G, \qquad (1 + e) = V_0^2/RG.$$

Substitution shows e to be a number of seven digits, hence with sufficient accuracy we can write $e = V_0^2/RG$ and

$$1/e = RG/V_0^2. \tag{1}$$

As e is large, the path is a hyperbola. The asymptotes to the hyperbola make with the axis $\theta = 0$ the angle $\tan^{-1} \sqrt{e^2 - 1}$. With sufficient accuracy we can substitute e^2 for $(e^2 - 1)$, which gives for the value of the angle $\tan^{-1} e$. Half of the total deflection we desire is 90° minus this angle; that is, denoting the total deflection by D_0:

$$D_0/2 = \cot^{-1} e = \tan^{-1} (1/e) = \tan^{-1} (RG/V_0^2).$$

Since D is very small, we can express it in seconds of arc, as

$$D_0''/2 = RG/(V_0^2 \tan 1''),$$
$$D_0'' = 2RG/(V_0^2 \tan 1''). \tag{2}$$

An inspection of the derivation shows this formula will give the deflection at any distance from the center of the sun, if the distance is denoted by R, and the sun's attraction at that distance by G.

3. The Mass of the Sun a Length. From Newton's law, the acceleration at a distance r from the center of the sun is

$$\alpha = k^2M/r^2, \tag{3}$$

where k is a constant depending on the units adopted, and M is the mass of the

[1] H. S. Uhler, MONTHLY (*1922*, 47).

[2] Moulton, *Celestial Mechanics*, revised edition, page 93, and elsewhere.

Ibid., page 18, Problem 3. The A of this problem corresponds to the h of page 81.

Ibid., page 81.

sun. The mass m used by the writers on relativity is in units so chosen as to reduce the k^2 to unity. Further, the unit of time is the time in which light travels unit distance. Denoting the velocity of light by V, the unit of time is $1/V$ and the acceleration in this unit [1] is α/V^2. When we make these substitutions, equation (3) becomes $\alpha/V^2 = m/r^2$, whence

$$m = \alpha r^2/V^2. \tag{4}$$

With the second as the unit of time, the kilometer as the unit of distance, and using values of α and r for the surface of the sun, we find for m:

$$m = \frac{(0.2738 \text{ km.})(695600 \text{ km.})^2}{(299860 \text{ km.})^2} = 1.47 \text{ km.}$$

The value for m is kilometers, not square kilometers, or cubic kilometers. In this sense m is a one-dimensional quantity, a length, and can be expressed in any unit of length.[2]

4. The Relativity Deflection at the Limb of the Sun. In works on relativity,[3] the deflection of a ray of light is expressed in the form

$$D_0 = 4m/R, \tag{5}$$

where D_0 is the deflection of a ray just grazing the limb of the sun, R is the radius, and m the mass in relativity units.

The quantity m is evaluated in (4). For the surface of the sun this becomes, using the previous notation, $m = R^2G/V^2$. Substituting in (5) and expressing D_0 in seconds of arc, we have

$$D_0'' = 4RG/V^2 \tan 1''. \tag{6}$$

Comparing this with (2), the Newtonian deflection, we see that it is approximately double. In the denominator of (6) we have V, the velocity of light in a vacuum, free from gravitational influence. In the Newtonian expression (2), we have V_0, the velocity of light at the limb of the sun after travelling subject to the attraction of gravitation from a distant star. The relation between the two is $V_0 = V + \sqrt{2RG}$. Using for the astronomical constants values taken from the *American Ephemeris and Nautical Almanac*, we obtain:

$$D_0'' \text{ (Newtonian)} = 0''.8702,$$
$$(1/2)D_0'' \text{ (Relativity)} = 0''.8737.$$

[1] If α is the acceleration per second per second, and V is the velocity of light per second, the acceleration per second in $1/V$ seconds would be α/V; and the acceleration per $1/V$ seconds in $1/V$ seconds would be α/V^2.

[2] Using the day as the unit of time and the mean distance of the earth from the sun as the unit of distance:

$$m = \frac{2.959 \times 10^{-4} \times 1}{3003} = 9.85 \times 10^{-9} \times \textit{mean distance} = 1.47 \text{ km.} = 0.915 \text{ mile.}$$

[3] Eddington, *Mathematical Theory of Relativity*, p. 91.

These values are so nearly equal that one naturally wonders whether theoretically they should be exactly equal. Presumably not, for in this paper, the approximations in the derivation of the Newtonian deflection affect the seventh significant figure. Dr. F. D. Murnaghan, after deriving the relativity deflection using elliptic integrals, concludes that the form $D_0 = 4m/R$ is correct to five significant figures.[1] Since the results differ in the third significant figure, the difference can hardly be attributed to neglected quantities in the approximations.

5. The Deflection Near the Limb of the Sun. In practice, a star exactly at the limb of the sun cannot be seen. The relativity deflection for a distance r from the center of the sun, where the acceleration is g, is from (6)

$$D'' = 4rg/(V^2 \tan 1'').$$

From the law of inverse squares we have $g = R^2G/r^2$. Substituting and rearranging:

$$D'' = \frac{4RG}{V^2 \tan 1''} \cdot \frac{R}{r} = D_0'' \frac{R}{r}.$$

The expression for the Newtonian deflection, in terms of that at the limb, would obviously reduce in the same way to $D_0''(R/r)$.

This is a convenient form for use in the measurement of eclipse plates. Expressing r, the distance from the center of the sun, in units of the sun's semidiameter, we have the form $D'' = D_0''/r$. The relativity deflection for each star would be $D'' \doteq 1''.75/r$ and the Newtonian deflection, $D'' = 0''.87/r$.

NOTE ON NEUBERG'S CUBIC CURVE.[2]

By FRANK MORLEY, Johns Hopkins University.

Fixing points on a plane by means of the values of a complex variable, the squared distance λ_{12}' between two points a_1' and a_2' is $\lambda_{12}' = (a_1' - a_2')(\bar{a}_1' - \bar{a}_2')$, where $\bar{a}_1'$, $\bar{a}_2'$ denote, as usual, the conjugate complex numbers to a_1' and a_2' respectively. Starting out with three points a_1', a_2', a_3' in the plane, Neuberg's cubic is the locus of points a_4' satisfying the equation[3]

$$\Delta' \equiv \begin{vmatrix} \lambda_{23}'\lambda_{14}' & \lambda_{23}' + \lambda_{14}' & 1 \\ \lambda_{31}'\lambda_{24}' & \lambda_{31}' + \lambda_{24}' & 1 \\ \lambda_{12}'\lambda_{34}' & \lambda_{12}' + \lambda_{34}' & 1 \end{vmatrix} = 0.$$

[1] *Philosophical Magazine*, **43**, 580, 1922.

[2] When I called Professor Morley's attention to the unsolved problem proposed by Messrs. T. W. Moore and J. H. Neelley in this MONTHLY (*1925*, 246), he communicated to me the solution contained in this note. I have taken the liberty of amplifying somewhat his remarks so as to make the matter more easily intelligible to a wider circle of readers of the MONTHLY. F. D. MURNAGHAN.

[3] *Cf.* Brown, B. H. The 21-point Cubic. This MONTHLY (*1925*, 110–115).

Let us now subject the points a_1', a_2', a_3', a_4' to an inversion with center at the point a_5. If k is the radius of inversion, any point u' inverts into a point v found from the equation $(v - a_5)(\bar{u}' - \bar{a}_5) = k^2$. From this it follows at once that if we denote the points into which a_1', a_2', a_3', a_4' invert by a_1, a_2, a_3, a_4, respectively,

$$(a_1' - a_2')(\bar{a}_1' - \bar{a}_2') = k^4(a_1 - a_2)(\bar{a}_1 - \bar{a}_2)/(a_1 - a_5)(\bar{a}_1 - \bar{a}_5)(a_2 - a_5)(\bar{a}_2 - \bar{a}_5)$$

or

$$\lambda_{12}' = k^4\lambda_{12}/\lambda_{15}\lambda_{25}.$$

Proceeding similarly with the other squared distances we find

$$\Delta' = k^{12}I_2/\lambda_{15}^2\lambda_{25}^2\lambda_{35}^2\lambda_{45}^2, \quad \text{where} \quad I_2 \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23}\lambda_{15}\lambda_{45} + \lambda_{14}\lambda_{25}\lambda_{35} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31}\lambda_{25}\lambda_{45} + \lambda_{24}\lambda_{35}\lambda_{15} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12}\lambda_{35}\lambda_{45} + \lambda_{34}\lambda_{15}\lambda_{25} & 1 \end{vmatrix}.$$

If we expand this determinant and use the notation $(1\ 2\ 3\ 4\ 5)$ for the product $\lambda_{12}\lambda_{23}\lambda_{34}\lambda_{45}\lambda_{51}$, it is seen at once to be a sum of 12 terms of the type $(1\ 2\ 3\ 4\ 5)$. Each of these terms is cyclic and reversible, *i.e.*,

$$(1\ 2\ 3\ 4\ 5) = (2\ 3\ 4\ 5\ 1) = (3\ 4\ 5\ 1\ 2) = (4\ 5\ 1\ 2\ 3) = (5\ 1\ 2\ 3\ 4)$$

and

$$(1\ 2\ 3\ 4\ 5) = (1\ 5\ 4\ 3\ 2),$$

etc. This reversibility reduces the $4! = 24$ cyclic arrangements of 5 things to 12 and it is these twelve that occur in the expansion. The sign to be attached to any term is $+$ or $-$ according as the arrangement is of the same class as $(2\ 3\ 1\ 4\ 5)$ or not. For example, the sign attached to the term $(1\ 5\ 2\ 3\ 4)$ is $-$. The important thing to notice is that I_2 involves the five points a_1, a_2, a_3, a_4, a_5 symmetrically. If we put $I_2 = 0$, we may regard this as the equation of the locus of a point a_4, the other four points a_1, a_2, a_3 and a_5 being given. The inverse of this locus in an inversion whose center at a_5 is the Neuberg curve of the three points a_1', a_2' and a_3'. We shall call, then, the curve whose equation is $I_2 = 0$ the inverted Neuberg curve with respect to one of the points (say a_5) of any three of the remaining four (say a_1, a_2 and a_3). The fifth point a_4 is the variable point which traces out the locus.

We shall now choose the point a_5 so that the points (a_1, a_2, a_3), into which the original triad (a_1', a_2', a_3') invert, are the vertices of an equilateral triangle. To see how this is done let us introduce, for the moment, homogeneous coördinates by writing the complex variable z in the form (z_1/z_2) and multiplying through by the proper power of z_2. To get back to the original complex variable, make $z_2 = 1$, $z_1 = z$. Then the triad (a_1', a_2', a_3') may be regarded as given by the zeros of the cubic polynomial

$$f'(z_1', z_2') \equiv \alpha_0'z_1'^3 + 3\alpha_1'z_1'^2z_2' + 3\alpha_2'z_1'z_2'^2 + \alpha_3'z_2'^3.$$

Any inversion is equivalent to a linear transformation of the form

$$z_1' = l\bar{z}_1 + m\bar{z}_2; \qquad z_2' = n\bar{z}_1 + p\bar{z}_2,$$

and we wish to find l, m, n and p so that (a_1, a_2, a_3) are the zeros of a polynomial $f(z_1, z_2)$ of the type $\alpha_0 z_1^3 + \alpha_3 z_2^3$, the middle terms being missing. Now the Hessian of f, *i.e.*, the quadratic polynomial

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial z_1^2} & \dfrac{\partial^2 f}{\partial z_1 \partial z_2} \\ \dfrac{\partial^2 f}{\partial z_1 \partial z_2} & \dfrac{\partial^2 f}{\partial z_2^2} \end{vmatrix},$$

has for its zeros in this instance $z_1 = 0$ and $z_2 = 0$, *i.e.*, the points 0 and ∞. Since the Hessian of $f' = (\bar{l}\bar{p} - \overline{mn})^{-2}$ times the Hessian of f, it follows that the triad a_1', a_2', a_3' and its Hessian points (h_1', h_2') invert into a triad (a_1, a_2, a_3) and its Hessian points (h_1, h_2). Conversely if we can fix the Hessian points h_1 and h_2 of a triad at 0 and ∞, we know the triad is equilateral (for the Hessian points of $\alpha_0 z_1^3 + 3\alpha_1 z_1^2 z_2 + 3\alpha_2 z_1 z_2^2 + \alpha_3 z_2^3$ are given by

$$(\alpha_0\alpha_2 - \alpha_1^2)z_1^2 + (\alpha_0\alpha_3 - \alpha_1\alpha_2)z_1z_2 + (\alpha_1\alpha_3 - \alpha_2^2)z_2^2 = 0$$

and we are therefore given that $\alpha_0\alpha_2 - \alpha_1^2 = 0$ and $\alpha_1\alpha_3 - \alpha_2^2 = 0$. If $\alpha_0 \neq 0$, this forces $\alpha_1 = 0$, $\alpha_2 = 0$ so that the triad is equilateral). If then we invert the original triad (a_1', a_2', a_3') from one of its Hessian points h_1', the triad will invert into an equilateral triad and the other Hessian point h_2' will invert into its center $h_2 = 0$.[1] Since a circle and a pair of inverse points invert into a circle and a pair of inverse points, we see that the Hessian points of any triad are inverse points with respect to the circumcircle of the triad. Thus the circumcenter and the two Hessian points lie on a line.

Since we are now separating out the variable point a_4 on the locus and the center of inversion a_5 from the five points which enter symmetrically the expression I_2, we write out I_2 in the form

$$\begin{aligned} I_2 = (\lambda_{23}\lambda_{31}\lambda_{45} + \lambda_{34}\lambda_{35}\lambda_{12})(\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42}) \\ + (\lambda_{31}\lambda_{12}\lambda_{45} + \lambda_{14}\lambda_{15}\lambda_{23})(\lambda_{24}\lambda_{53} - \lambda_{25}\lambda_{43}) \\ + (\lambda_{12}\lambda_{23}\lambda_{45} + \lambda_{24}\lambda_{25}\lambda_{31})(\lambda_{34}\lambda_{51} - \lambda_{35}\lambda_{41}), \end{aligned}$$

where each term follows from the preceding one by a cyclic interchange of 1, 2 and 3. We further denote the variable point a_4 on our locus by x and the fixed center of inversion by y.

The points (a_1, a_2, a_3) now form an equilateral triad whose center is at the origin and we may so choose the radius of the inversion whose center is at $a_5 = y$

[1] A figure readily shows that this is equivalent to saying that the Hessian points of a triad are the points whose pedal triangles, with respect to the triangle formed by the triad, are equilateral.

that the three points (a_1, a_2, a_3) lie on the unit circle whose center is at the origin.[1] We have then $a_1 = t$, $a_2 = \omega t$, $a_3 = \omega^2 t$, where ω, ω^2 are the complex cube roots of unity and t is a turn, *i.e.*, a complex number whose conjugate is its reciprocal. It follows at once that

$$\lambda_{12} = \lambda_{23} = \lambda_{31} = 3, \qquad \lambda_{45} = (x - y)(\bar{x} - \bar{y}),$$
$$\lambda_{14} = 1 + x\bar{x} - t\bar{x} - (x/t), \qquad \lambda_{15} = 1 + y\bar{y} - t\bar{y} - (y/t),$$

and the expressions for λ_{24}, λ_{34}, etc., follow on replacing t by ωt and $\omega^2 t$, respectively, in these. The first of the three terms in the expression for I_2 is accordingly $3(3\lambda_{45} + \lambda_{34}\lambda_{35})(\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42})$ and we have at once

$$\begin{aligned}\lambda_{34}\lambda_{35} = {} & (1 + x\bar{x})(1 + y\bar{y}) + x\bar{y} + y\bar{x} + (\omega^2 xy/t^2) + \omega t^2\bar{x}\bar{y} \\ & - \{x(1 + y\bar{y}) + y(1 + x\bar{x})\}(\omega/t) - \{\bar{x}(1 + y\bar{y}) + \bar{y}(1 + x\bar{x})\omega^2 t, \\ \lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42} = {} & (\omega^2 - \omega)[\{\bar{x}(1 + y\bar{y}) - \bar{y}(1 + x\bar{x})\}\omega^2 t \\ & + \{y(1 + x\bar{x}) - x(1 + y\bar{y})\}(\omega/t) + x\bar{y} - y\bar{x}].\end{aligned}$$

In developing the product $(3\lambda_{45} + \lambda_{34}\lambda_{35})(\lambda_{14}\lambda_{52} - \lambda_{15}\lambda_{42})$ we need not consider the terms in t, t^{-1}, t^2, t^{-2} since in each of these t has to be replaced by ωt, $\omega^2 t$ in turn, the resulting expressions then being added to the original one, and since $1 + \omega + \omega^2 = 0$. We find, then,

$$I_2 = 9(\omega^2 - \omega)\{A + (B/t^3) - \bar{B}t^3\},$$

where

$$\begin{aligned}A &= \{3(x - y)(\bar{x} - \bar{y}) - (1 + x\bar{x})(1 + y\bar{y}) + x\bar{y} + \bar{x}y\}(x\bar{y} - y\bar{x}), \\ B &= xy^2(1 + x\bar{x}) - yx^2(1 + y\bar{y}).\end{aligned}$$

The Apollonian circles of any triad are the three circles one through each point *about* the other two, *i.e.*, having these two as inverse points. Hence a triad and its Apollonian circles inverts into a triad and its Apollonian circles. In the case of the equilateral triad the Apollonian circles are the straight lines one through each point bisecting perpendicularly the join of the other two. The three remaining points where the Apollonian circles cross the circumcircle of the triad form the "counter-triad" and in the case of the equilateral triad given by the polynomial $x^3 - t^3 = 0$ the counter-triad is given by $x^3 + t^3 = 0$. This counter-triad has the same circumcircle and Hessian points as the original triad, a result holding, therefore, for every triad and its counter-triad. The equation of the inverted Neuberg cubic for the original triad being

$$I_2 = 0 \quad \text{or} \quad A + (B/t^3) - \bar{B}t^3 = 0,$$

that of the inverted cubic for the counter-triad is $A - (B/t^3) + \bar{B}t^3 = 0$ and

[1] It is easy to see that if d_1 and d_2 are the distances of the circumcenter and the second Hessian point from the center of inversion and R is the radius of the circumcircle, then $k^2 = d_1 d_2/R$.

upon adding these we see that the curve $A = 0$ goes through their points of intersection. Now $A = 0$ when either of its factors $x\bar{y} - y\bar{x}$ or $3(x - y)(\bar{x} - \bar{y}) - (1 + x\bar{x})(1 + y\bar{y}) + x\bar{y} + y\bar{x}$ is zero. The first factor gives the straight line joining the origin to the center of inversion, *i.e.*, the straight line through the center of inversion and the Hessian points of the equilateral triad. The other factor gives the circle whose equation is

$$C \equiv 2(x - y)(\bar{x} - \bar{y}) - 1 - y\bar{y}x\bar{x} = 0.$$

It will now be convenient to take the axis of reals through the center of inversion so that y is a real number μ. If we look at the expression for B, we see that $B - \overline{B} = \mu(x - \bar{x})\{\mu(1 + x\bar{x}) - (x + \bar{x})(1 + \mu^2)\}$ and $A = \mu(x - \bar{x})C$. Hence in the special cases $t^3 = \pm 1$ the inverted Neuberg curve degenerates, one of the factors giving the axis of reals and the other factor giving the circles $C \pm C_1 = 0$, where $C_1 \equiv \mu(1 + x\bar{x}) - (x + \bar{x})(1 + \mu^2)$. If we suppose these degenerate inverted Neuberg curves, $C \pm C_1 = 0$, known, then the circle $C = 0$ is described as belonging to the pencil determined by them and harmonic to the circle $C_1 = 0$. It is seen at once that $C_1 = 0$ goes through the points $-\omega\mu$ and $-\omega^2\mu$ and it is orthogonal to the circumcircle $x\bar{x} - 1 = 0$. It is readily seen that the two points $-\omega\mu$, $-\omega^2\mu$ invert into the vertices of the two equilateral triangles described upon the join of the two Hessian points of the original triad. Hence the inverse of the circle $C = 0$ is the circle through the two vertices of these equilateral triangles and orthogonal to the circumcircle.

The geometrical significance of the relations $t^3 = \pm 1$ follows at once from a consideration of the expressions given for λ_{12}', etc. If $t^3 = \pm 1$, two of the three squared distances λ_{15}, λ_{25}, λ_{35} are equal and so two of the original squared distances λ_{23}', λ_{31}', λ_{12}' are equal, *i.e.*, the original triangle is isosceles. In this case the Neuberg cubic reduces to a circle and a line, namely, the join of the Hessian points (which is the perpendicular bisector of the base of the isosceles triangle). As we vary the t in the equation $x^3 - t^3 = 0$ of our equilateral triad we get a series of inverse triads having a common circumcircle and Hessian points. Out of all these triads two are isosceles and these furnish, through their degenerate Neuberg cubics, the two circles which are the inverses of the pair $C \pm C_1 = 0$ and which determine the pencil to which the sought for circle belongs.[1]

[1] It may be mentioned that the Neuberg cubic of a triangle may be most easily studied starting from the property that it passes through the isogonal conjugate of each of its points and that the join of each such pair of isogonal conjugates is parallel to the Euler line of the triangle. Mr. B. C. Patterson, one of Dr. Morley's students, has studied the covariants of the Neuberg cubic under the inversion group.

APPLICATION OF THE THEOREM OF RESIDUATION TO THE 21-POINT CUBIC.

By O. M. THALBERG, Oslo, Norway.

With great interest I have read the paper by B. H. Brown in the MONTHLY, *1925*, pp. 110–115. Perhaps it may prove instructive to see the application of the Brill-Noether theorem [1] of residuation to the 21-point cubic or to any cubic of the pencil (through A_i, K, K_i and the two circular points) considered by Mr. Brown (the notation refers to the figure, *l.c.*, p. 111, which is here reproduced).

Since the points A_i, K, K_i are collinear and the two circular points and the points A_j, A_k, K, K_i are concyclic, the point A_i must be coresidual with the pointgroup consisting of the two circular points and A_j, A_k, and thus we may write down the following

THEOREM: *Any straight line through A_i meets the cubic again in two points concylic with A_j and A_k.*

As special cases of this theorem we at once deduce the following ones:

If the circumcircle of the triangle $A_1A_2A_3$ again meets the cubic in I, then IA_i is tangent to the cubic at A_i (*cf.* Brown, p. 112).

A circle exists touching the cubic at K and at K_i. There exists also a circle passing through J_k and A_k and touching the cubic at A_j.

The line A_iJ_i is parallel to the real asymptote.

The point K is coresidual to the pointgroup consisting of the two circular points and A_i, K_i (the two points A_j, K_j forming the common residual points). Hence:

THEOREM: *Any straight line through K meets the cubic again in two points concyclic with A_i and K_i.*

In the same way we obtain the following

THEOREM: *Any straight line through K_i meets the cubic again in two points concyclic with A_j and K_k as well as with A_i and K.*

As special cases of these theorems the following ones are of particular interest:

The tangents at K_i (K) are parallel to the real asymptote (*cf.* Brown, p. 112).

There exists a circle touching the cubic at A_i and K_i, or at A_j and K_k, or at A_i and K.

As the third (real) cubic point at infinity is coresidual to the pointgroup K, K_i, A_j, A_k as well as to the pointgroup A_i, A_j, K_i, K_j (the two circular points being the common residual points), we obtain the following

THEOREM: *Any line parallel to the real asymptote meets the cubic again in two points, lying on a conic through the pointgroup K, K_i, A_j, A_k as well as on a conic through the pointgroup A_i, A_j, K_i, K_j.*

From this follow several special theorems, of which we mention only the following:

[1] See *Math. Annalen*, vol. 7, 1874, pp. 272–3.

A_i, J_i, A_j, A_k, K_j and K_k lie on the same conic.

A_i, J_i, K, K_i, A_j and A_k lie on the same conic.

There exists a conic through K, K_i, A_j, A_k touching the cubic at K_j, and a conic through A_i, A_j, K_i, K_j touching the cubic at K_k.

The two circular points at infinity are coresidual to A_i, J_i as well as to twice the point $K_i(K)$ (the third cubic point at infinity is the common residual point). Hence:

THEOREM: *Any circle cuts the cubic again in four points lying on a conic through A_iJ_i as well as on a conic touching the cubic at $K_i(K)$.*

By the theorem of residuation we further derive the following

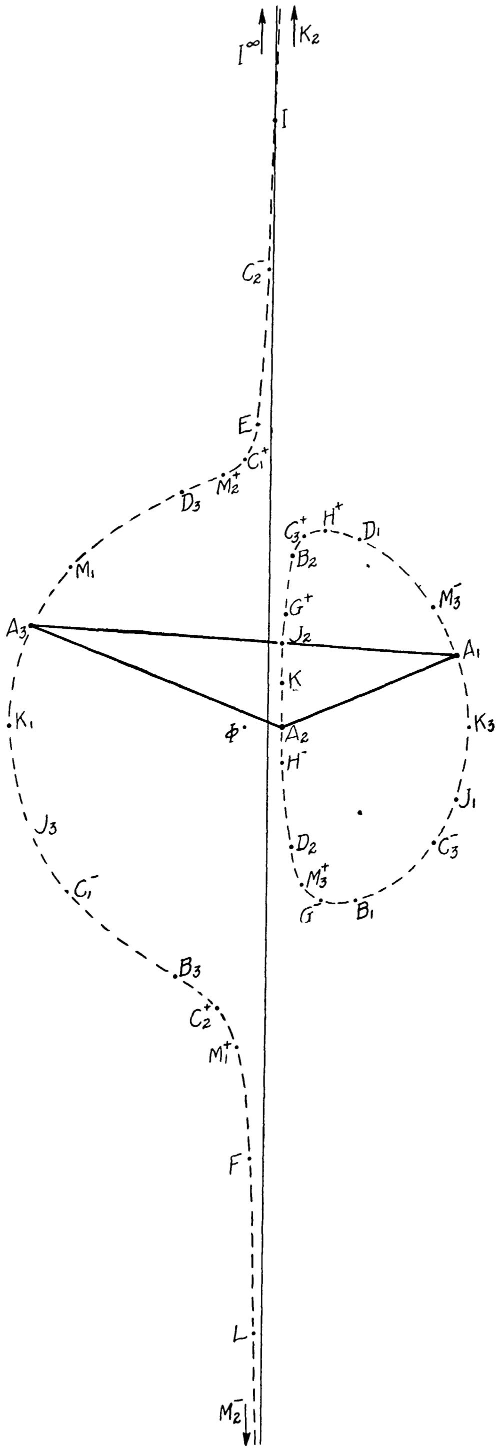
THEOREM: *Any circle through A_i, K_j cuts the cubic again in two points lying on a circle through A_j, K_i as well as on a circle through A_k, K.*

Finally we wish to show that it is convenient here to make use of the following well-known theorem:

Let T_1, T_2, T_3 and T_4 be the four points of contact of the four tangents from the point P on the cubic. Then the three diagonal points S_1, S_2 and S_3 of the quadrangle $T_1T_2T_3T_4$ lie on the cubic, and these three points and the point P have the same tangential point.

From this theorem and from the above property of the line A_iJ_i [or from the property of the tangent at $K_i(K)$] now the following theorem follows:

The common tangential I of the points A_i is the third intersection of the real asymptote with the cubic. The three points J_i and the point I have the same tangential point L (cf. Brown, pp. 112, 114). *The three diagonal*



points of the quadrangle $J_1J_2J_3I$ *lie on the cubic and have common tangential point with* L.

All the above theorems relate to any cubic of the pencil considered by Mr. Brown.

If we particularly consider the 21-point cubic, the theorem of residuation in connection with the results in Mr. Brown's paper will give us several new theorems as for instance the following:

Any circle cuts the cubic again in four points, lying on a conic through B_i, D_i *—on a conic through* $C_i^{\pm}$, $M_i^{\pm}$*—on a conic through* E, F *as well as on a conic through* $G^{\pm}$, $H^{\mp}$ (*cf.* Brown, p. 115).

AMERICAN CONTRIBUTIONS TO MATHEMATICAL SYMBOLISM.

By FLORIAN CAJORI, University of California.

1. Maya Number-System. Probably seven or eight centuries prior to the introduction of the zero in the Hindu-Arabic numerals, the Maya of Central America had a fully developed number system on the scale of 20 (except in one step). This system, as it appears in Maya codices, had a symbol for zero, and an extended application in the wonderful system of Maya chronology.[1]

2. Peruvian Knot Records. The use of knots in cords for reckoning, and recording numbers, early practised by the Chinese, had a most remarkable development among the Inca of Peru, from the eleventh to the sixteenth century of our era. Upon a twisted woolen cord (quipu) other smaller cords of different colors were tied. The color, length and number of knots, and distance of one from the other, all had their significance.[2] Quipu-like string records have been found in North America among the Indians of the Northwest.[3]

3. Dollar Mark. An extended study of manuscripts has led to the conclusion that our dollar mark descended, during the last quarter of the eighteenth century, from the Spanish-American abbreviation "p^s" for "pesos." [4]

4. Sporadic Notations for Radicals. In an anonymous publication, *The Columbian-Arithmetician. By an American,* Haverhill, Mass., 1811, there is added to the usual exponental notation 4^2, 2^m the following bold innovation (p. 13): ^{24} to mean $\sqrt{4}$, ^{38} to mean $\sqrt[3]{8}$, ${}^m8$ to mean $\sqrt[m]{8}$. This symbolism found no favor.

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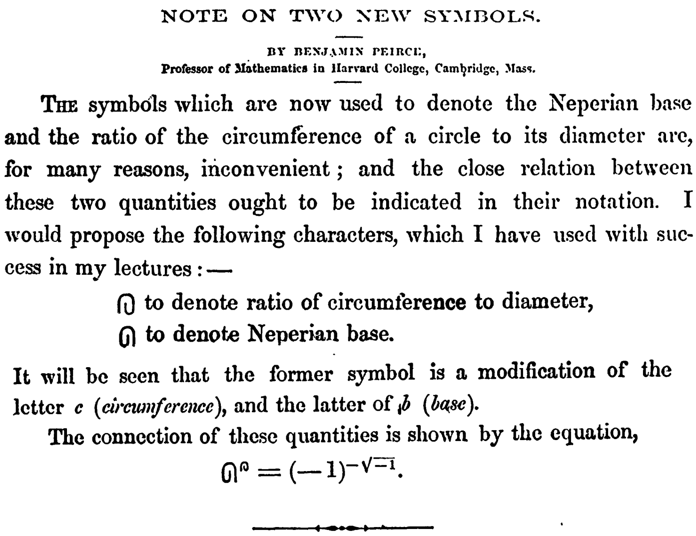
[1] See S. G. Morley, *An Introduction to the Study of the Maya Hieroglyphs.* Government Printing Office, Washington, 1915.

[2] L. Leland Locke, *The Ancient Quipu or Peruvian Knot Record.* 1923.

[3] J. D. Leechman and M. R. Harrington, *String Records of the Northwest, Indian Notes and Monographs,* 1921.

[4] F. Cajori in *Popular Science Monthly,* vol. 81, 1912, p. 521; *Science,* N.S., vol. 38, 1913, p. 848.

5. B. Peirce's Signs for our π, *e* and *i*. These are shown [1] in Figs. 1 and 2; they were used by his sons, J. M. Peirce and C. S. Peirce.


NOTE ON TWO NEW SYMBOLS.

BY BENJAMIN PEIRCE,
Professor of Mathematics in Harvard College, Cambridge, Mass.

THE symbols which are now used to denote the Neperian base and the ratio of the circumference of a circle to its diameter are, for many reasons, inconvenient; and the close relation between these two quantities ought to be indicated in their notation. I would propose the following characters, which I have used with success in my lectures: —

to denote ratio of circumference to diameter,

to denote Neperian base.

It will be seen that the former symbol is a modification of the letter *c* (*circumference*), and the latter of *b* (*base*).

The connection of these quantities is shown by the equation,

$= (-1)^{-\sqrt{-1}}$.

FIG. 1. B. Peirce's signs for π and e in the *Mathematical Monthly*, 1859.

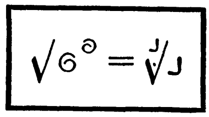


FIG. 2. From J. M. Peirce's *Tables*, 1871.

6. Equivalence in Geometry. This was expressed by the sign $\backsimeq$ which occurs in C. Davies' *Elements of Geometry and Trigonometry*,[2] 1851. For a quarter of a century following 1885, the sign enjoyed considerable popularity; the two signs = and $\backsimeq$ were used in Geometry to express congruence and equivalence.

7. Approaching the Limit was designated by $\doteq$. The sign is due to J. E. Oliver of Cornell who at first used it in the sense "is nearly equal to." In 1880, W. E. Byerly [3] gave it the meaning "approaches as a limit." This same symbol $\doteq$ was used in 1875 by A. Steinhauser [4] of Vienna in the sense "nahezu gleich," but to the best of our knowledge Oliver is in no way indebted to Steinhauser.

8. A Symbolism in Vector Analysis was invented by Josiah Willard Gibbs of Yale, a pioneer in this field. In 1881 he marked a scalar or "direct" product by $\alpha\cdot\beta$, a vector or "skew" product by $\alpha\times\beta$, where small Greek letters represent

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[1] B. Peirce in Runkle's *Mathematical Monthly*, vol. 1, No. 5, 1859, pp. 167, 168, "Note on two new symbols."

EDITOR'S NOTE:—Professor W. E. Story of Clark University used these symbols in this sense in 1907 in a course of lectures on the calculus of finite differences. NORMAN ANNING.

[2] Charles Davies, *Elements of Geometry and Trigonometry*, from the works of A. M. Legendre, New York, 1851, p. 87.

[3] W. E. Byerly, *Elements of the Differential Calculus*, Boston, 1880, p. 7. Oliver himself used the symbol in print in the *Annals of Mathematics*, Charlottesville, Va., vol. 4, 1888, pp. 187, 188. It is used also in Oliver, Wait and Jones' *Algebra*, Ithaca, 1887, pp. 129, 161.

[4] A. Steinhauser, *Lehrbuch der Mathematik*. Algebra. Vienna, 1875, p. 292.

vectors. He let a_0 stand for the magnitude of vector α. He[1] marked triple products, $\alpha \times \beta \cdot \gamma$, $(\alpha \cdot \beta)\gamma$, $\alpha[\beta \times \gamma]$.

9. Symbolic Logic was developed by C. S. Peirce. "He understood how to profit by the work of his predecessors, Boole and De Morgan, and built upon their foundations, and he anticipated the most important procedures of his successors even when he did not work them out himself."[2] C. S. Peirce[3] introduced a considerable number of new symbols, for instance, $-\!\!<$ for "inclusion in" or "being as small as"; x, y signifies commutative multiplication; in multiplication, "identical with—" is *1*; etc. Other symbols were proposed for symbolic logic by Mrs. Christine Ladd Franklin and O. H. Mitchell.[4]

10. General Analysis. Of notations introduced in America in the present century, I mention only the symbolism used by E. H. Moore in his general analysis.[5] He takes some of his logical signs from Peano's *Formulario mathematico*, 1906, and uses them approximately in the sense of Peano. Among Moore's other signs are $\neq$ for logical diversity, $\equiv$ for definitional identity, $\ni$ for "such that," x^P for "x has the property P." Moore's aim is different from that of Peano, Whitehead and Russell whose object was to proceed with absolute certainty in difficult, abstract studies of the foundations of mathematics, and who for that purpose used elaborate notations. Moore aims to meet the needs of the working mathematicians who consider extreme logical complications relatively unimportant, but desire simplicity and flexibility of notation.

None of the ten notations cited as originating in America has thus far found general acceptance in Europe, except, perhaps, the dollar mark.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. DERIVATION OF ANNUITY FORMULAS WITHOUT SERIES.

BY F. R. MORRIS, Fresno State College.

It is customary in deriving the annuity formulas to treat an annuity as a geometrical progression and to use the formula for the sum of n terms of this

[1] *The Scientific Papers of J. Willard Gibbs*, vol. 2, 1906, pp. 22, 23.

[2] C. I. Lewis, *A Survey of Symbolic Logic*, Berkeley, 1918, p. 79.

[3] C. S. Peirce, "On an Improvement in Boole's Calculus of Logic" in *Proceedings of the American Academy of Arts and Sciences*, vol. 7, Cambridge and Boston, 1868, pp. 250–261; "Description of a Notation for the Logic of Relatives" in *Memoirs Amer. Acad. of Arts and Sciences*, Cambridge and Boston, N.S., vol. 9, 1867, pp. 317–378.

[4] See *Studies in Logic*. By Members of the Johns Hopkins University. Boston, 1883.

[5] E. H. Moore, Introduction to a Form of General Analysis. *New Haven Mathematical Colloquium*. New Haven, 1910, p. 150.

series.[1] However, it is not necessary in the development of these formulas to make use of the geometrical progression, and it is the purpose of this paper to give a development based upon the definitions of annuities and compound interest.

The compound interest law states that $(1 + i)^n$ is the amount on 1 for n years at rate i. It is easy to see that the application of this law gives results which are in agreement with those commonly used, providing n is zero or a positive integer. If n is a negative integer, the function gives the value of 1 n years before it is due. Then *by definition* $(1 + i)^n$ is said to be the value of 1 for any real value of n.[2]

If the interest is converted into principal more than once each year, n may be considered as the number of periods and i the rate per period; but, to avoid confusion in this case, it is better to introduce new symbols and definitions. Let m be the number of conversion periods per year. The ratio of the entire interest for a year to the investment at the beginning of that year is called the effective rate of interest and is designated by i. The nominal rate is defined as m times the ratio of the interest for a period to the investment at the beginning of that period, which is usually the rate named in the document, and is designated by j. Hence j/m is the rate per period and $(1 + j/m)^m$ is the amount on 1 for one year. But $1 + i$ is also the amount on 1 for one year. Therefore

$$1 + i = (1 + j/m)^m. \tag{1}$$

An ordinary annuity may be defined as a number of equal payments made at the end of equal periods. It is convenient to use as a basis the annuity for which the payment is 1 at the end of each year. The annual payment is called the annual rent. If the rent is not 1, results may be easily obtained by multiplying the corresponding results for the basic annuity by the given rent. The amount of an annuity of 1 per year at the end of n years where the payments bear interest at the uniform effective rate i is designated by the symbol $s_{\overline{n|}}$. It is desirable to express this amount as a function of i and n.

The income of 1 at rate i is an ordinary annuity the annual rent of which is i. The amount of this annuity at the end of n years is the compound interest on 1 for n years, which is $(1 + i)^n - 1$. If this expression is divided by i, the quotient is the amount of an annuity the annual rent of which is 1 instead of i. As a result we have the fundamental formula

$$s_{\overline{n|}} = \{(1 + i)^n - 1\}/i. \tag{2}$$

The next problem is to derive the formula for the amount of an annuity of 1 per year paid in p equal installments. This amount is given the symbol $s_{\overline{n|}}^{(p)}$.

[1] For an exception, see the Discussion by C. N. Reynolds, Jr., in this MONTHLY (*1922*, 122).—EDITOR.

[2] Although this definition is generally used in the theory of investment, it does not quite agree with the common practice of computing compound interest for the integral number of years and simple interest for the remaining fraction of a year.

It is evident that the p payments, each of which is $1/p$, will have a value greater than 1 at the end of the year. We may think of 1 as the nominal annual rent, while the value of the p payments at the end of the year is the effective annual rent. Then $s_{\overline{n}|}^{(p)}$ is the product of $s_{\overline{n}|}$ and the effective annual rent.

The effective annual rent may be obtained from equation (1). Let m be replaced by p. The rent per period which will produce i in one year is j/p. Hence $1/p$ per period will produce an annual income of i/j, which is the effective annual rent. From the same equation j is found to be $p[(1 + i)^{1/m} - 1]$. Therefore the desired formula is

$$s_{\overline{n}|}^{(p)} = \frac{i}{p((1 + i)^{1/m} - 1)} \cdot s_{\overline{n}|}. \tag{3}$$

We see from the compound interest law that multiplying any amount by $1 + i$ gives the accumulated value a year later, and that dividing by $1 + i$ gives the value a year in advance. Making use of this principle we obtain the present value of the basic annuity. The present value, which is denoted by $a_{\overline{n}|}$, is the value of an annuity which n years later has a value $s_{\overline{n}|}$. Hence

$$a_{\overline{n}|} = (1 + i)^{-n} s_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}. \tag{4}$$

Likewise we obtain the formula for $a_{\overline{n}|}^{(p)}$ which is the present value with p installments per year:

$$a_{\overline{n}|}^{(p} = \frac{i}{p((1 + i)^{1/m} - 1)} \cdot a_{\overline{n}|}. \tag{5}$$

By this same principle the various formulas for annuities due and for deferred annuities may be derived.

If the interest is converted more than once a year, the annuity formulas are modified by substituting for i the value given by equation (1), thus giving functions of j, m, n and p. For the case $m = p$, formulas (3) and (5) reduce to forms similar to formulas (2) and (4). This completes the system of essential formulas for annuities.

II. Errata in Steinhauser's 20-Place Logarithm Table.

By E. B. Escott, Oak Park, Illinois.

One of the most convenient tables for computing logarithms to 20 places is Anton Steinhauser's *Hilfstafeln zur präcisen Berechnung zwanzigstelliger Logarithmen zu gegebenen Zahlen und der Zahlen zu 20stelligen Logarithmen* (Wien, 1880). In the first edition, a list of 46 errata was given and about five years later an additional list of 17 errata was given. Although the author says that he is convinced that the table contains no more errata, the following additional errata will show that this statement is far from being true. It is to be hoped

Table I.

Columns *A*.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
3	1153	3	9294	7294	59	6718	3	9405	9505
3	1192	4	7624	7625	60	6884	3	2655	1655
5	1302	3	6232	4232	62	7009	4	5606	5605
8	1635	4	1755	1775	62	7009	3	0083	9983
8	1697	1	9242	9742	63	7173	3	0825	0835
17	2536	4	3924	4924	64	7249	4	7849	7809
18	2658	4	7497	5497	66	7467	3	1428	1328
23	3108	3	0828	0128	66	7496	3	7779	7879
24	3247	3	8726	8626	69	7772	3	1727	1927
26	3442	3	6991	5991	69	7778	3	8711	8811
29	3748	3	2115	2215	70	7810	5	8426	8926
30	3886	3	6163	6263	71	7916	3	5434	5534
33	4187	3	0691	0891	72	8046	3	8135	8235
35	4304	3	2558	2658	73	8136	3	9814	9914
36	4402	3	8117	8217	75	8374	3	6355	6555
37	4534	3	5890	5790	75	col. *n*		5310	8310
40	4884	4	5565	7565	78	8608	3	8222	8322
44	5256	3	6451	6551	78	8662	3	9493	9393
44	5292	3	5715	5515	79	8786	3	8829	8929
45	5304	3	5496	5396	80	8804	3	3781	3881
46	5427	3	1679	1579	81	8972	3	4810	4910
47	5516	3	7703	7503	81	8974	3	5297	5197
47	5526	3	0680	0580	82	9068	3	1554	1454
54	col. *n*		3255	6255	83	9166	3	2808	2908
	"		3265	6265	83	9175	3	0224	2924
	"		3275	6275	84	9202	3	8828	8928
	"		3285	6285	85	9363	3	3484	3384
	"		3295	6295	86	9416	3	9735	9835
54	6216	3	6492	5792	86	9456	3	3437	3537
54	6269	4	8926	9826	86	9496	3	5846	5946
55	6319	3	5463	5363	87	9566	3	6435	6535
55	6394	3	1835	1935	87	9574	3	3045	3145
56	6494	3	4389	4289	89	9741	3	3445	3345

Table II.

Columns *B*.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
100	0321	1	9389	9398	240	7305	3	6399	6299
148	2716	4	94	93	240	7306	3	9796	9696
148	2717	4	99	98	240	7307	3	3194	3094
148	2718	4	1803	1802	240	7308	3	6592	6492
148	2719	4	07	06	240	7309	3	9990	9890
160	3313	3	7033	7933	240	7327	3	0048	1048
160	3313	last	7–	7	241	7392	4	90	91
166	3624	4	36	35	249	7795	4	3839	3840
191	4888	3	1377	1277	264	8505	5	4	3
193	4980	5	1	2	272	8911	4	72	82
197	5197	4	60	64	275	9065	5	4	3
222	6411	4	53	33	276	9103	3	0437	8437
229	col. *n*		7690	6790	288	9730	5	3	4
233	6983	3	1900	2000					

that the table may be checked against the manuscript table to 28 places of Dr. Edward Sang, which gives logarithms as far as the number 20,000, and which is in the possession of the Royal Society of Edinburgh.

It will be noticed that the errata in the logarithms in Column A are largely in the 10th figure, and most of these were discovered by comparing them with the 10 place logarithms in Vega's *Thesaurus*. Many of the errata in the logarithms in Columns B, C and D were discovered by comparing logarithms in the three columns on the same line. The differences in these logarithms would be about the same and any variation was investigated. For this reason, it is hoped that most of the errata in the first 10 figures of the logarithms in Columns A and the first 7 or 8 figures in the logarithms in Columns B, C and D have been found.

TABLE II.

COLUMNS C.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
101	0381	3	6629	6619	160	3346	3	34	31
101	0382	3	0059	0049	179	4291	3	3758	5758
101	0383	3	3488	3478	214	6033	3	9353	9853
101	0384	3	6918	6908	229	6755	3	9337	9336
101	0385	3	0347	0337	229	6756	3	41	40
101	0386	3	3776	3766	229	6757	3	46	45
101	0387	3	7206	7196	229	6758	3	50	49
101	0388	3	0635	0625	229	6759	3	54	53
101	0389	3	4065	4055	232	6929	3	97	92
135	2053	3	11	16	254	8025	4	2	3

TABLE II.

COLUMNS D.

Page	Log. of	Figure Group	Instead of	Should be	Page	Log. of	Figure Group	Instead of	Should be
99	0266	2	85	55	208	5729	3	7298	7308
115	1066	3	5891	5791	219	6289	2	62	12
121	1399	2	25	75	231	6859	3	5285	2585
128	1707	3	4067	4068	242	7445	3	2	3
130	1817	3	1207	1307	256	8145	3	2	3
139	2296	3	4003	4013	262	8420	3	8	3
179	4291	1	2762	5762	270	8809	2	56	57
186	4600	3	7461	5461	276	9115	2	3585	9585
191	4895	3	1	2	282	9405	3	0390	3960
203	5455	3	7239	7639	286	9611	3	0406	0426
204	5546	2	95	85	291	9887	3	6054	6954

xii	$N = 71$	1	3202 –	2302 –
263	col. 1	line 1	(11)(7)	(7)(11)

The figure groups in the above are numbered from the right.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Cours complet de Mathêmatiques Spêcielles, Vol. II, Geometrie. By J. HAAG. Paris, Gautier-Villars, 1921. 651 pages.

Volume I of this work, entitled *Algèbre et analyse,* appeared in 1914 and the author explains that the delay in the appearance of the second volume was due to the war. This book is divided into two parts, the first treating general problems in geometry, while the second part is concerned wholly with conics and quadrics. The method used is principally analytic but the author does not hesitate to use synthetic methods when it seems to him desirable to do so. Plane and solid geometry are treated in close proximity to each other. We find one chapter entitled the straight line in the plane while the next is entitled the plane in space, and so for many other topics.

The presentation is clear, concise and appealing. The arrangement is admirable. It has long been considered in this country that conic sections should be the first subject in geometry to present to the student. This course usually precedes calculus. Here we find Professor Haag presenting a course in geometry after calculus and placing the study of conics and quadrics at the end. Most of the elementary theorems of geometry are presented. I shall not attempt to give a detailed discussion of the contents; but in order to give an idea of its scope, shall mention a few chapters.

In Part I, chapter VIII, on vectors, treats sums, differences, scalar and vector products of vectors. Chapter IX is on anharmonic ratio and involution; chapter X, on trilinear and tetrahedral coördinates. Chapter XIV presents a study of a curve in the neighborhood of a point. Both plane and twisted curves are considered and the appearance of the latter when looked at along the tangent, normal or binormal, is given. Chapter XVIII is on unicursal curves and surfaces; chapter XXIII, on curvature of lines traced on a surface; chapter XXIV, on geometric problems which lead to differential equations; chapter XXVI, on ruled and developable surfaces; chapter XXVII, on systems of lines; chapter XXVIII, on transformations. Part II comprises 220 pages, and is all devoted to curves and surfaces of the second order.

C. L. E. MOORE.

Die Geburt der modernen Mathematik. II. Die Infinitesimalrechnung. By HEINRICH WIELEITNER. Karlsruhe in Baden, G. Braun, 1925. 72 pages. Price, 1 gold mark.

The second part of Dr. Wieleitner's little essay *Die Geburt der modernen Mathematik* carries out the policy of the first brochure. It consists of six chapters: the first relating to the introduction of the infinite series into elementary mathe-

matics; the second, to the concept of limits and of integrals; the third, to the development of the integral concept; the fourth, to the development of the differential calculus; the fifth, to the Leibniz and Newton contributions and to the dispute concerning priority; and the sixth, to a general survey of the problem.

There is a brief bibliography of works on the history of the calculus, and a valuable index of names, with the dates. With respect to the latter the reader will have to use caution in following the dates given. Some are questioned, which is quite proper, but others, which are equally doubtful, are stated as if they were certain. The reader will also need to observe that Dr. Wieleitner has not followed any international scheme of spelling of proper names, and, indeed, seems not to have read the proof with his usual care, as witness the name of Diophantus.

As to the work itself, it accomplishes in a satisfactory manner the purpose which the author and editors have in mind; namely, the setting forth of the critical points in the development of the infinitesimal calculus. The reader will find here in brief form a statement of the steps that were taken in reaching the stage which may be characterized by the name of Cavalieri, and then the subsequent stages from there on to the completion of the work in the period of Leibniz. He will also find a subsequent statement of the priority question with respect to Leibniz and Newton, a question which probably would never have arisen if it had not been for certain busybodies who stirred up the strife. As a brief summary the pamphlet will be of considerable service.

DAVID EUGENE SMITH.

Studies in Human Biology. By R. PEARL. Baltimore, Williams and Wilkins, 1924. 650 pages, 123 charts. Price $8.00.

This is essentially a volume of collected papers, as only a very few of the chapters have not previously appeared in the periodical literature. A review of this book in a sense is a review of the human activities of one biologist. Raymond Pearl is Professor of Biometry and Vital Statistics in the School of Hygiene and Public Health of the Johns Hopkins University. Consequently we find, as we would expect, that this book is limited to those phases of human biology that lend themselves well to statistical analysis and have bearing on problems of health and the questions that may be answered from the data of vital statistics. A glance through the table of contents shows that such a delimitation still permits of the study of almost all of the aspects of human biology, for we find man treated both as an animal and as a population problem.

The first chapter presents a detailed statistical study of five series of brain weights taken from different countries. Definite racial types are found. Brain weight correlates very poorly with body weight. The sexes are equally variable. Brain weight does not appear to be sensibly correlated with intellectual ability. These are only a few of the questions that are answered by the array of statistical data of this chapter. The next chapter tells us that there is very slight correlation between intelligence and the size of the head. Two methods for obtaining

correlation coefficients were used, the contingency method giving $0.14 \pm .04$ and the Pearson four-fold method $0.103 \pm .034$. This data would mean more to many readers if the long German statement of the material used (page 94) had been briefly summarized in English for the benefit of those who do not read German fluently.

The following two chapters on the sex ratio complete the first part of the book, which considers man as an animal. Data which show whether the children's parents are Italian, Spanish, Argentine, or combinations of these, from the vital statistics of Buenos Ayres, are used to show that the sex ratio, or the number of males per hundred females, is slightly higher from cross than from pure matings. Environmental and demographic conditions were analyzed, but failed to explain this difference in the sex ratio. Evidence is given to show that there is no relation between the time of fertilization and the sex ratio.

Part II is concerned with the biological aspects of vital statistics. Chapter five shows that if we measure variability of the two sexes by means of the congenital malformations which lead to the death of the individual, we find the female sex is slightly more variable than the male. This is shown by a study of the means, standard deviations and coefficients of variation. This chapter brings to the reader's attention the fact that vital statistics may be used to study evolution in the actual elimination of the unfit in the struggle for existence.

For deaths under one month and under one year the center of mortality occurs at 0.3 month and 0.3 year respectively. For the interval of under five years the center occurs at one year.

Next Pearl turns to the vitality of the people of America. This vitality is measured by the percentages of marriages producing children during 1919, the number of deaths, and the vital index or the ratio of the births to the deaths. He shows that there is a very considerable actual fusing going on between the foreign and native population. The vitality of the foreign born is greater than that of the native born. During 1919 the parents of 65% of the children born were native, of 10% one parent was native, and of only 25% were both parents foreign. Such data is more cheerful and helpful than the speculations of the sociologist. The trends of four vital indices,

$$\frac{100\ (\text{Births of whites of native parents})}{\text{deaths of native whites}},$$

$$\frac{100\ (\text{Births of whites both parents foreign})}{\text{deaths of foreign born whites}},$$

$$\frac{100\ (\text{Births of negroes})}{\text{deaths of negroes}};$$

and

$$\frac{100\ (\text{Births of whites})}{\text{deaths of whites}},$$

are presented using data from both rural and urban groups.

Following the treatment of vitality we find an analysis of the incidence of tuberculosis and how it is related to the constitution of the individual. Pearl seems to favor the hereditary factors rather than the infection factor though his study shows, however, that we must know more before we can justly reach any conclusion. Another chapter studies the longevity of the parents of the tuberculous and cancerous.

One of the most interesting chapters in the book is the one wherein various occupations of the British are classified into five groups based on the amount of physical activity involved. Then the mortality of each group at various ages was studied by Pearl in order to find out what relation, if any, existed between physical activity and mortality. After occupational accidents and hazards were excluded, he found that up to the age of about 40 there was no relation, but after age 40 a higher mortality was associated with the more strenuous activities. Outdoor occupations were found to be more healthful than indoor occupations. Pearl analyzed his data by means of differences in mortality after dividing the data into groups and the groups into quintiles.

Mortality tables are in existence for three animals only, man, the fruit fly and a rotifer. When these results are placed on a comparable scale, we find the highest mortality in the fruit fly, man is next and the rotifer has the least mortality. The curves are very similar.

Part III deals with public health and epidemiology. First Pearl presents an analysis of our food consumption in terms of primary food and secondary food. The secondary food includes those sources that are produced by the use of primary food, such as meat, etc. Of our food 47% was primary and 53% secondary. By studying the graphs for the different foods we discover, perhaps with surprise, that wheat furnished the greatest percentage of both our protein and carbohydrate. Also that we consume about 120 grams of protein a day when we actually need only 50 to 75 grams. Another chapter shows that the food conservation campaign during the war was really successful in that it cut down the amount of garbage produced in some 96 cities. The reviewer wonders if the price of food and its obtainability might not also have had some effect.

Before studying epidemics Pearl derives an index for the age distribution of a population. A very long chapter analyzes the influenza pandemic of 1918 showing how greatly it varied in different cities. Besides the duration of an epidemic, its standard deviation and number of peaks, the author suggests a number of indices for use in studying an epidemic.

The last part of the book is concerned with the population problem. The treatment starts by going back to Malthus and then pointing out that modern statistics show that population is outstripping food production though it is not following the particular mathematical expression of Malthus. Then the author stops to discuss "biology and the war," reviewing the causes of war as part of the inherent make-up of man. Pearl devotes several pages to the German military use of Darwinism to develop their war philosophy. He concludes from his statistical study of the war that a nation neither gains nor loses by war.

If one is interested in the stability of population, the data from England may be investigated. Here the ratio of 100 times the births divided by the deaths gradually increased from 1840 until the war with two exceptions due to influenza. During the war it dropped, but by 1920 it had risen to a higher place than the projection of the previous curve would indicate. The last two chapters of the book present Pearl's curve for population growth and its application to the growth of several countries. The first empirical curve of Pearl's suitable for this type of growth was a logarithmic parabola. While this curve gave good results, it was not in a general form. His general curve came from a study of the following five fundamental considerations: that a population must have a finite limit of area, an upper limiting asymptote, a lower limiting asymptote $= 0$, cycles of growth—successive ones being additive, and that the general shape of growth curves is that of an S. The general curve is of the form $x = k/(1 + me^{f(x)})$, where the $f(x) = a_1x + a_2x^2 + a_3x^3 \cdots$. This curve represents well the population growth in several countries and in a city. Methods for the actual fitting of the curve are given with the probable errors of the constants. For the growth of the United States $k = 197.27 \pm 0.55$, $m = 67.32 \pm 0.17$, and $a_1 = -0.0313 \pm 0.00013$. Pearl, after discussing the errors involved in the procedure, extrapolates the curve and predicts that we will reach the limiting value of our population of about 197 millions very shortly after 2100 A.D. The curves for population growth in 12 European countries and in Baltimore, Maryland, are given as well as that for the world. The limiting asymptote for the world population is given at approximately 2026 million.

Probably few books present as great a cross section through modern life and living conditions. For a group of collected papers it is quite coherent. This type of book is open to the objection that many of the chapters are so detailed as to detract from their interest to anyone not doing special work in the particular subject. Apparently the chapters are reprinted with a minimum of change from their original form of publication. One may, of course, skip that which does not interest him or is too detailed, so it is a moot point whether greater revision would have been desirable. The book should awaken greater interest in the field of vital statistics, both toward further analysis and toward making the reporting of them more comprehensive and accurate. With possibly a few exceptions the subjects treated are of great enough importance to merit the attention and careful consideration of all modern thinkers, and the evidence from this form of study should be welcomed and suitably evaluated. Notwithstanding the fact that much of the content of these studies is in the nature of a first approximation, they are truly a contribution to human biology which amply warrants their being brought together in this form. O. W. RICHARDS.

Medical Biometry and Statistics. By R. PEARL. Philadelphia, Saunders, 1923. 379 pages, illustrated. Price $5.00.

The mathematician will be interested and, perhaps, surprised to find a book devoted entirely to medical and biological statistics among the several new books

on this subject. Since the biological and medical sciences are fast becoming users of mathematics, it is unfortunate that more mathematicians are not interested in this new field which offers many problems of a new and more complex order than those of chemistry and physics.

This work on statistics is a new contribution to the subject, based on many years of teaching experience, and represents a novel method of teaching the subject. Raymond Pearl, Professor of Biometry and Vital Statistics in the School of Hygiene and Public Health at Johns Hopkins, has presented the subject with a minimum of formal mathematical proof, essentially for non-mathematical readers, but in such a logical manner that they can obtain an accurate knowledge of the limitations of the various methods and measures. Since the book is designed for the medical student, we find a considerable portion of the book devoted to vital statistics. To this extent it is specialized and does not represent the broader field of biological statistics. If the mathematician will keep this limitation in mind, he may turn to Pearl's book to see how rapidly this field is growing and for a further proof that mathematics is fast developing and is not the stationary subject that his college major students sometimes find in their texts.

The author first shows the need that the medical man has for statistical aid and then defines the subject and its general terms. Then he shows the student very briefly how vital statistics has grown from the first weekly bills of mortality in 1532 to its present status. This is followed by a short history of biometry. Chapter three presents the "raw data of biostatistics" according to the three ways of obtaining it, the census, registration, and the medical case record. This chapter is illustrated with census forms and records and the international list of the causes of death. The next chapter discusses, with many examples, the ways that such data may best be tabulated. The question of size of the class interval is well treated. A short chapter familiarizes the student with mechanical tabulation with regard to hospital records by use of Hollerith machines.

The sixth chapter contains an exceptionally fine treatment of graphical methods. Besides a complete discussion of all the common forms of graphs, we find methods for the preparation of nomograms. The need for arithlog graphs is indicated by the use of death rates. The following chapter teaches the student a proper conception of rates and ratios by showing him the differences between crude death rates and specific death rates and how the several vital indices are used. Should the reader desire to know just what a life table contains and something of the methods used in its preparation, he will find the answer in chapter eight. Then follows a chapter on "standardized and corrected death-rates" which indicates methods for determining the effect of age and sex on death as well as the relative health of the group considered.

At this point Pearl introduces the probable error concept showing how sampling may affect the results and also how it may be used as an aid in determining the reliability of the various measures used. This leads to an elementary consideration of probability, which is related to the life of the student by discussing the results of tossing a coin. The treatment is entirely limited to the

probability of experience that may be used in practical application. From this he leads to permutations and combinations, and then to the binomial theorem. The probability of concurrent events is briefly but adequately presented. From these concepts the author develops the normal curve as the limit of the point binomial when the probabilities are equal. Such development is perhaps suitable for the use made of the normal curve but a broader treatment of the binomial curve and its relation to the normal curve which can be developed by shifting the origin and by the use of Stirling's formula for large factorials would have been of use to the student of biometrics. The use of the normal curve and tests of sampling are presented with data from vital statistics.

The thirteenth chapter of the book takes up the methods of measuring variation such as the mean, mode, standard deviation, skewness, etc. It is interesting to note that while the standard deviation has appeared before, using this method of approach, these measures are left to almost the end of the book. The subject of correlation is treated with considerable completeness both for linear and non-linear regression. Another chapter contains an excellent brief discussion of partial correlation. Biological data are used for illustrating all of these methods. Several of the type curves of Pearson are illustrated by the variation of infant mortality in different population groups.

The last chapter is devoted to simple curve fitting by the method of least squares and is illustrated by fitting a straight line, a parabola, and a logarithmic curve to data representing the sitting heights of embryos.

The first appendix gives age and sex specific death rates. The second lists books and tables useful in the calculations; and the third contains an extensive list of formulæ from algebra, trigonometry, and calculus, as well as certain constants. The last two appendices contain a short table of the normal probability integral and sums of logarithms respectively. Bibliographic references are given at the end of each chapter and as occasional footnotes.

The book is well written and sufficiently non-technical to permit the advanced student or medical practitioner to use it profitably and with some appreciation of the logic back of the methods. Since most of the first part of the book pertains to vital statistics, and since most of the illustrative material is taken from that field, only parts of the book will interest the biology student. The book might have been made more useful to the biologist by the addition of methods for curve analysis and fitting other than least squares, and of a statistical discussion of problems of growth and of dynamic equilibria. The mathematician will find the book useful in preparing courses for students of public health and biostatistics, as well as those of medicine.

O. W. RICHARDS.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS.

Functional Equations.

By B. H. Brown, Dartmouth College.

Cauchy [1] showed that the only real continuous solutions of the functional equations:

$$
\begin{array}{ll}
I & \phi(x+y) = \phi(x) + \phi(y), \\
II & \phi(x+y) = \phi(x) \cdot \phi(y), \\
III & \phi(xy) \quad\ = \phi(x) + \phi(y), \\
IV & \phi(xy) \quad\ = \phi(x) \cdot \phi(y);
\end{array}
$$

are ax, a^x, $a \log x$, x^a respectively.[2] These derivations are simple and are well adapted for a mathematical club talk.

Discontinuous solutions of these equations exist [3]; but the proof is neither simple nor suitable for our purposes. What is of very great interest is the totally depraved nature of these discontinuous solutions and the very slight restrictions which suffice to establish continuity. Thus a solution of Equation I is continuous under any one of the following restrictions:

(*a*) If $\phi(x)$ is continuous in the neighborhood of one value of x,

(*b*) if $\phi(x)$ is positive when x is positive,

(*c*) if there is any rectangle in the xy plane which does not contain a point of the locus $y = \phi(x)$.

Finally [4] a fifth functional equation

$$V \quad \phi(x+y) + \phi(x-y) = 2\phi(x) \cdot \phi(y)$$

has for continuous solutions: $\cos ax$; $(A^x + A^{-x})/2$.

Application A. Equation V is of importance in statics in establishing the theorem of the parallelogram of forces.[5] The same theorem is established by Darboux [6] by the use of Equation I; the statical hypothesis which suffices for continuity is worthy of attention.

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[1] *Cours d'Analyse*, Paris, 1821, pp. 103ff.; for a more recent treatment *cf.* Tannery, *Théorie des fonctions d'une variable*, sec. ed., Paris, 1904, p. 275.

[2] There are, of course, the trivial particular solutions $\varphi(x) \equiv 0$ for Equations II and IV.

[3] Hamel, *Math. Ann.*, vol. 60, 1905, p. 459; Schimmack, *Diss.* Halle, 1908.

[4] Other functional equations arising from problems in permutations and combinations, applying to integral values of the argument may be found in treatises on algebra. *Cf.* C. Smith, *A treatise on algebra*, London, 1888, pp. 286, 287.

[5] Voss, *Enc. der Math. Wiss.*, IV, 1, p. 66. *Cf.* d'Alembert, Paris, *Mem. de l'Acad.*, 1769, p. 278.

[6] *Bull. sci. math.*, vol. 9, 1875, p. 281.

Application B. Equation V serves as a definition of the circular functions, and shows the close relationship between these functions, spherical trigonometry, the plane Euclidean and non-Euclidean trigonometries, and statics.[1]

Application C. THEOREM: *If there exists a correspondence between two series of elements which preserves harmonic ratio, the correspondence is projective.* This theorem, without any assumption of continuity for the correspondence, is established [2] by the aid of Equation I and restriction (b) which is a consequence of our hypothesis.

Application D. Equation II has been used to establish Laguerre's projective definition of angle.[3] If OP and OQ are the minimal lines through the vertex of the angle $\theta = AOB$, Laguerre showed that

$$\theta = (\log \{AB;\, PQ\})/2i,$$

where $\{AB;\, PQ\}$ is the cross-ratio of the four lines OA, OB; OP, OQ. To prove this [4] we first show that the cross-ratio is independent of translation and rotation and hence is actually some function $f(\theta)$ of the angle θ. The well-known identity

$$\{AB;\; PQ\} = \{AC;\, PQ\}\cdot\{CB;\; PQ\}$$

is then equivalent to $f(\theta_1 + \theta_2) = f(\theta_1)\cdot f(\theta_2)$, or Equation II. Hence

$$f(\theta) = a^{\theta},$$

$$\theta = a' \log \{AB;\; PQ\},$$

and a' may be shown to be $1/2i$ for the special case $AOB = \pi/2$.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF COOPER UNION, New York City.
[*1924*, 496.]

The officers for the year 1924–1925 were: President, Samuel Lubkin '27; Vice-president, Fred Miller '26; Secretary-Treasurer, Sydney Helprin '27. Meetings were held at intervals of three weeks throughout the year. The slide-rule presented by the Club to the member of the first year class having the highest standing in mathematics was won by Peter Douglas '28.

Following is the program for the year:

October 28, 1924: "Mathematics in industrial research"—Thornton C. Fry of the Western Electric Company.

November 18, 1924: "Introduction to the solution of modern geometric problems"—Aaron Rabinowitz '27.

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[1] *Cf.* Andrade, *Bull. Soc. math.*, vol. 28, 1900, p. 58.

[2] Darboux, *Math. Ann.*, vol. 17, 1880, p. 54.

[3] Laguerre, *Nouv. Ann.*, (1) vol. 12, 1853, p. 64; *Oeuvres* 2, Paris, 1905, pp. 12–13; but see also *Enc. der Math. Wiss.*, III AB9, p. 901 where Laguerre's result as a *definition* is credited to Cayley.

[4] The spirit if not the letter of this method is due to Klein, *Math. Ann.*, vol. 4, 1871, p. 584. In the same spirit Equations I, II, and V are frequently employed in connection with angle and distance in the non-Euclidean geometries.

December 9, 1924: "Notions about vectors"—Thaddeus Slonczewski '26.
January 6, 1925: "Some original methods of solving geometrical problems"—Samuel Lubkin '27.
January 27, 1925: "The mathematics of linkages, with an exhibition of models"—Samuel Lubkin '27.
February 17, 1925: "Some results in the theory of numbers"—Fred Miller '26.
March 10, 1925: "When is a check not a check?"—Sydney Helprin '27.
March 31, 1925: "History of the development of the slide-rule"—Edwin Schwarz '27.
April 21, 1925: "Crinkly curves"—Professor H. W. Reddick. Election of officers.

(Report by Sydney Helprin, Secretary.)

MATHEMATICAL CLUB, NEW JERSEY COLLEGE FOR WOMEN, New Brunswick, N. J. [*1924*, 352.]

At the regular monthly meetings of the Mathematical Club at the New Jersey College for Women, Rutgers University, the following papers were read:
"Life and work of Archimedes" by Elizabeth Sinton.
"Life of George Boole" by Janey Kudlich.
"The Solution of some problems" by Dorothy McFarland.
"Invariants and covariants"
"Problems involving complex numbers" } by Professor Richard Morris.
"The discriminant"
"History of complex numbers" by Katherine Mitchell.
"A geometrical fallacy" by Marjorie Cadwallader.
"Problems involving the law of cosines" by Professor Starke.
"Mathematics in the industries" by Dorothy Brown.
"Taylor's theorem for 2 and 3 variables" by Vera Joslin.
"Euler's theorem for 3 or more variables" by Myrtle Hughes.
"Life of John Landen" by Ada Salmon.

(Report by Professor Morris.)

MATHEMATICAL CLUB, RUTGERS UNIVERSITY, New Brunswick, N. J. [*1924*, 352.]

The officers of the Men's Mathematical Club of Rutgers University for the past year were: President, R. M. Walter; Vice-president, L. J. Paradiso; Secretary-Treasurer, R. J. Seeger.

The following papers were read at the regular meetings during the year:
"Complex numbers" by Professor Brasefield.
"Time solids" by W. H. Mitchell, Jr.
"Method of least squares" by Harry Rolnick.
"Applications of invariants and covariants" by Professor Morris.
"Orthogonal transformations" by L. J. Paradiso.
"Theorems of Pascal and Brianchon" by W. H. Mitchell, Jr.
"Hyperbolic functions" by R. M. Walter.
Statement of the problem of three bodies by R. J. Seeger.
Transformation of the general elliptic integral by L. J. Paradiso.
Dimensional units by Harry Rolnick.

(Report by Professor Morris.)

WHITE MATHEMATICS CLUB, UNIVERSITY OF KENTUCKY, Lexington, Ky. [*1924*, 452.]

Oct. 10, 1924. Election of the following officers: President, Professor J. M. Davis; Secretary, Professor H. H. Downing. It was decided to make Eddington's book on *Einstein's theory of relativity* the basis for the year's club work and to have joint meetings with the physics club to consider the problem from a physical as well as a mathematical point of view.
Oct. 16. "Charts of mathematical history" by Professor E. L. Rees.
Oct. 30. "What is geometry? Relativity" by Dr. P. P. Boyd.

Nov. 11. "Demonstration of the Michelson-Morley light experiment" by Professor W. S. Webb, physics department. "Discussion of the Michelson-Morley light experiment" by Dr. Otto Koppius, physics department.
Nov. 18. "Lorentz's equations in the theory of relativity" by Dr. M. N. States, physics department.
Dec. 18. "Applications of the Lorentz equations to certain problems" by Professor J. M. Davis.
Jan. 15, 1925. "Time, the fourth dimenson" by Professor E. L. Rees.
Feb. 5. "Fields of force" by Professor H. H. Downing.
Feb. 19. "Kinds of space" by Dr. F. Elizabeth Le Stourgeon.
Feb. 26. "Postulates for the Lorentz transformations" by Mr. J. C. Nixon, instructor.
Mar. 5. "The Bucherer experiment" by L. A. Pardue, '25.
Mar. 11. "The mathematics of Einstein's law of gravitation" by Professor E. L. Rees.
Mar. 16. "Geodesics in general relativity" by Professor H. H. Downing.
Apr. 16. "Mathematical wrinkles" by all present.
May 7. "Properties of congruences" by Mr. D. E. South, instr.
May 14. "Vector treatment of the motion of a rigid body in a plane" by Mr. T. Andrew, instr. "Algebraic solution of quartic equations" by R. S. Park, '25.
May 21. "Some mathematicians who have contributed to the theory of equations" by Grace Richards, '25. "Proof of Pascal's theorem" by M. C. Brown, gr.
May 27. "Algebraic equations" by Mary H. Cooper, '25. "Determinants" by Eva Weller, '25.

(Report by Professor H. H. Downing.)

The Mathematics Club of the University of Kansas, Lawrence, Kansas. [*1921*, 88.]

The officers of the Mathematics Club of the University of Kansas for the year 1924–25 were: President, Mildred Woodside '25; Vice-president, Maude Long '25; Secretary-Treasurer, Violet Shoemaker '25; Faculty Adviser, H. E. Jordan.

The following topics were presented at the meetings:
October 6, 1924: "The greatest word in mathematics" by Professor G. W. Smith.
October 20: "Railroad curves" by Professor F. A. Russell.
November 3: "Various proofs of the Pythagorean theorem" by Violet Shoemaker '25.
November 17: "The slide rule" by Professor H. E. Jordan.
December 1: "Magic squares" by Lucille Heil '25.
December 15: "Graphical solution of cubic equations" by Forrest Noll '25.
January 12, 1925: "Newton and Leibniz" by Maude Long '25; "Properties of the catenary" by Mildred Woodside '25.
February 23: "Japanese mathematics" by Leta Galpin '25.
March 5: "Schuster's periodogram" by Professor Dinsmore Alter.
March 16: "Solution of equations by the iteration process" by Elizabeth Bolinger '26.
April 6: "A new method of extracting roots of numbers" by C. A. Reagan, Gr.
April 20: "Interpolation formulae" by Wesley M. Roberds '25.
May 4: "Normals to a parabola" by Lester Lehnberg '25.
May 18: Annual picnic.

(Report by Miss Shoemaker.)

Goucher College Mathematics Club, Baltimore, Maryland. [*1925*, 91.]

Nov. 6. "The trisection of an angle."
History of the problem; solution by the hyperbola and quadratrix, by Sara Scott '27.
Proof of the incorrectness of a proposed construction, by Anna Grimm '26.
Analytic proof that trisection by ruler and compass is impossible, by Elizabeth Bauernschmidt '25 and Louise Kinnamon '25.
Nov. 20. "Numerical calculation."
Calculating prodigies, by Amelia Frank '27.

History and construction of adding machines, by Margaret Heinzerling '26 and Gwendolyn Eichhorn '26.

Tricks with numbers by Ruth Farr '25.

Dec. 18. "The theory of relativity" by Professor F. P. Lewis.

Feb. 12. "Typical theorems of pure projective geometry" by Catherine Snyder '25.

"The slide rule" by Marvell Weinberg '27 and Lillian Peper '27.

Report on the December meetings of the Mathematical Association in Washington, Professor Bacon and Professor Lewis.

March 12. "Mathematics in sculpture and painting" by Professor Hans Froelicher, Department of Art, Goucher College.

April 16. "The nine-point circle and Feuerbach's theorem" by Leah Seidman '26.

"A ruler and compass construction for the regular pentagon and decagon" by Doris Heineman '27.

May 21. Annual club picnic.

(Report by Professor Lewis.)

Mathematics Club of the State University of Iowa, Iowa City, Iowa. [*1924*, 496.]

The following officers were elected for the year 1924–1925: President, Howard K. Hughes, G; Secretary-Treasurer, Arthur H. Blue, G; Faculty Adviser, Professor E. W. Chittenden. The attendance has been about thirty at each of the nine meetings. A membership fee of twenty-five cents has amply provided for tea and cakes, served before each program. The following topics were presented at the meetings:

"Pencils of conics" by H. K. Hughes, G.

"Duhamel's theorem" by R. J. Hannelly, G.

"An octaval number system" by Frances E. Baker, G.

"Accuracy of interpolation" by A. H. Blue, G.

"Applications of determinants to geometry" by Eva M. Schillig, G.

"Hyperbolic functions" by W. J. Davidson, '25.

"The steel square" by W. L. Hunter.

"Quadrature formulas" by J. Van S. Longenecker, G.

"The incomplete gamma function" by Clair Kirkpatrick, G.

"What are the stars?" by Dr. D. H. Menzel.

(Report by Mr. Blue.)

Mathematics Club of the University of Arkansas, Fayetteville, Arkansas.

During 1924–25 the following topics were treated:

"Some elementary number theory" by F. E. Taylor.

"Continued fractions" by Jewell Hughes.

"Modern geometry" by A. D. Campbell.

"Mathematical recreations" by T. T. Spitzberg.

"History of geometry" by Emily Heston.

"Determinants" by C. T. Willis.

"Relation of mathematics to physics" by S. R. Parsons.

"Theory of probability" T. L. Edmiston.

"A broader idea of integers" by Jewell Hughes.

"The fourth dimension" by A. Campbell.

"Conic sections" by F. E. Taylor.

"The Game of Nim" by Grace Harrison.

(Report by Professor Campbell.)

Simpson Mathematical Society, University of Florida, Gainesville, Florida.

The Simpson Mathematical Society of the University of Florida was organized this year. All interested in mathematics are eligible to membership. The officers of the Society for the spring of 1925 were as follows:

Z. M. Pirenian, President; V. C. Steen, Vice-president; S. H. Huffman, Secretary; W. Stanwix-Hay, Reporter.

The following topics have been discussed in the regular weekly meetings:

March 10. The equation $e^{i\pi} = \cos \pi + i \sin \pi = -1$.
March 18. Mathematical induction; Euclid's fifth postulate.
March 25. Cardan's solution of the cubic; Mathematical puzzles.
April 1. The teaching of mathematics; Mathematical tricks.
April 8. Magic squares; Complex numbers.
April 15. Theory of investment problem; Solution of the quartic.
April 22. The fourth dimension; Hyperbolic functions.
April 29. Einstein theory; Mathematical puzzles.
May 6. Theory of probability; Mathematics of biology.
May 13. Life of Leibniz; The theory of iteration.
May 20. Logarithms; The slide rule.

(Report by S. H. Huffman, Secretary.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3144. Proposed by H. S. CARSLAW, The University, Sydney, Australia.

Let

$$y = \frac{1}{n}\sum_{1}^{n} (n - r + 1) \frac{\sin (2r - 1)x}{2r - 1} = S_n(x). \qquad (0 \leqq x \leqq \tfrac{1}{2}\pi)$$

For a given positive integer n, it is easy to show that the maxima occur when x is $\frac{\pi}{n+1}$, $\frac{2\pi}{n+1}$, $\frac{3\pi}{n+1}$, $\cdots$, the last being at $\frac{1}{2}\pi$, if n is odd, and at $\frac{1}{2}\pi - \frac{1}{2(n+1)}\pi$, if n is even. Also the minima occur when x is $\frac{\pi}{n}$, $\frac{2\pi}{n}$, $\frac{3\pi}{n}$, $\cdots$, the last being at $\frac{1}{2}\pi$, when n is even, and at $\frac{1}{2}\pi - \frac{1}{2n}\pi$, when n is odd.

It is required to show that, for a given n, the ordinates of the maxima continually increase as x increases from 0 to $\frac{1}{2}\pi$, and similarly for the minima. Also that the ordinate of the last maximum continually diminishes as n increases, and that its limit is $(\pi/4)$ when $n \to \infty$.

3145. Proposed by W. D. CAIRNS, Oberlin College.

The center of gravity of any zone of a certain surface of revolution lies midway between the bases of the zone. What is the surface?

3146. Proposed by W. C. EELLS, Whitman College.

For what integral values of s and n is it true that the nth power of any integer leaves the same remainder when divided by s as does the integer itself when divided by s? (Generalized from Problem 873, *School Science and Mathematics*, March, 1925.)

3147. Proposed by J. A. BULLARD, U. S. Naval Academy.

The cylinder $(x/a)^2 + (y/b)^2 = 1$, passes through the sphere, $x^2 + y^2 + z^2 = a^2$. If e is the eccentricity of the ellipse $(x/a)^2 + (y/b)^2 = 1$, show that
(a) the surface cut from the sphere $= 8a^2 \arccos e$,
(b) the volume cut from the sphere $= 8a^3[e^2\sqrt{1-e^2} + \arccos e]/3$.

2677 [1918, 75]. Proposed by R. K. MORLEY, Worcester, Mass.

A quarter-mile track is to be constructed, having semi-circular ends and straightaway sides. It is required to have the rectangular part of enclosed field referred to in No. 12, Granville's *Calculus*, page 116, as large as possible. Find length of the straightaways.

Also, required to inscribe the maximum rectangle in a track of length l, with semi-circular ends and straightaway sides, assuming that two sides of the rectangle are parallel to the straightaways. Find the length of the straightaways and the dimensions of the rectangle.

2762 [1919, 70]. Proposed by N. P. PANDYA, Amreli, India.

$ABCD$ is a cyclic quadrilateral inscribed in an ellipse. $AB = 2BC$ and $CD = 2DA$. Find the eccentricity of the ellipse in terms of the sides of the quadrilateral.

2769 [1919, 171]. Proposed by B. J. BROWN, Kansas City, Mo.

Expand in powers of x as far as x^2 the function $\dfrac{\cosh \lambda x}{\cosh \lambda} - x\dfrac{\sinh \lambda x}{\sinh \lambda}$ in which λ is a positive constant.

Prove that, if $\lambda \tanh \lambda > 2$, the function has only one maximum value for $x > 0$ and that the value of x for which the maximum occurs is less than 1. (India Civil Service. 1912.)

2770 [1919, 171]. Proposed by A. M. HARDING, University of Arkansas.

Solve the differential equation

$$\frac{d^3x}{dt^3} - 2(\mu t + \lambda)\frac{dx}{dt} + (1 - \mu)x = 0,$$

where λ and μ are constants.

SOLUTIONS.

2874 [1921, 37]. Proposed by J. L. RILEY, Stephenville, Texas.

Show that the equation,

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + k = 0,$$

has some imaginary roots if $a^2 - 2b < n\sqrt[n]{k^2}$; $a, b, \cdots, k$ are supposed real.

Note. This result is a particular case of a theorem contained in a paper published a few years ago.—EDITORS.

SOLUTION BY OTTO DUNKEL, Washington University.

A proof of the theorem mentioned in the note above may be found in the paper by Dunkel on "Generalized Geometric Means and Algebraic Equations" in *Annals of Mathematics*, 2d ser., vol. 11, no. 1, October 1909, pp. 24–25.

3103 [1924, 498]. Proposed by OTTO DUNKEL, Washington University.

Given the equation

$$y = \sum_{i=1}^{i=\infty} \frac{(i+1)^{i-1}}{i!}x^i, \qquad |x| < 1/e,$$

find x in terms of y in finite form.

Solution by the Proposer.

It will be more convenient to write

$$y = \sum_{i=0}^{i=\infty} \frac{(i+1)^{i-1}}{i!} x^i. \tag{1}$$

This change in the lower limit does not alter the problem in any essential respect.

The following identity in a will be proved which will be used later:

$$\begin{aligned}(t+a+1)^t &= \sum_{i=0}^{i=t} (a+i)^i(t-i+1)^{t-i-1}{}_tC_i, \left({}_tC_i = \frac{t!}{i!(t-i)!} \right), \\ &= \sum_{i=0}^{i=t} (a+t-i)^{t-i}(i+1)^{i-1}{}_tC_i.\end{aligned} \tag{2}$$

Proof. In the second form of (2) set $(a+t-i)^{t-i} = [(a+t+1)-(i+1)]^{t-i}$, and develop this last expression by the Binomial Theorem. The right side of (2) then becomes the double sum

$$\sum_{i=0}^{i=t} \sum_{j=0}^{j=t-i} (a+t+1)^j(i+1)^{t-j-1}{}_tC_i\ {}_{t-i}C_j(-1)^{t-i-j}.$$

Since ${}_tC_i\ {}_{t-i}C_j = {}_tC_j\ {}_{t-j}C_i$, we may write the above double sum in the form

$$\sum_{j=0}^{j=t} (a+t+1)^j\ {}_tC_j(-1)^{t-j} \sum_{i=0}^{i=t-j} (i+1)^{t-j-1}\ {}_{t-j}C_i(-1)^i.$$

Now the last single sum (with respect to i) is zero, if $j \neq t$ (see the solution of **3108** [**1925, 388**]); but, if $j = t$ and hence $i = 0$, it is equal to unity. Thus the double sum reduces to the single term $(a+t+1)^t$ and this is what we wished to prove. The first form of the right member of (2) is obtained from the second form by replacing i by $t-i$, and is therefore equal to the second form.

Turning now to (1) we readily verify for the range of values given for x that

$$xy' + y = \sum_{i=0}^{i=\infty} \frac{(i+1)^i}{i!} x^i.$$

Hence

$$\begin{aligned}y(xy'+y) &= \left(\sum_{i=0}^{i=\infty} \frac{(i+1)^{i-1}}{i!} x^i \right)\left(\sum_{t=0}^{t=\infty} \frac{(t+1)^t}{t!} x^t \right), \\ &= \sum_{t=0}^{t=\infty} \frac{x^t}{t!} \sum_{i=0}^{i=t} (i+1)^{i-1}(t-i+1)^{t-i}{}_tC_i.\end{aligned} \tag{3}$$

Setting now in the identity above $a = 1$, we may replace the second single sum in the double summation by $(t+2)^t$. We have then

$$y(xy'+y) = y' \qquad \text{or} \qquad yd(xy) = dy. \tag{4}$$

It now follows that $xy = \log y + A$. In this case $A = 0$, since in (1) if $x = 0$, $y = 1$. We then find the final result

$$x = \frac{\log y}{y} \qquad \text{or} \qquad e^{xy} = y. \tag{5}$$

Remarks. The second equation in (5) occurs in the solution of **3071** [**1925, 139**] with the notation $e^{mh} = m$. Here h is given, and hence (1) gives that solution of the equation which was used in the solution of the problem but which was not written in this explicit form. The analysis in the solution of 3071 with certain additions may be used to give another derivation of the result above. We have thus found the values of the coefficients a_i which occurred in that solution,

$a_i = (i+1)^{i-1}$. The equation (6) of that solution may now be written

$$m = \sum_{i=0}^{i=\infty} \frac{(i+1)^{i-1}}{i!} h^i = \frac{\sum_{i=0}^{i=\infty} \frac{[x+(i+1)h]^i}{i!}}{\sum_{i=0}^{i=\infty} \frac{[x+ih]^i}{i!}} \tag{6}$$

where x may be any number.

It may be of interest to write a second identity which may be proved in a similar manner,

$$\frac{(t+a+1)^t}{a} = \sum_{i=0}^{i=t} (a+i)^{i-1}(t-i+1)^{t-i}\, {}_tC_i. \tag{7}$$

This identity could have been used in place of (1), after setting $a = 1$, but (1) was chosen as corresponding in its first form to (4) [**1925, 139**].

3109 [1925, 46]. Proposed by the late J. W .NICHOLSON.

Find two rational numbers which separate the roots of the equation $x^3 - ax^2 + bx - c = 0$.

SOLUTION[1] BY A. A. BENNETT, University of Texas.

Given the cubic $f(x) = x^3 - ax^2 + bx - c$, we have on differentiating $f'(x) = 3x^2 - 2ax + b$, $f''(x) = 6x - 2a$. The discriminant of f' (according to one definition) may be written as $9l = a^2 - 3b$, and the value of $f(x)$ for $f''(x) = 0$ may be denoted by $q = -2a^3/27 + ab/3 - c$. We may remark that by the translation $u = x - (a/3)$, $f(x)$ may be put in the reduced form $u^3 - 3lu + q$. The discriminant of the cubic is $\Delta = 27(4l^3 - q^2)$. The problem has a meaning only if the roots are real and distinct and therefore only if Δ, and hence also l, is positive.

If l is the square of a rational number, an explicit solution is always afforded by the pair $\frac{1}{3}a + \sqrt{l}$, $\frac{1}{3}a - \sqrt{l}$, since these are the roots of $f'(x) = 0$, which by Rolle's theorem separate the roots of the original equation. If $\sqrt{l}$ is irrational, rational numbers can always be found sufficiently close to the roots of $f'(x) = 0$, so as to still lie in the given intervals. In particular if $q = 0$, $27f(x)$ factors into $(3x - a)[(3x-a)^2 - 27l]$. Thus if r is any nonvanishing rational number for which $r^2 < 3l$, we shall have as a solution the pair of numbers $\frac{1}{3}a + r$ and $\frac{1}{3}a - r$.

We are interested however in an explicit rational formula for the general case, namely for $q \neq 0$ and l not a perfect square. The following pair is in fact such a solution, $\frac{1}{3}a$, $(ab - 9c)/(2a^2 - 6b)$. The latter expression may also be written as $\frac{1}{3}a + q/(2l)$. Indeed $f(\frac{1}{3}a) = q$, and $f(\frac{1}{3}a + q/(2l)) = -q\Delta/(6l)^3$, which, since Δ and l are positive, is of opposite sign to q. To complete the proof, we shall consider separately the two cases of $q > 0$ and of $q < 0$. For $q > 0$, we have $-\infty < \frac{1}{3}a < \frac{1}{3}a + q/(2l) < +\infty$, and $f(-\infty) < 0$, $f(\frac{1}{3}a) > 0$, $f(\frac{1}{3}a + q/(2l)) < 0$, $f(+\infty) > 0$. On the other hand, for $q < 0$, we have $-\infty < \frac{1}{3}a + q/(2l) < \frac{1}{3}a < +\infty$, and $f(-\infty) < 0$, $f(\frac{1}{3}a + q/(2l)) > 0$, $f(\frac{1}{3}a) < 0$, $f(+\infty) > 0$. Hence the two given values separate the roots in either case. The proof is also obvious geometrically. One has merely to compare the points of intersection with the X-axis of the vertical line through the point of inflexion, and of the straight line which joins the point of inflexion with both of the bend-points.

3110 [1925, 46]. Proposed by J. L. RILEY, Stephenville, Texas.

If the curves $a_1x^m + b_1y^m + c_1 = 0$ and $a_2x^m + b_2y^m + c_2 = 0$ touch at a point, find the ratio of their radii of curvature at that point.

NOTE BY NORMAN ANNING, University of Michigan.

On page 280 of Loria, *Spezielle alg. u. transz. ebene Kurven*, the following theorem is proved. "Wenn zwei auf dieselben Axen bezogene Lamé'schen Kurven mit den Indices m und m' sich in einem Punkte berühren, so wird das Verhältnis der Krümmungen in diesem Punkte durch $(m-1)/(m'-1)$ ausgedrückt." In this problem, $m = m'$ and the required ratio has the value 1.

Also solved by J. A. BULLARD, ALICE A. GRANT, HARRY LANGMAN and H. A. SIMMONS.

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[1] A solution of this problem by C. F. Gummer has already been published (*1925*, 389). EDITOR.

3113 [1925, 94]. Proposed by A. S. Wiener, Cornell University.

A man is paying off a mortgage of N dollars, interest i per cent. annually, by monthly installments of a dollars (where $a > iN/1200$, the first month's interest). How many months will it take him to pay off the mortgage?

SOLUTION BY C. C. WYLIE, University of Iowa.

It will be assumed that the first month's interest is due at the end of the first month when the first payment is made. If $r = 1 + i/1200$, the N dollars at the end of n months will amount to Nr^n, while the n payments of a dollars each will amount to

$$a(1 + r + r^2 + \cdots + r^{n-1}) = a(r^n - 1)/(r - 1).$$

Setting these two equal and solving for n we have

$$n = -\frac{\log [1 - N(r - 1)/a]}{\log r}.$$

If $N = 100$, $i = 6$, $a = 1$, then $n = 139$ months approximately.

Also solved by W. S. BARLOW, J. A. BULLARD, L. A. EASTBURN, J. M. EARL, PHILIP FITCH, ALICE A. GRANT, MICHAEL GOLDBERG, ELMER LATSHAW, HARRY LANGMAN, C. H. LEHMANN, G. A. LYLE, and A. W. RICHARDSON.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. KUHN, Ohio State University, Columbus, Ohio.

At the University of Washington, Associate Professor R. M. WINGER has been promoted to a professorship.

Mr. H. B. LEMON of Denison University and Mr. W. O. MENGE have been appointed instructors at the University of Michigan.

Professor ANNA J. PELL of Bryn Mawr was married on July 6, 1925, to Professor A. L. WHEELER of Princeton University. Mrs. Wheeler has been made head of the mathematics department at Bryn Mawr to succeed Professor CHARLOTTE A. SCOTT.

Assistant Professor A. L. NELSON of the University of Michigan has been appointed professor of mathematics at the Detroit Junior College.

Professor T. P. NUNN of the University of London gave a number of courses at the summer session of Columbia University. Afterward he visited Cornell, Purdue, University of Chicago, and University of Michigan.

Professor B. M. WALKER has recently been made President of the Mississippi Agricultural and Mechanical College. He was president of the new Mississippi-Louisiana Section of the Association during its first year just closed.

Miss EMMA L. KONANTZ, for a number of years connected with the department of mathematics of Ohio Wesleyan University, but recently on the faculty

of Peking University, Peking, China, has been advanced to a full professorship in mathematics in that institution.

Dr. D. L. HOLL, for the last two years assistant professor of mathematics at Ohio Wesleyan University, has accepted a like position at Iowa State College. Mr. R. L. NEWLIN, formerly of Guilford College, N. C., has been appointed to succeed Dr. Holl at Ohio Wesleyan.

At the University of Arkansas, Assistant Professor A. D. CAMPBELL has been promoted to an associate professorship of mathematics.

Professor L. M. HOSKINS, of the department of applied mathematics at Stanford University, has retired from active teaching.

Dr. H. B. CURTIS, of Northwestern University, has been appointed associate professor of mathematics at Marquette University.

Dr. L. L. SMAIL, of the University of Oregon, and Mr. H. S. VANDIVER, of Cornell University, have been appointed associate professors of mathematics at the University of Texas.

Professor MAX MASON, of the department of mathematics at the University of Wisconsin, has been elected president of the University of Chicago.

The death is announced of Professor B. H. KERSTEIN, of the State Teachers College, Silver City, New Mexico.

Professor W. F. SHENTON, of the United States Naval Academy, has been appointed professor of mathematics and physics at the American University, Washington, D. C.

Professor H. L. SMITH, of the University of the Philippines, has been appointed to an assistant professorship at the University of Minnesota.

At the University of Pennsylvania the following promotions and new appointments are announced in the mathematics department: Dr. M. J. BABB from assistant professor to professor, Dr. J. D. ESHLEMAN from instructor to assistant professor, and Mr. N. E. RUTT from assistant to instructor; Dr. P. A. CARIS, assistant professor; Mr. W. A. BRISTOL, instructor; Mr. H. M. LUFKIN, instructor; and Mr. M. BROOKS and Mr. L. ZIPPIN, assistants.

At the University of Iowa, Associate Professor E. W. CHITTENDEN has been promoted to a full professorship. Dr. ROSCOE WOODS has been promoted to an assistant professorship.

Dr. D. H. MENZEL, University of Iowa, has been appointed assistant professor of astronomy at Ohio State University.

Dr. L. E. WARD of Harvard University has been appointed instructor in mathematics at the University of Iowa.

Mr. E. E. ERICKSON of the University of Iowa has been appointed assistant professor in Miami University.

Mr. H. K. HUGHES, University of Iowa, has been appointed instructor in mathematics, University of Kansas.

Mr. D. O. STREYFFELER of the University of Iowa has been appointed instructor in mathematics, University of Kentucky.

Dr. ISRAEL MAIZLISH has been appointed associate professor of physics at Centenary College, Shreveport, La.

Miss DORA E. KEARNEY has been appointed professor of mathematics at Iowa State Teachers College, Cedar Rapids.

Assistant Professor F. M. WEIDA, of Montana State College, has been appointed assistant professor of mathematics at Lehigh University.

The following appointments to instructorships are announced:

University of Illinois, Dr. R. M. MATHEWS.

Northwestern University, Mr. O. E. BROWN.

College of the City of Detroit, Mr. WILLIAM BORGMAN and Miss LUCILLE CHALMERS.

Assistant Professor J. W. MILLER, of the University of Pennsylvania, died Sept. 11, 1925.

On September 11, 1925, Prof. ERNST VON HAMMER of the Technische Hochschule of Stuttgart, Germany, died at the age of 67 years. For some forty years he had filled the chair of geodesy and practical astronomy in that institution. He was the author of numerous books and articles.

The University of Strasbourg, in addition to the standard courses on analysis, astronomy, and mechanics, is offering this year the following advanced courses which are particularly fitted to prepare candidates for the "diplôme d'études supérieures de Mathématiques" and for the Doctorates:

First Semester (November, 1925, to February, 1926)—By Professor BAUER, Lorentz theory, Theory of quanta; by Professor CERF, Outer multiplication and derivation; by Professor FRÉCHET, Integration of functionals, Introduction to nomography.

Second Semester (March 1 to June 15)—By Professor BAUER, Spectra and dynamics of atoms; by Professor FRÉCHET, Integration of functionals, Advanced nomography; By Professor VALIRON, Normal families of meromorphic functions; by Professor VILLAT, Conformal representation of minimal surfaces; by Professor THIRY, Advanced analytic geometry, Remarkable curves and surfaces.

THE CHAUVENET PRIZE FOR MATHEMATICAL EXPOSITION.

In March 1925, PRESIDENT COOLIDGE proposed that the Association establish a prize for special merit in mathematical exposition. The proposition was sanctioned by mail vote of the Trustees and a committee consisting of Professors A. J. KEMPNER, Chairman, LOUISE D. CUMMINGS and D. R. CURTISS was

appointed to formulate the details. This committee presented a report at the Ithaca meeting which in somewhat modified form was adopted by the Trustees. The substance of the report is as follows:

The committee believe that the proposed prize will exert a desirable influence on the production of high-grade exposition articles. They adopted the name suggested by President Coolidge, namely, "The Chauvenet Prize for Mathematical Exposition." For a study of the life and influence of WILLIAM CHAUVENET, 1820–1870, Professor of mathematics in the U. S. Navy, 1847–1859, President of the Academic Board of the Navy, 1847–1850, Professor of mathematics and natural philosophy at Washington University, St. Louis, Mo., 1859–1869, Author of many works and treatises, they refer to an article by Professor W. H. ROEVER, in *Washington University Studies*, Vol. XII, *Scientific series*, No. 2, 1925.

The Chauvenet Prize is to be awarded every five years for the best article of an expository character dealing with some mathematical topic, written by a member of the Mathematical Association of America and published in English in a journal during the five calendar years preceding the award. This prize will not be awarded for books, even though a large portion of mathematical books are mainly or completely expository in character, such as textbooks. They bring their own reward in the form of royalties.

The amount of the prize was fixed at one hundred dollars, an amount which the committee deemed sufficiently large to be attractive apart from the honor of the award. The cash for the first award has been provided by a friend of the Association. Thereafter, it will be supplied from the Association treasury, one fifth of the amount being set aside each year for the purpose.

The first award is to be made at the annual meeting in December 1925, covering the five-year period ending with the calendar year 1924.

It is provided that the award shall be determined at each quinquennial period by a scrutinizing committee of three to be appointed by the president of the Association and that this committee should be restricted as little as possible, aside from the specifications mentioned in the foregoing paragraphs. President Coolidge appointed the scrutinizing committee for the first award as follows:

E. B. VAN VLECK, Wisconsin, *Chairman*,
ANNA J. P. WHEELER, Bryn Mawr,
W. C. GRAUSTEIN, Harvard.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Ninth Summer Meeting of the Association, Ithaca, N. Y., September 8-9, 1925;
Tenth Annual Meeting, Kansas City, Mo., December, 1925.
The following are dates of Section Meetings of the Association in 1925 (unless otherwise specified):

ILLINOIS, Decatur, May 7–8, 1926
INDIANA, Bloomington, May 8–9, 1925
IOWA, Coe College, Cedar Rapids, April 30–May 1, 1926.
KANSAS, Topeka, February 7
KENTUCKY, Univ. of Kentucky, April or May
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 20–21
MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, George Washington Univ., Washington, Dec. 5, 1925.
MICHIGAN, Ann Arbor, April 1, 1926
MINNESOTA, St. Johns Univ., Collegeville, May 16
MISSOURI, Kansas City, December, 1925
NEBRASKA, Creighton Univ., Omaha, May 2
OHIO, Ohio State Univ., Columbus, April 3
ROCKY MOUNTAIN, Laramie, April
SOUTHEASTERN, Birmingham, Ala., Spring
SOUTHERN CALIFORNIA, February 28
TEXAS, Dallas, November 27–28, 1925

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THE MAY MEETING OF THE IOWA SECTION.

The fourteenth regular meeting of the Iowa Section of the Mathematical Association of America was held at Iowa State Teachers College, Cedar Falls, Iowa, on May 1 and 2, 1925, in affiliation with the annual meeting of the Iowa Academy of Science.

Among those attending were the following twenty-four members of the Association: Julia T. Colpitts, I. S. Condit, Marian E. Daniells, R. D. Daugherty, R. M. Deming, C. W. Emmons, Annie W. Fleming, C. Gouwens, H. K. Hughes, Daniel Kreth, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, E. A. Pattengill, J. F. Reilly, Fred Reusser, H. L. Rietz, Maria M. Roberts, E. R. Smith, G. W. Snedecor, J. S. Turner, C. W. Wester, W. H. Wilson, Roscoe Woods.

The chairman of the section, Professor E. R. Smith, presided at both the Friday afternoon and Saturday morning sessions. Dinner was enjoyed together Friday evening at Bartlett Hall. At the business meeting the following were elected officers for 1925–1926: Chairman, I. S. CONDIT, Iowa State Teachers College; Vice-Chairman, MARIAN E. DANIELLS, Iowa State College; Secretary-Treasurer, J. F. REILLY, University of Iowa.

The next regular meeting will be held at Cedar Rapids, April, 1926.

The following papers were presented:

(1) "Experiments with small numbers of observations" by Professor G. W. SNEDECOR, Iowa State College.

(2) "On the numerical effect of certain variations in the assumptions relating to the distribution of unknown probabilities in the inversion of the Bernoulli Theorem" by Professor H. L. RIETZ, University of Iowa.

(3) "Notes on algebraic deficiencies" by Professor J. V. McKELVEY, Iowa State College.

(4) "The members of the system of conics passing through four real points" by Mr. H. K. HUGHES, University of Iowa.

(5) "The locus of the centers of the conics through four real points" by Mr. H. K. HUGHES.

(6) "An application of Rolle's theorem to functions of finite genus" by Professor MARIAN E. DANIELLS, Iowa State College.

(7) Address by the retiring chairman: "The determination of mathematical ability" by Professor E. R. SMITH, Iowa State College.

(8) "Sufficient conditions for the periodicity of the solutions of certain functional equations" by Professor W. H. WILSON, University of Iowa.

(9) "The rule of double position" by Professor J. S. TURNER, Iowa State College.

(10) "Some properties of prolate spheroids" by Professor J. S. TURNER.

(11) "On a line similar to Simson's line" by Professor ROSCOE WOODS, University of Iowa.

(12) "Two theorems on annuities" by Professor J. F. REILLY, University of Iowa.

Abstracts of papers, numbered as in the above list of titles, follow:

1. Work with small numbers of observations, as done by "Student," Fisher and Pearson, has been limited to samples drawn from normal distributions. In his experiments Professor Snedecor's object was to develop empirical information as to the behavior of means and standard deviations of small groups of observations made on a non-normal population. Three experiments were reported in this paper, each consisting of thirty-one samples of five observations, drawn respectively from (*a*) an approximately normal distribution with a coefficient of variability of thirteen per cent, (*b*) an approximately symmetrical distribution, non-normal, C equal fifteen per cent, (*c*) a markedly skew distribution, C equal forty-eight per cent.

The conclusions reached were that with observations as few as five in a population with coefficient of variability ten per cent or above the ordinary method of estimating reliability is entirely unreliable, and that "Student's" method gives quite trustworthy results unless the coefficients of variability and skewness become too large.

2. The reasoning involved in the inversion of the theorem of Bernoulli is dependent on the principle of Bayes. This principle in turn involves an assumption to the effect that the unknown probability of success before any trials are made may be regarded as uniformly distributed in the interval 0 to 1. Much criticism has been directed against the lack of generality of this assumption. In the present paper Professor Rietz gave the results of an examination into the numerical effects in certain practical problems of statistics of replacing the above assumption by certain other assumptions.

3. In this paper Professor McKelvey presented data comparing the progress in college mathematics of students who are well qualified in high school mathematics with the progress of students who are not thus qualified. The data indicated that any attempt to major in mathematics or to carry the mathematics of an engineering course without first doing some thorough work in elementary algebra would almost certainly result in failure.

4. The kinds of conics included in a system passing through four fixed points depend on the relative positions of these points. All relative positions can be obtained by taking three points fixed and a fourth variable. In his study of such systems Professor Hughes found the following: that the system contains a circle when and only when the four points lie on a circle; that it always contains one real rectangular hyperbola, and will consist entirely of rectangular hyperbolas if the variable point be at the orthocenter of the three fixed points; that there are, in general, two parabolas in the system, which are real and coincident if three of the points are collinear, real and distinct if the quadrilateral formed by the points is convex, and imaginary if the quadrilateral is reëntrant; and finally that the system contains only "double lines" when the four points become two pairs of coincident points.

5. The locus of the centers of the conics passing through four real fixed points is itself a conic, whose type depends upon the relative position of the four points. Professor Hughes found that this center-locus is a rectangular hyperbola if the points lie on a circle, a circle if the variable point is at the orthocenter of the three fixed points, a degenerate parabola if three of the points are collinear, a hyperbola if the quadrilateral formed by the points is convex, and an ellipse if the quadrilateral is reëntrant.

6. In her paper Professor Daniells proved that if an entire function $F(z)$ of genus p has a finite number q of imaginary roots, then $F'(z)$ has in addition to the number of roots located by means of Rolle's theorem at most $p+q$ other roots, real or imaginary.

7. Professor Smith reported the results of a statistical study which led him to conclude that the achievement of students in college mathematics bears but little relation to the grades obtained in high school courses in mathematics. The same conclusion holds for the standard psychological tests. However, an entrance examination covering the élements of secondary school mathematics may be used to predict, with a fair degree of success, the mathematical ability of the student. It was also pointed out that the high schools could well afford to introduce in the student's senior year a course primarily designed to test out mathematical tastes and aptitudes.

8. Professor Wilson discussed the periodicity of functions satisfying some of the well-known functional equations. Sufficient conditions for periodicity were found and some of the consequences of periodicity were exhibited. As special cases the periodicity of the continuous functions was discussed.

9. In this paper the history of "The rule of double position" was briefly presented, with extracts from old arithmetics, and its disappearance from modern arithmetics was remarked. An algebraic proof of the rule was given.

10. In his second paper Professor Turner gave a geometric construction for the tangent cone from an external point T to the prolate spheroid whose foci are S and H. He proved that this cone is a quadric cone whose focal lines are TS and TH, that the curve of contact C is a plane curve, and that the cones with vertices S and H, and passing through C are right circular cones. He also showed that if the sum of the focal angles of the tangent cone is a right angle, the locus of its vertex is the director sphere of the spheroid. Other properties were stated without proof.

11. In his paper Professor Woods gave the equation of the line defined in the following way: if from any point P on the circumcircle of a given triangle lines are drawn so as to make a given angle θ with the sides (the angle taken in the same sense in each instance) the three points of intersection so obtained are collinear. The name θ-line was assigned to this line. When θ is a right angle the line is known as the pedal (Wallace or Simson) line of the point P. Several properties of θ-lines were given and each result was specialized for Simson lines.

12. In the first part of his paper Professor Reilly presented a general formula that included the amount and present value of an annuity, the amount and

present value of an annuity due, and the present value of a deferred annuity as special cases. In the latter part he showed that the formulas for constructing tables of the amount and present value of annuities by the continuous process were linear difference-equations of the first order, with constant coefficients, whose solutions gave the usual formulas for the amount and present value of annuities certain.

J. F. REILLY, *Secretary-Treasurer.*

IN THE SURNAMED CHOSEN CHEST.

By DAVID EUGENE SMITH, Columbia University.

III. Numismatica Mathematica.[1]

It might naturally be supposed that few mathematicians would ever have been honored by portrait medals struck in recognition of their services to the world. The issuing of such medals was common in the case of kings and of those who fought their battles, and the church has recognized her prelates and saints in the same manner. In recent years the stage and the societies of the fine arts in general have followed the older custom, and certain scientists of great repute have been similarly honored. These tokens of esteem have usually, however, been bestowed upon men who have had some influence with those in power, either the power of office or that of wealth. Only in the nineteenth century did governments, through learned societies, make a real beginning in honoring the deceased scholars of their countries, and in this way many of the medals of mathematical interest came to be struck.

In this collection there are between 125 and 150 medals of this nature, not counting ancient coins selected to represent the development of the numerals, and counters (jetons) that were used in computation in medieval times. Some of those which represent what might be called an indirect or a casual interest in mathematics, such as medals of Abélard, Pestalozzi, Vittorino da Feltre, and Erasmus are not mentioned in the following list, but those relating to workers in pure mathematics or to those who applied a high order of mathematics to their work in astronomy are included. It is thought that such a list may be of service to those who are interested in the history of mathematics and, to a less degree, of mathematical astronomy. All the medals are originals except in the case of a few electrotypes of specimens in the British Museum, and these cases are specified. In each case the approximate diameter of the medal is given in millimeters. The abbreviations *Ob.* and *Rev.* refer to the obverse (face) and reverse of the medal, and r. and l. are used for right and left respectively. Since the list is not primarily for numismatists, the full description has not generally been given, but enough has been included to allow of easy identification.

[1] This is the third of a series of articles relating to certain historical material in the library of the author. The first dealt with association copies of mathematical books (*1925*, 287–294), and the second with manuscripts of oriental mathematical works (*1925*, 393–397).

1. *Arago*, François (1786–1853). Bronze. Ob., bust by A. Bovy, to r. Rev., "A/Arago/ les auditeurs/ de son cours/d' astronomie. 1843." 56 mm.

2. *Arago*. Bronze. Medallion by David d'Angers. Signed "David 1832." 149 mm.

3. *Arago*. Bronze. Ob., bust by A. Dubois to l. Rev., "Institvt/ de France / Academie / des Sciences / Medaille Arago." 67 mm.

4. *Archimedes* (*c.* 287–212 B. C.). Bronze. Ob., bust by Catenacci, to r. "Archimedes Mathesis ac Mechanices Prodigivm." Rev., "Syracvsis N. A. A.C. CIↃ IↃCCCIX in Vrbis Expvgnatione peremptvs/ A. Arnaud Scvl." 41 mm.

5. *Aristotle* (384–322 B. C.). Bronze. Ob., bust to r. Greek inscription. Rev., pegasus. (Electrotype facsimile from original in the British Museum. Probably 16th century.) 50 mm.

6. *Bacon*, Roger (1214–1294). Bronze. Ob., bust by Gayrard, to r. One of the "series Numismatica," "Durand editit," 1818. 41 mm.

7. *Bailly*, Jean Sylvain (1736–1793). Bronze. Ob., bust by Montagny, to r. One of the "Galerie Metallique des Grands Hommes Français," issued in 1821. 41 mm.

8. *Bailly*. Bronze gilt. Ob., bust to r. "J. Sylvain Bailly né à Paris le XV. Sept. MDCCXXXVI." Rev., "Astronome,/ Autheur de l'Histoire / de l'Astronomie / Membre des trois Academies/ Francaise, des Belle Lettres / et des Sciences / Président de l'Assembleé / Nationale le 17. Juin / elu/er Maire de Paris / le 15. Juillet 1789 / et Hélas (figure of a headsman's axe) . . . 11 Nov. 1793." 41 mm.

9. *Bailly*. Bronze. Ob. as in No. 8, but under the bust "Offert a la Ville." Rev. as in No. 8 but beginning with the words, "Merite Reconnu / Membre / des Trois Académies," the rest as in No. 8, but without the last line. 41 mm.

10. *Bailly*. Bronze. Ob., bust to 1. "J. Sylvain Bailly né à Paris en 7bre. 1736. / Décapité le 12 9bre. 1793." Rev., "Premier Président de/ l'Assemblée Natle. / Ier. Maire de Paris / Il périt / douloureusement / sur un échaffaud / victime / de l'ingratitude / populaire." 32 mm.

11. *Bertrand*, L. F. Joseph (1822–1900). Bronze. Ob., bust by J. C. Chaplain, to l. Rev., "A Joseph Bertrand / Mbre de. l'Academie / Française / Secretaire. Perpetual / de l'Academie. des. sciences / pour. honorer. 50. années / de. Devouement. a. la. Science / et. a. l'Enseignement / ses. Eleves. ses. Admirateurs / et. ses. Amis / Mars / 1844–1894." 67 mm. (One of the finest portrait medals of a mathematician ever issued.)

12. *Brahe*, Tycho (1545–1601). Bronze. Ob., bust by Rogat, to r. One of the same series as No. 6. 41 mm.

13. *Buniakovsky*, Victor Yakovlevich (1804–1889). Bronze. Ob., bust by V. Alexeyev. "Doctor of mathematics of the University of Paris since May 19, 1825" (in Russian). Rev., "In memory of fifty years of service to science, May 19, 1875" (in Russian). 70 mm.

14. *Cardan*, Jerome (1501–1576). Bronze. Ob., bust to r. "Hier. Cardanus. Aetatis. A N. XLVIII." Rev., Landscape and muses. "ONEIPON" (Electrotype facsimile from original in the British Museum. 16th century.) 50 mm.

15. *Cassini*, Giovanni Domenico (1625–1712). Ob., bust by Peuvrier, to r. Rev. "Natus / An. M.DC.XXV. / Perinaldo / in Nicaeae / Massiliensium / Comitatu / Obiit / An. M.DCC.XII." One of the same series as No. 6. Dated 1823. 41 mm.

16. *Cassini*, G. D. Ob., bust by Gatteaux, to r. One of the same series as No. 7.

17. *Cassini*, G. D. Ob., bust to r. "Io. Dom. Cassinvs. Archigym. Bonon. Primar. Astron. et R. Acad." Rev., "Facta. copia. coeli." Dated "Bonon. / M.DC.VC." 60 mm.

18. *Cauchy*, Augustin Louis (1789–1857). Bronze. Medallion by David d'Angers. Head to r. Signed "David 1843." 174 mm.

19. *Cavalieri*, Bonaventura (1598–1647). Bronze. Ob., bust by Nesté, to l. "Bonaventura Cavalerivs." Rev., "Pr. Id. Septemb. / An. MDCCCXLIV / Mediol." 43 mm.

20. *Copernicus*, Nicolaus (1473–1543). Bronze. Ob., bust by Durand, to l. One of the same series as No. 6. Dated 1820. 41 mm.

21. *Copernicus*. White metal. Ob., half figure slightly to r. Legend in Polish. Rev., legend in Polish. Dated 1873. 65 mm.

22. *Copernicus*. Bronze. Ob., bust by Petit, to l. One of the same series as No. 6. Dated 1818. 41 mm.

23. *D'Alembert*, Jean le Rond (1717–1783). Bronze. Ob., bust by Gatteaux, to l. "J. D'Alembert." "F. Gatteaux F. 1785." Rev., laurel wreath inclosing "A l'Immortalité." 59 mm.

24. *D'Alembert*. Bronze. Ob., bust by Depaulis, to l. "Jean le Rond D'Alembert." Rev., "Né / à Paris / en M.DCC.XVII./Mort en M.DCC.LXXXIII." One of the same series as No. 7. Dated 1824. 41 mm.

25. *De Moivre,* Abraham (1667–1754). Bronze. Ob., bust by Dassier, to r. "Abrahamus de Moivre." Rev., "Utriusque / Societatis Regalis / Lond. et Berol. / Sodalis. / M.DCCXLI." 55 mm.

26. *Descartes,* René (1596–1650). Bronze. Ob., bust by Henrionnet, to l. "Renatus Cartesius." Rev. "Natus / An. M.D.XCVI. / Hagae Turonicae / in Gallia / Obiit / Holmiae / An. M.DC.L." One of the same series as No. 6. Dated 1822. 41 mm.

27. *Descartes.* Bronze. Ob., bust by Galle, to r. "Rene Descartes." Rev. "Né à la Haye / en Touraine / en M.D.XCVI./ Mort / en M.DC.L." One of the same series as No. 7. Dated 1819. 41 mm.

28. *Descartes.* Bronze. Ob., bust by Dassier, to r. "Rene Descartes." Rev., recumbent figure, two globes, book, two trumpets. "Philosophe / M. 1650." 27 mm.

29. *Descartes.* Bronze. Ob., bust by Rogat, to r. "René Descartes;" below, "E. Rogat 1846." Rev.,: "Lycée Descartes / Concours Général." Presentation medal to L. J. B. Toussaint, 1848.

30. *Descartes.* Bronze gilt. Ob. same as No. 26. Rev., "Né à la Haye (Indre et Loire) / 1596./ Mort à Stockholm 1650 / Son corps / ramené en France / 1666." 50 mm.

31. *Dürer,* Albrecht (1471–1528). Bronze. Bust to r. "Imago. Alberti. Dvreri. Aetatis. sva. LVI." No rev. This would make the date 1521. The workmanship resembles that of a medal in the South Kensington Museum dated 1528.

32. *Dupin,* Charles, Baron (1784–1873). Ob., bust by A. Bovy, to l. "Baron Charles Dupin de l'Académie des Sciences." Rev., "A Charles Dupin / après 50 années / d' heureuse confraternité / les membres de l' Académie des Sciences / ses collaborateurs / ses amis / —— 1818–1868." 51 mm.

33. *Dupin,* Charles. Bronze, lighter color than No. 29, but ob. from the same die. Rev., "Né à Varzy 1784./ De l' Adad.^{e} des Sc.^{es} 1818./Deputé du Tarn 1827./" . . . "Senateur 1852." 51 mm.

34. *Euler,* Léonard (1707–1783). Silver. Ob., bust by Abramson, to l. Rev., "Radio describit orbem / Natus / MDCCVII." 41 mm.

35. *Fermat,* Pierre (*c.* 1608–1665). Bronze. Ob., bust by Desboeufs, to l. Rev., "Né à Toulouse / en M.D.XCV. / Mort / en M.DC.LXV." One of the same series as No. 7. Dated 1822. The date of his birth (M.D.XCV) is, according to the inscription on his tomb, incorrect. 41 mm.

36. *Fermat.* Bronze. Ob., bust by Dubois, from De Puy Maurin's drawing, to l. Rev. "Académie Roy.^{le} des Sciences Inscrip.^{ons} et Bel.^{es} Let.^{es} de Toulouse / 1822." 35 mm.

37. *Fermat.* Same as No. 36, but in silver. 35 mm.

38. *Fossombroni* Vittorio (1754–1844). Bronze. Ob., bust by "Fabrs d'Udine," to r. "Vittorio Fossombroni Politico e Matematico." Rev., "MDCCCXXXXIIII / Perduta / tanta cagione / I Toscani / La Monumentano / L. Muzzi F." 54 mm.

39. *Galileo Galilei* (1564–1642). Bronze. Ob., bust by Gayrard, to r. "Galilæus Galilæi." Rev., "Natus Pisis / in Italia./ An. M.D.LXIV. / Obiit / An. M.DC.XLII." One of the same series as No. 6. Dated 1818. 41 mm.

40. *Galileo Galilei.* Bronze. Ob., bust by Cinganelli, to l. Rev. "Pisa / che lo vide nascere / ne / celebro / il trecentesimo natalizio / à XVIII Febraio / M.D.CCC.LXIV." 55 mm.

41. *Galileo Galilei.* Bronze. Imperfectly cast piece showing the obverse of No. 40.

42. *Galileo Galilei.* Bronze. Ob., bust by Cerbara, to l. Rev., "Conditori / disciplinae / ad. leges. motvs / et. astron. certo / cognoscendas." 41 mm.

43. *Galileo Galilei.* Silver. Ob. the same as No. 40. Rev., "S.P.Q.P / Academia / pristino decori restitvta / omnibvsqve disciplinis aperta/ III non Decem. MDCCCLIX / Vict. Emm. Sabavd. II P.F.A / Ital. Med. Rege Electo." 55 mm.

44. *Galileo Galilei.* Bronze. Ob., bust to r. "Galilevs. Galilei. Patr. Flor. Mathe.^{m} Cele.^{b}" Rev., two graces with astronomical and geometric models. 88 mm.

45. *Galileo Galilei.* Brass. Ob., bust to r. "Galilevs. de. Galilei. Flor." This legend is engraved. No reverse. 64 mm.

46. *Galileo Galilei.* Bronze. Ob., bust to l. "G. Galilei." Cast bronze piece. No reverse. 66 mm.

47. *Galileo Galilei.* Bronze. Ob., bust to l. "Galilevs. de. Galileis. Florentinvs." Rev., plain, with the single word "Archimedes." 58 mm.

48. *Galileo Galilei.* Bronze. Ob., bust to r. "Galilevs Lyncevs." Reverse, figures (telescope, etc.) "Naturamqve novat." 71 mm.

49. *Galileo Galilei.* Lead, bronzed. Ob., bust slightly to l. "G. Galilei." Rev. "RP." Cast piece. 90 mm.

50. *Gassendi,* Pierre (1592–1655). Bronze. Ob., bust by Vatinelle, to r. Rev., "Né / à Chantersier / près Digne / en M.D.XCVIII. / Mort / en M.DC.LVI." One of the same series as No. 7. Dated 1818. The dates of birth and death are both incorrect. 41 mm.

51. *Gassendi.* Bronze. Ob., bust by I.D., to r. Rev., on a tablet. "Philosophe / M. 1653," with winged figure, globe, etc. The date of death is incorrect. 28 mm.

52. *Gauss,* Carl Friedrich (1777–1855). Bronze. Ob., bust by Brehmer, to r. "Carl Friedrich Gauss / M.DCCLXXVII April XXX / MDCCCLV Februar XXIII." Rev., "Königliche Gesellschaft der Wissenschaften zu Göttingen." Dated 1877. 70 mm.

53. *Gerbert* (Pope Sylvester II) (940–1003). Bronze. Papal medal, 16th (?) century. Ob., bust to r. Rev., two papal keys. Inscription probably "Claves Regni Celorvm," but not very legible. Electrotype from the original in the British Museum. 37 mm.

54. *Grandi,* Guido (1671–1742). Ob., profile by Sulvi, to r. "D. Gvido. Grandvs. Abbas. Camald. Mathem. Pis. Vniv." Rev. "EYPHKA / Inveni / An. MDCCXXXVIII / A. Sulvi. F." Various figures. Electrotype from the original bronze piece in the British Museum. 88 mm.

55. *Halley,* Edmund (1656–1724). Bronze. Ob., bust by Dassier, to r. "Edmundus Halley." Rev. "Astronomus / Regis Magnae / Britanniae. / MDCCXLIV." 55 mm.

56. *Hutton,* Charles (1737–1823). Bronze. Ob., bust by B. Wyon, to l. "Carolus Hutton, LL.D., R.S.S. Æt. LXXXV/ 1821." Rev., various figures; "Fulmina belli / Pondusq. terrae / aestimata." 44 mm.

57. *Huygens,* Christiaan (1629–1695). Bronze. Ob., bust by Henrionnet, to r. "Christianus Hugenius." Rev., "Natus/ Hagae Comitum / in Batavia / An. M.DC.XXIX./ Obiit / An. M.DC.XCV. / " One of the same series as No. 6, 41 mm.

58. *Kepler,* Johann (1571–1630). Bronze. Ob., bust by Caqué, to r. "Joannes Keplerus." Rev., "Natus / An M.D.LXXI,/ Viel in / Regno Wirtembergensi / Obiit / An. M.DC.-XXX." One of the same series as No. 6. Dated 1823. 41 mm.

59. *Lacroix,* Sylvestre François (1765–1843). Bronze. Medallion by David d'Angers. Dated 1841. 169 mm.

60. *Lagrange,* Joseph Louis (1736–1813). Bronze. Ob., bust by Donadio, to l. "Joh. Ludovicus Lagrange." Rev. "Natus / Taurini / in Pedemontis / An. M.DCC.XXXVI. / Obiit / An. M.DCCC.XIII." One of the same series as No. 6. Dated in 1822. 41 mm.

61. *Lagrange.* Bronze. Ob., bust by Donadio, to l. "Joseph Louis Lagrange." Rev., "Né / à Turin / en M.DCC.XXXVI. / Mort / en M.DCCC.XIII." One of the same series as No. 7. Dated 1818. 41 mm.

62. *Lagrange.* Bronze. Medallion by David d'Angers. Bust to l. "Louis de la Grange, David." 152 mm.

63. *Lagrange.* Bronze. Ob., bust by Galeazzi, to l. Aloisivs Lagrange." Rev. "Geometras aevi svi / svperavit / antiqvorvm famam / aeqavit." Although bearing the name Aloisius, the head is that of Joseph Louis. 43 mm.

64. *Lalande.* Joseph Jerome François de (1732–1807). Bronze. Ob., bust by Gatteaux, to l. "Jos. Hier. le François de la Lalande. N. Burgt 1732." Electrotype from the original in the British Museum. 42 mm.

65. *Lalande.* Bronze. Medallion, bust by David d'Angers, to l. "De la Lande / ancien directeur de l'observatoire." 165 mm.

66. *Laplace,* Pierre Simon (1749–1827). Cast iron medallion, no reverse. Ob., bust to l. "Pierre Simon Laplace. Né le 23 Mars 1749." 100 mm.

67. *Laplace.* Bronze. Medallion, bust by David d'Angers, to l. "Laplace." 172 mm.

68. *Le Verrier,* Urbain Jean Joseph (1811–1877). Bronze gilt. Ob., bust by Alphée Dobois, to l. "U. J. J. Le-Verrier de l' Académie des Sciences. 1811–1877." Rev., Helios with four horses; planets with figures and names. Dated, 1884. 68 mm. One of the finest medals ever struck in memory of a mathematician and astronomer.

69. *Leibniz,* Gottfried Wilhelm (1646–1716). Bronze. Ob., bust to r. "Godofr. Wilh. L. B. de Leibniz natvs D.XXI.IVN. / I.IVL / MDCXXXXVI." Rev., Fame placing a wreath on an altar. "Academia Regia Borvss. Scient. Primo Praesidi svo. / MDCCCXXXXVI / D II. IVL." 53 mm.

70. *Leibniz.* Same as No. 69, but in silver.

71. *Leibniz.* Same as 69, but of different color.

72. *Leonardo da Vinci* (1452–1519). Bronze. Ob., bust by Hérard, to l. "Leonardvs-

Vincivs-Florentinvs." Rev. Brush, pen, crown. "Scribit. qvam. svscitat. Artem./ 1665." 54 mm.

73. *Leonardo da Vinci.* Bronze. Ob., bust by Cossa, to l. "Leonardo da Vinci. / L. Cossa. F. 1820." Rev., "Vinci I Vaciti primi / Gallo Monarca gli ultima respiri / cloux le mortali spoglie / A. A. /" An electrotype facsimile. 48 mm.

74. *Lobachevsky,* Nicolaï Ivanovitch (1793–1856). Bronze. Ob., bust ¾ to r. Name in Russian letters. Rev. Russian inscription of the Physico-Mathematical Society at the University of Kasan, 1895. 42 mm.

75. *Maurolico,* Francesco (1494–1575). Bronze gilt. Ob., bust by Catenacci, to r. "Franc. Mavrolycvs Archimedes alter." "V. Catenacci scvlp. / L. Taglioni con. Neap." Rev., "Messanae nat. Ann. A.C. CIƆCIƆXCIV Ibiqve obiit CIƆIƆLXXV. / A. Arnaud scvlp." Tablet with further inscription. 41 mm.

76. *Maurolico.* Same as No. 75 but in bronze.

77. *Monge, Gaspard* (1746–1818). Bronze. Ob., bust by Gatteaux, to l. "Gaspard Monge." Rev., "Né /à Beaune en M.DCC.XLVI. / Mort en M.DCCC.XVIII." One of the same series as No. 7. Dated 1822. 41 mm.

78. *Monge.* Bronze. Medallion, bust by David d'Angers, to r. "Monge." "David" 171 mm.

79. *Monge.* Bronze gilt. Ob., bust by Vauthier Galle, to l. "Gaspard Monge." Rev., Né / à Beaune 1746 / Membre de l' Académie /" etc. . . . "Mort 1818." 50 mm.

80. *Neudorffer,* Johann, the elder (1497–1563). Silver. Ob., bust to r. "Ioann Nevdorffer Arith. Æ 57." Rev., "Ditat servata fides." 22 mm.

81. *Neudorffer,* Johann, the younger (1543–1581). Silver. Ob., bust to r. "Ioann Nevdorffer F. Arith. Æ 36." Rev. as in No. 80. 21 mm.

82. *Newton,* Isaac (1642–1727). Bronze. Ob., bust by Croker, to l. "Isaacvs Newtonvs." "I.C." Rev., a seated figure, winged head, holding a tablet with a plan of the solar system. "Felix. Cognoscere. Cavsas / M.DCC.XXVI./" The date of death (1726) is according to the old style, still in use at that time in England.

83. *Newton.* Same as in No. 82, but in silver.

84. *Newton.* Silver. Ob., bust ¾ to r. "Isaacus Newtonius." Rev., wreath surrounding inscription: "Eq. Aur. / Philosophus. / Obiit 31. Mart. / 1727. / Natus Annos / 85." 33 mm.

85. *Newton.* Bronze. Ob., bust by Dassier, to r. "Isaacus Newtonius./" "I. Dassier F./" Rev., his tomb in Westminster Abbey. "Nat. 1642. / M. 1726./" 42 mm.

86. *Newton.* Silver. Ob., bust to r. by Roëttiers. "Isaacvs Newtonivs" "Jac. Roëttiers." Rev. "Quæritur / Huic / Alius / Verg. Æneid: / M.DCC.LXXIV." 54 mm.

87. *Newton.* Bronze. Ob., bust to l. by Petit. "Isaacus Newtonius" "Petit. F." Rev., "Natus / Volstropii / in Anglia / An. M.DC.XLII. / Obiit / An. M.DCCXXVII. /" One of the same series as No. 6. 41 mm.

88. *Newton.* A poor reproduction of No. 85, cast from the original, in bronze.

89. *Newton.* Bronze. Same as No. 84, but in bronze.

90. *Newton.* Bronze. Same as No. 86, but in bronze.

91. *Newton.* Bronze gilt. Same as No. 85, but in bronze gilt.

92. *Newton.* Silver. Same as No. 85, but in silver.

93. *Newton.* Tin. Ob., bust to r. "Sir Isaac Newton." "L.B." "Died 20 March 1727." Rev., "Sir Isaac was born at Woolsthorpe in the Soke of Grantham Dec$^{r.}$ 25$^{th.}$ 1642 and educated at the Free Grammar School of King Edward the Sixth. Grantham." 45 mm.

94. *Newton.* Tin. Ob. Bust to r. A calendar medal. "A Calendar 1822. S. Isaac Newton." Rev., calendar. 42 mm.

95. *Newton.* Bronze. A halfpenny struck at the mint in 1793. Ob., bust to l. "Sr. Isaac Newton." Rev. "Half Penny / 1793." Design in the center. 27 mm.

96. *Newton.* Bronze. Similar to No. 95. Date, 1793. 27 mm.

97. *Newton.* Bronze. Similar to No. 96. Date, 1793. 27 mm.

98. *Newton.* Bronze. Similar to No. 97. Date, 1793. 27 mm.

99. *Newton.* Bronze. Similar to No. 98. Date, 1793. 27 mm.

100. *Newton.* Bronze. Similar to No. 98. Date, 1793. 27 mm.

Nos. 95–100 are all of the same design but are not all struck from the same die, as slight differences show. Four have milled edges. Two have inscriptions

on the edges. One reads, "Payable in Hull and in London;" the others, "Payable in London Bristol & Lancaster."

101. *Newton*. Bronze. A farthing struck at the mint in 1793. Ob., bust to l. of the same design as in Nos. 95–100. "Sr. Isaac Newton." Rev., "Farthing / 1793." Design in the center. 21 mm.

102. *Newton*. Bronze. Same as No. 101 but 0.5 mm. larger. They seem to have been struck from the same die, but this is not certain. 21.5 mm.

103. *Newton*. Bronze. A farthing struck in 1793. Ob., bust to l. of the same design as in Nos. 100 and 101, but not from either of the above dies. Legend, "Is. Newton." Rev. "Farthing." Monogram F.H. (T.H.?). "1793."

104. *Newton*. Bronze. A farthing struck in 1793. Ob., bust to l., differing considerably from those in Nos. 101–103. "Isaac Newton." Rev. "Farthing." Britannia seated. "1793." 20.5 mm.

105. *Newton*. Bronze. A farthing struck in 1793. Ob., bust to l. "Ic. Newton." Rev., "Farthing." Britannia seated. "1793." 20.5 mm.

106. *Newton*. Bronze. Similar to No. 105, but not from the same die. 21 mm.

107. *Newton*. Bronze. Similar to Nos. 105 and 106, but not from the same die. 22 mm.

108. *Pascal*, Blaise (1623–1662). Bronze. Ob., bust by Dubois, to r. Rev. "Né / à Clermont / en M.DC.XXIII./ Mort en M.DC.LXII." One of the same series as No. 7. 41 mm.

109. *Pascal*. Bronze gilt. Ob., bust by Dantzell, to r. Rev. "Né/à Clermont / (Auvergne)/ 1623./ Mort/ à Paris/ 1662. 50 mm.

110. *Pascal*. Bronze. Ob., bust by Pingret, to l. "Blasius Pascal." On the edge, "Monachii." Rev., "Natus / Claromontio / Arverniae / An. M.DC.XXIII. / Obiit / An M.DC.-LXII./" One of the same series as No. 6. Dated 1823. 41 mm.

111. *Pascal*. Bronze. Ob., bust by I.D., to l. "Blaise Pascal." Rev. "Philosophe / M. 1662." Winged figure. 28 mm.

112. *Poinsot*, Louis (1777–1859). Bronze. Medallion by David d'Angers, profile to r. Dated 1843. 161 mm.

113. *Poisson*, Simeon Denis (1781–1840). Bronze. Medallion by David d'Angers, profile to l. Dated 1830. 169 mm.

114. *Pythagoras* (c. 572–c. 501 B.C.). Bronze coin of Samos. Obv., seated figure of Pythagoras. "[P]ythagores Sam Ion." in Greek. Electrotype from the original in the British Museum, and so for Nos. 115–119. 29 mm.

115. *Pythagoras*. Bronze coin of Samos. Obvs., similar to No. 114. "Pythag . . . Sam. Ion." 31 mm.

116. *Pythagoras*. Bronze. Similar to No. 115, but of different coinage. "[. . .] thagores Sa . . . Ion." 31 mm.

117. *Pythagoras*. Bronze. Similar to No. 116, but of different coinage. "Pythagores. Sam. Ion." 30 mm.

118. *Pythagoras*. Bronze. The same in style as No. 117, but smaller. The inscription is illegible. 20 mm.

119. *Pythagoras*. Bronze. The same in style as No. 118, but still smaller. "[. . .] Pythag." 19 mm.

120. *Quetelet*, Lambert Adolphe Jacques (1796–1874). Bronze. Medallion by David d'Angers. Profile to r. 160 mm.

121. *Secchi*, Angelo (1818–1878). Bronze. Ob., reclining figure of Father Tiber; wolf at right nursing Romulus and Remus, by C. Voigt. "Alterivs sic Altera / Poscit opem." Rev. "Angelvs / Secchi / Pont. Acad. Tiberinae / Praeses / A. Ab Acad. Inst. LXV / V.C.MMDC-XXIX." 44 mm.

122. *Steiglehner*, Cölestin (George Christoph) (1738–1819). Bronze. Ob., bust by Losch, to r. "Colest. Steiglehner letzter Furst Abt V. St. Emer. in Regensb." Rev. "Geb. 1738 Gest. 1819 / Dem Vielgeliebten Lehrer / dem Steten Beförderer/ der Wissenschaft/ und Kunst / widmeten dies Denkmal / seine Freunde / und Verehrer." 41 mm.

123. *Stevin*, Simon (1548–1620). Bronze. Ob., bust by De Hondt, to r. "Simon Stevin / Inaug. MDCCCXLVI. F. De Hondt." Rev., arms, "S.P.O.B." 48 mm.

124. *Traversa*, Giulio. Bronze. Ob., bust by Restelli, ¾ to r. "Giulio Traversa." Rev., "Al / Geometra / Giulio Traversa / " . . . 19 Giugno 1887." 41 mm.

125. *Thales.* (*c.* 640–*c.* 546 B.C.). Bronze. Renaissance medal. Ob., bust to r. "Thaleuos Miletiou" in Greek letters. Rev., standing figure. Electrotype from the original in the British Museum. 30 mm.

126. *Viviani,* Vincentius (1622–1703). Bronze. Ob., bust to r. "Vinc. Viv. Noviss. Magni Galilæi Discipvlvs æt. LXXIX. Qvi primvs, A. Sal. M.D.CIIIII. Rev. "Ostendit æqvas. et sphæricas svperficies nil recti habentes notis rectangvlis." Electrotype from the original in the British Museum. 90 mm.

127. *Wolf,* Christian (1679–1754). Bronze. Ob., bust by J. Dassier, to r. "Christianus Wolfius." Rev. "Sedes Fructusque Perennis." Seated figure of woman, altar, cornucopia. 42 mm.

128. *Wren,* Christopher (1632–1723). Bronze. Ob., bust by W. Wilson, to r. Rev., St. Paul's cathedral, "Christopher Wren Architect—MDCCX / Si monumentum requiris / circumspice." 58 mm.

There are numerous other medals of mathematical interest in the collection, but only one will be mentioned:

129. *Medal commemorating the Success of the Metric System,* struck in 1874, bronze, from a die engraved by J. C. Chaplain. Ob., four seated figures, one holding the standard meter. "Popvlorvm. Concordiae. Sacrum. / Paris. 1872/ J. C. Chaplain fecit / MDCCCLXXIV." Rev., "Nova Pondervm. ac. Mensvrarvm. Ratio. in. Gallia. Institvta. M. Germ. A. Reip. Conditae III./" surrounded by a further inscription. 100 mm.

VELOCITIES IN THE EINSTEIN THEORY.

By WILLIAM LOWELL PUTNAM.[1]

Before dealing with Einstein's hypothesis let us recall the discovery of Michelson in regard to light. He found that if a flash of light be sent to a mirror and reflected, the time from the starting of the flash to its getting back is always the same, if the distance is the same, no matter in what direction it is sent.

As we know, if a farmer is walking behind his hay rigging and hurries up to the horse's head and then hurries back to his place behind the hayrigging, he has not only gone much further but has taken much more time than if the hayrigging had been standing still.

But though the earth is rushing at times through the ether at a speed of at

[1] William Lowell Putnam was born in Roxbury, Mass., November 22, 1861, and graduated from Harvard in 1882, doing very distinguished work in mathematics. Although he followed the law as a life career, his enthusiasm for mathematics never flagged. For years he was chairman of the committee appointed by the Board of Overseers to visit the mathematical department at Harvard. His sympathetic interest and coöperation in all problems of mathematical teaching were a help and inspiration to those engaged in the work. All of them felt a sense of keen personal loss at his death in July, 1924.

Shortly before his death, he had read, among others, my book on the theory of relativity (*Relativity and Modern Physics*). This seems to have led him to write the present article which was found among his papers by Mrs. Putnam. He had talked over with her his desire to make the theory clearer to the layman in as simple language as possible. His very interesting exposition appears here practically as written by him, save for two explanatory footnotes kindly added by Professor H. P. MANNING.

What a pity it is that whereas in England brilliant amateurs like Sir Thomas Heath and Major MacMahon have added very greatly to mathematical scholarship, in America the non-professional student of mathematics is so rare as to be almost unknown.—G. D. BIRKHOFF.

least 30 miles a second, light takes the same time to go to the mirror and back in the direction in which the earth is travelling as it takes in a direction across this motion.

Michelson's proof of this fact is as convincing as is that which supports our belief in most truths of physics. It is about as convincing as the proof that the earth moves around the sun or that ice is lighter than water.

Let us then assume it to be true and by its aid study Newton's laws.

We begin with bodies moving in a single straight line with uniform velocities, and we assume them to act as if their masses were concentrated at their centers of gravity.

In order to use Michelson's discovery we must assume that one of the bodies, A, emits flashes of light which are reflected to it from other bodies B, C, etc.

Suppose an observer at A draws a chart in which the distance measured along the horizontal axis represents the time in seconds (from some selected epoch O) at which light rays passing to the right leave A. The distance measured along the vertical axis represents the time in seconds at which rays passing to the left reach A. All these times can be taken by A.

Suppose now that A has chosen for his epoch, noon, January 1, 1924, and that exactly at the epoch he sends a flash of light to the right towards B, C, and suppose he gets the reflection back from B at two seconds past twelve and from C at five seconds past, and let the sending of the reflections from B, C be called events which we will designate on our time chart as P_1, P_2 respectively. The coördinates of P_1 are (0, 2), of P_2, (0, 5).

In general every event P on any body Q may be designated by two coördinates on A's chart: t_1, the time at which a ray of light going to the right would have to leave A in order to reach Q at the time of the event and t_2 the time at which the ray of light leaving Q at the time of the event P and going to the left reaches A.

Let us consider a series of events for which the coördinates are equal, (0, 0), (1, 1), (2, 2), (3, 3). Clearly they lie on a straight line making angles of 45° with the axes.

For all events on this line the times of sending the original flash and the times of receiving the return flash are the same. These events must all occur on A itself. It is therefore the life line of A, or more briefly A's line. The events (1, 1), (2, 2), (3, 3), etc. occur respectively 1 second, 2 seconds, and 3 seconds later than the epoch but they all occur on A.

In general, the interval $t_2 - t_1$ between the time of sending a flash to the right from A to reach an event P, and the time of receiving the return flash to the left is proportional to the distance from A to P.

If a series of flashes are sent to the right from A to B, and the reflections always return after the same interval, then the distance from A to B is not changing.

Suppose flashes sent to B at the epoch and at intervals of a second afterward are reflected so as to reach A respectively at one second after the epoch, at $2\frac{1}{2}$ seconds after, at 4 seconds after, and at $5\frac{1}{2}$ seconds after. Then the reply to

the one sent at the epoch was received 1 second after the flash was sent, the reply to the one sent 3 seconds after the epoch was received $2\frac{1}{2}$ seconds after the flash was sent. This shows that the bodies are getting uniformly further from each other.

If the intervals between the sending of the flash and the receipt of the reply are growing longer, then the bodies are getting further apart. If the intervals are growing shorter, then the bodies are getting nearer to each other.

As all the bodies we are considering are moving in one straight line in space, each with a uniform velocity, they must either be moving at the same speed and in the same direction as A or they must be approaching A at a uniform rate or receding from A at a uniform rate.

Therefore since A's line on his chart is a straight line the lines of all other bodies as shown on A's chart must also be straight lines. Assume two particles A and B starting from the same epoch and moving each with uniform velocity to the right, B moving faster than A. These are shown in two positions in Fig. 1.

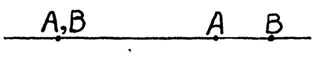

FIG. 1.

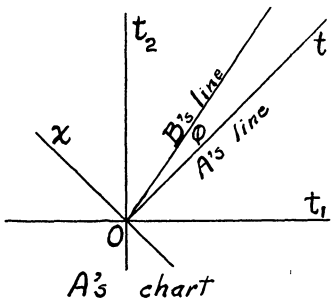

FIG. 2.

In the second position B is some distance to the right of A and the speed with which this distance increases, that is, the distance divided by the time from the epoch is the velocity of B with respect to A or, in a reverse direction, of A with respect to B.

Evidently the speed with which B is receding from A is proportional to the tangent of the angle between the life lines of A and B on A's chart(Fig. 2).[1] If φ represent this angle and v this velocity then $v = k \tan \varphi$ where the value of the constant k depends on our unit of velocity.

The vertical axis of A's chart represents a ray of light travelling to the right from the epoch. Let us assume the velocity of light as our unit; hence for this particular line $v = 1$. For the same line $\varphi = 45°$ and therefore $\tan \varphi = 1$. Therefore, with the unit chosen $k = 1$ and $v = \tan \varphi$.

[1] This statement by Mr. Putnam is easily verified but perhaps not entirely evident. It will be found that the time and distance of an event can be represented as coördinates with axes bisecting the angles formed by the t_1 and t_2 axes. That is, time is measured along A's line, while the distance of an event is the distance from this line of the point which represents the event, (*Relativity and Modern Physics*, page 22). Hence the speed of a particle, *i.e.*, the speed with which it is receding from A, is proportional to the slope of its line in the coördinate system formed by these axes.

Let us now consider the addition of velocities.

Assume that A, B, and C start from the same epoch with different velocities. Let

v_1 = velocity of B with reference to A,
v_2 = velocity of C with reference to B,
v_3 = velocity of C with reference to A.

Let

OA be A's line on A's chart,
OB be B's line on A's chart,
OC be C's line on A's chart.

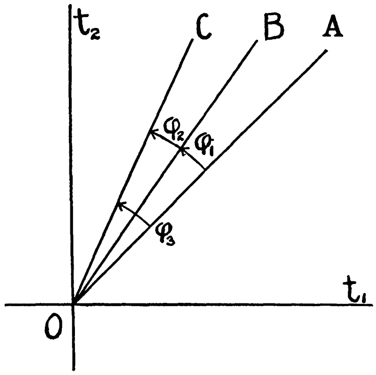

FIG. 3.

Let angle $AOB = \varphi_1$, $BOC = \varphi_2$, $AOC = \varphi_3$. Then $v_1 = \tan \varphi_1$, $v_3 = \tan \varphi_3$.

For v_2, two entirely distinct values are suggested by A's chart. First $v_2 = v_3 - v_1$, second $v_2 = \tan \varphi_2 = \tan (\varphi_3 - \varphi_1)$. These two values are about equally plausible but they are entirely different from each other. As Einstein shows v_2, the relative velocity of B and C, can be ascertained only from the chart of either B or C.

For this purpose a means of transformation from A's chart to B's is necessary.

As A is any particle we can draw a similar chart for B. But as B is moving to the right faster than A the rays of light moving to the right passing A at intervals of one second will pass B at intervals of more than a second and fewer such rays will pass B in a given time.

If t_1 is the time for A at which a given ray going to the right passes A, and t_2 the time for B at which the same ray passes B, then $t_1' = k_1 t_1$, where k_1 is a constant multiplier; and, by the same reasoning, if t_2 is the time at which a given ray going to the left passes A and t_2' is the time at which the same ray passes B, then $t_2' = k_2 t_2$ where k_2 is another constant multiplier.[1]

Let us now take any event P having on A's chart the coördinates t_1, t_2.

Draw the line OP making an angle θ with the horizontal axis.

Now show the same event on B's chart calling it P' with coördinates t_1', t_2', and draw the line OP' making an angle of θ' with the horizontal axis.

That is, $t_1' = k_1 t_1$ and $t_2' = k_2 t_2$, and $t_2'/t_1' = k_2 t_2/k_1 t_1$. But $t_2/t_1 = \tan \theta$ and $t_2'/t_1' = \tan \theta'$. Therefore $\tan \theta' = (k_2/k_1) \tan \theta$, where k_2/k_1 is the same for all values of θ.

If β is the angle which B's line on A's chart makes with the horizontal axis, and if we take our event P on B's line, then $\theta = \beta$.

On B's chart B's line makes an angle of 45° with the horizontal axis so that $\theta' = 45°$.

In this particular case therefore the equation $\tan \theta' = (k_2/k_1) \tan \theta$ becomes $\tan 45° = (k_2/k_1) \tan \beta$, or $k_2/k_1 = \cot \beta$.

[1] This statement by Mr. Putnam is also easily verified but perhaps not entirely self-evident. (*Relativity and Modern Physics*, pages 26–29.)

Therefore any particle, whose life line on A's chart passes through the epoch and makes the angle θ with the horizontal, will on B's chart have a life line making an angle θ' with the horizontal, where $\tan\theta' = \cot\beta\tan\theta$.

Applying this formula to find the position of A's line on B's chart we have $\theta = 45°$, $\tan\theta = 1$, so that $\tan\theta' = \cot\beta$ and $\theta' = 90° - \beta$.

If we let φ be the angle between A's line and B's line on A's chart and φ' be the angle between B's line and A's line on B's chart, then $\varphi = \beta - 45°$, $\varphi' = \theta' - 45°$, from which $\varphi' = (90° - \beta) - 45° = 45° - \beta = -\varphi$. That is, the angle between A's and B's line is the same on both charts.

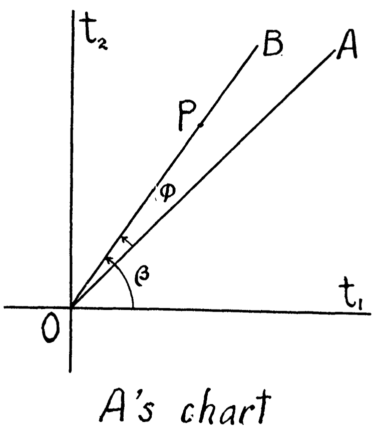

FIG. 4.

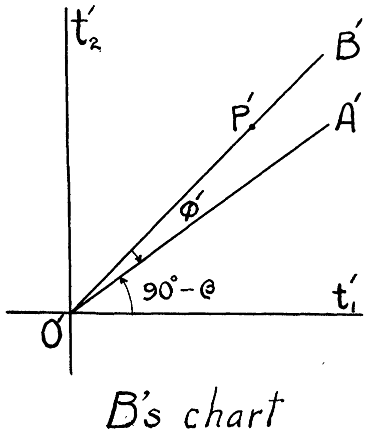

FIG. 5.

Let us now take three bodies whose lines pass through the epoch, shown on A's chart as OA, OB, OC whose angles with the horizontal are respectively 45°, β and γ.

Let the relative velocity of B to A be v_1, and of C to A, v_3. Then

$$v_1 = \tan(\beta - 45°) = \frac{\tan\beta - 1}{\tan\beta + 1},$$

$$v_3 = \tan(\gamma - 45°) = \frac{\tan\gamma - 1}{\tan\gamma + 1}.$$

From the first of these we deduce

$$\tan\beta = \frac{1 + v_1}{1 - v_1},$$

and from the second

$$\tan\gamma = \frac{1 + v_3}{1 - v_3}.$$

If v_2 is the relative velocity of C to B, to ascertain it we must transpose to B's

chart. On this chart let γ' be the angle of C's line with the horizontal then

$$v_2 = \tan(\gamma' - 45°) = \frac{\tan\gamma' - 1}{\tan\gamma' + 1}.$$

But $\tan\gamma' = \cot\beta\tan\gamma$, so that

$$v_2 = \frac{\cot\beta\tan\gamma - 1}{\cot\beta\tan\gamma + 1} = \frac{\tan\gamma - \tan\beta}{\tan\gamma + \tan\beta}$$
$$= \frac{(1+v_3)(1-v_1) - (1-v_3)(1+v_1)}{(1+v_3)(1-v_1) + (1-v_3)(1+v_1)} = \frac{v_3 - v_1}{1 - v_1 v_3},$$

from which Einstein's equation follows,

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}.$$

It is obvious that Newton's equation $v_3 = v_2 + v_1$ is inconsistent with our chart and therefore with our understanding of the Michelson-Morley discovery.

When we remember that our unit of velocity is 183,000 miles per second we see that all ordinary velocities are such small fractions that their cubes may be neglected and Einstein's equation reduces to Newton's.

CERTAIN RECURSION FORMULAS CONNECTED WITH THE SOLUTION OF $X^2 + Y^2 = Z^2$.

By P. H. DAUS, University of California, Southern Branch.

1. Introduction. In the MONTHLY for 1914, Professor G. A. Osborne considered the relations connecting the different solutions of

$$x^2 + y^2 = z^2 \qquad \text{when} \qquad y = x + 1. \tag{A}$$

Let us consider the case when

$$y = m + x, \quad m, \text{ a positive integer.} \tag{B}$$

The actual solution of the equation subject to this condition has been considered by Fermat, Frenicle, T. Pepin and others. (See Dickson's *History of the Theory of Numbers,* vol. II, pp. 183–184.) The purpose of this paper is to show how each solution can be determined in terms of the three preceding ones, indicating a marked similarity to ternary continued fractions.

Without loss of generality, we may confine our attention to the case when x, y, z are relatively prime.

2. Sets of Solutions. If in

$$x^2 + y^2 = z^2 \tag{1}$$

we put $y = x + m$ and solve for x, we get

$$x = \frac{-m \pm \sqrt{2z^2 - m^2}}{2}.$$

In order that x shall be rational, it is necessary that $2z^2 - m^2 = r^2$, which gives

$$x = (r - m)/2, \qquad y = (r + m)/2, \qquad r = x + y = 2x + m, \tag{2}$$

using the positive sign with the radical. (It will be noted that the use of the negative sign is equivalent to interchanging x and y and changing the signs of both.) That is, we must determine z and r, so that

$$2z^2 - r^2 = m^2. \tag{3}$$

From the theory of quadratic forms,[1] it follows that we must find a substitution which transforms the form $(2, 0, -1)$ into (m^2, l, n), where $l^2 \equiv 2 \pmod{m^2}$. If m is a product of k distinct primes, each prime must be of the form $8t \pm 1$, and this congruence has 2^k incongruent solutions (M, p. 36), giving 2^k sets of solutions, and this theory determines methods for finding these solutions, the actual calculations being most readily made by means of continued fractions.[2]

3. Relations between Solutions of a Set. Let us suppose that a solution (z, r) of (3) is known. Then it follows (M, pp. 88–91) that all the solutions belonging to its set will be of the form (z_n, r_n) where $(r_n + z_n\sqrt{2}) = (r + z\sqrt{2})(t + u\sqrt{2})^n$ and (t, u) is the smallest positive solution of $t^2 - 2u^2 = +1$, viz.: $t = 3$; $u = 2$.

We shall find it convenient to indicate the solutions of one set by even subscripts and those of another by odd subscripts. From

$$(r_{2n+2} + z_{2n+2}\sqrt{2}) = (r_{2n} + z_{2n}\sqrt{2})(3 + 2\sqrt{2}),$$

it follows that $r_{2n+2} = 3r_{2n} + 4z_{2n}$ and $z_{2n+2} = 2r_{2n} + 3z_{2n}$, from which we find, using (2),

$$\begin{aligned} x_{2n+2} &= 3x_{2n} + m + 2z_{2n}, \\ y_{2n+2} &= 3x_{2n} + 2m + 2z_{2n}, \\ z_{2n+2} &= 4x_{2n} + 2m + 3z_{2n}. \end{aligned} \tag{4}$$

We may note here, since from (3) r must be odd, and m is odd, that x and y are integers.

If we combine the equations of (4) for different subscripts, we will obtain (where s is replaced by x, y, z in turn)

$$s_{2n+6} = 7s_{2n+4} - 7s_{2n+2} + s_{2n}. \tag{5}$$

[1] Mathews, *Theory of Numbers*, pp. 58 and 65. Future references to this book will be indicated by (M, p. —).

[2] See note to example No. 2 at the end of this paper.

For example

$$\begin{aligned} x_{2n+4} &= 3x_{2n+2} + m + 2z_{2n+2} \\ &= 3x_{2n+2} + m + 8x_{2n} + 4m + 6z_{2n} \\ &= 3x_{2n+2} + 5m + 8x_{2n} + 3x_{2n+2} - 9x_{2n} - 3m \\ &= 6x_{2n+2} - x_{2n} + 2m. \end{aligned}$$

Likewise $x_{2n+6} = 6x_{2n+4} - x_{2n+2} + 2m$ and by subtracting

$$x_{2n+6} = 7x_{2n+4} - 7x_{2n+2} + x_{2n}. \tag{5'}$$

Similarly (5) can be established for y and z, as well as the related expressions written for odd subscripts, viz.:

$$\begin{aligned} x_{2n+3} &= 3x_{2n+1} + m + 2z_{2n+1}, \\ y_{2n+3} &= 3x_{2n+1} + 2m + 2z_{2n+1}, \qquad (6) \\ z_{2n+3} &= 4x_{2n+1} + 2m + 3z_{2n+1}, \end{aligned}$$

$$s_{2n+5} = 7s_{2n+3} - 7s_{2n+1} + s_{2n-1}. \tag{7}$$

4. Arrangement of Initial Solutions. Let x_0, y_0, z_0 be the solution of

$$x^2 + y^2 = z^2, \qquad y = x + m, \tag{1}$$

obtained from the smallest value of z of a set. We will call this the fundamental solution and designate it by s_0. We obtain s_1 as follows: Determine d so that $(x_1y_1z_1)$, defined as $(x_0 + d, y_0 + d, z_0 + d)$, is a solution of (1). This is always possible in integers, for

$$(x_0 + d)^2 + (y_0 + d)^2 = (z_0 + d)^2$$

gives

$$d = 2(z_0 - x_0 - y_0) = 2(x_1 + y_1 - z_1). \tag{8}$$

Now define s_{-n} by $x_{-n} = -y_{n-1}$; $y_{-n} = -x_{n-1}$; $z_{-n} = +z_{n-1}$. We indicate the initial array below:

$$\begin{array}{cccc} -y_1 & -x_1 & z_1 & s_{-2} \\ -y_0 & -x_0 & z_0 & s_{-1} \\ x_0 & y_0 & z_0 & s_0 \\ x_1 & y_1 & z_1 & s_1. \end{array}$$

We will show that all solutions of (1) can be obtained in k sets from the three preceding ones by certain recursion formulas, which we write (s being replaced by x, y, z in turn)

$$s_{2n+1} = ps_{2n} - ps_{2n-1} + s_{2n-2}, \tag{9}$$

$$s_{2n+2} = qs_{2n+1} - qs_{2n} + s_{2n-1}, \tag{10}$$

for all values of n, where p and q, not necessarily integers, are

$$p = \frac{x_1 + y_1}{x_0 + y_0} \quad (11); \qquad q = \frac{2(x_0 + y_0 + z_0)}{d}. \tag{12}$$

5. Determination of the First Solutions. In order to establish our general results by mathematical induction, it will be necessary to determine the relations connecting s_1, s_2, s_3 with the preceding solutions.

As $y = m + x$, it is evident that, if (9) and (10) are true for the x's, they will be true for the y's. It is readily seen that, since $z_{-1} = z_0$, (9) is true for the z's, when $n = 0$, independent of p. If we are also to have (9) true for the x's,

$$x_1 = p(x_0 + y_0) - y_1 \qquad \text{or} \qquad p = \frac{x_1 + y_1}{x_0 + y_0}. \tag{11}$$

If (10) is true for $n = 0$,

$$x_2 = q(x_1 - x_0) - y_0 = qd - y_0; \qquad y_2 = qd - x_0; \qquad z_2 = qd + z_0; \tag{13}$$

must satisfy (1), *i.e.*, $(qd - y_0)^2 + (qd - x_0)^2 = (qd + z_0)^2$. Solving for q, using the fact that s_0 is a solution of (1), we get

$$q = \frac{2(x_0 + y_0 + z_0)}{d}. \tag{12}$$

We note that x_2, y_2, z_2 are integers, since qd is an integer, and also that x_2, y_2, z_2 can be written in the form of (4), identifying s_2 as obtained here with s_2 in **3**.

Now let $s_3 = p(s_2 - s_1) + s_0$. We will show that $(x_3y_3z_3)$ is a solution of (1). First we will show that $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are divisible by $x_0 + y_0$, which will make x_3, y_3, z_3 integers. $x_2 - x_1 = qd - y_0 - x_1 = qd - x_0 - y_0 - d = 2(x_0 + y_0 + z_0) - x_0 - y_0 - 2(z_0 - x_0 - y_0)$ or $x_2 - x_1 = 3(x_0 + y_0) = y_2 - y_1$. Similarly $z_2 - z_1 = qd + z_0 - z_1 = qd - d = 2(x_0 + y_0 + z_0) - 2(z_0 - x_0 - y_0) = 4(x_0 + y_0)$. This gives

$$x_3 = 3(x_1 + y_1) + x_0; \qquad y_3 = 3(x_1 + y_1) + y_0; \qquad z_3 = 4(x_1 + y_1) + z_0. \tag{14}$$

Equations (14) can be written in the form of (6). For example

$$x_3 = 3x_1 + 3y_1 + x_0 = 4x_1 + 3y_1 - d = 4x_1 + 3y_1 - 2x_1 - 2y_1 + 2z_1,$$

i.e., $x_3 = 3x_1 + m + 2z_1$; and similarly for y_3 and z_3. This identifies s_3 as defined here with s_3 as obtained in **3** from s_1, and shows that s_3 is a solution of (1), since s_1 is.

6. Let us define

$$s_{2n+2}' = q(s_{2n+1} - s_{2n}) + s_{2n-1}, \tag{15}$$
$$s_{2n+3}' = p(s_{2n+2} - s_{2n+1}) + s_{2n}. \tag{16}$$

We will establish the equivalence of s as defined in **3**, with s' as defined here,

by using equations (4) and (6). This equivalence has already been established for s_2 and s_3, and it is readily established for s_1.

$$\begin{aligned} x_1 &= x_0 + d = x_0 + 2(z_0 - x_0 - y_0) = -3y_0 + m + 2z_0 = 3x_{-1} + m + 2z_{-1}, \\ y_1 &= y_0 + d = y_0 + 2(z_0 - x_0 - y_0) = -3y_0 + 2m + 2z_0 = 3x_{-1} + 2m + 2z_{-1}, \\ z_1 &= z_0 + d = z_0 + 2(z_0 - x_0 - y_0) = -4y_0 + 2m + 3z_0 = 4x_{-1} + 2m + 3z_{-1}. \end{aligned}$$

This establishes the equivalence of s_1 as defined here with that as defined by equation (6).

Now let us assume the equivalence has been established for all values up to and including s_{2n+1}. Then, using (6) and (15),

$$\begin{aligned} x_{2n+2}' &= q(x_{2n+1} - x_{2n}) + x_{2n-1} \\ &= q(3x_{2n-1} + m + 2z_{2n-1} - 3x_{2n-2} - m - 2z_{2n-2}) + x_{2n-1} \\ &= 3[q(x_{2n-1} - x_{2n-2}) + x_{2n-3}] + x_{2n+1} - 3x_{2n-1} - 2z_{2n-1} \\ &\qquad + 2[q(z_{2n-1} - z_{2n-2}) + z_{2n-3}] \\ &= 3x_{2n} + m + 2z_{2n} \quad (\text{using } x_{2n-1} = 3x_{2n-3} + m + 2z_{2n-3}) \\ &= x_{2n+2}; \\ z_{2n+2}' &= q(z_{2n+1} - z_{2n}) + z_{2n-1} \\ &= q(4x_{2n-1} + 2m + 3z_{2n-1} - 4x_{2n-2} - 2m - 3z_{2n-2}) + z_{2n-1} \\ &= 4[q(x_{2n-1} - x_{2n-2}) + x_{2n-3}] + z_{2n+1} - 4x_{2n-3} - 3z_{2n-3} \\ &\qquad + 3[q(z_{2n-1} - z_{2n-2}) + z_{2n-3}] \\ &= 4x_{2n} + 2m + 3z_{2n} = z_{2n+2}. \quad \text{Likewise } y_{2n+2}' = y_{2n+2}. \end{aligned}$$

By advancing the subscripts one and changing q to p, we establish the corresponding relations for odd subscripts. It follows then that the solutions as defined by (15) and (16) are equivalent to the solutions defined by (4) and (6), or their equivalents (5) and (7).

7. Number of Expansions. We see then that each expansion, defined by equations (9) and (10) in **4**, accounts for two sets of congruent solutions of **2**. It follows then that if m is a prime or a power of a prime of the proper form, our expansion in **4** gives all the solutions, while if m has k distinct prime factors there will be k expansions. It is to be noted likewise, since the results of section **4** did not depend upon which solution we took as s_0, *that we can begin with any solution and will always get the same group to which it belongs,* the difference merely being the relative positions of the two subgroups of which it is composed. In particular, when m is a prime or a power of a prime, we get all solutions if we start with any one. This is illustrated by example 3 below. We conclude with

8. Numerical Examples. EXAMPLE 1.

If $m = 1$, $l^2 \equiv 2 \pmod 1$ really has only one solution and this leads to the trivial solution (0, 1, 1). No advantage is gained by the use of (9) and (10), while all solutions are readily obtained by finding s_2 and s_3 from (4) and then using (5).

	x	y	z	
	0	1	1	s_1
	3	4	5	s_2
	20	21	29	s_3
$-7, 7$	119	120	169	s_4
$-7, 7$	696	697	985	s_5
$-7, 7$	4059	4060	5741	s_6
		etc.		

EXAMPLE 2. $m = 7$.

The congruence $l^2 \equiv 2 \pmod{49}$ has the solutions $l \equiv \pm 10 \pmod{49}$. It will be noticed, if $(2, 0, -1) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (m^2, l, n)$, that $(2, 0, -1) \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} (m^2, -l, n)$. The difference between the two solutions is that the first yields $z = \alpha$, $r = \gamma$; the second $z = \alpha$, $r = -\gamma$, which is equivalent to using the negative sign with the radical. It will be convenient to use $l \equiv -10$, because the principal root of $(49, -10, 2)$ is positive. It is then found [1]

$$(2, 0, -1) \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix} (49, -10, 2), \qquad \text{or} \qquad z = 5, \quad r = 1.$$

This leads to $s_0 = (-3, 4, 5)$ and $d = 8$; $s_1 = (5, 12, 13)$, $p = 17$, $q = 3/2$, and we indicate the first 8 solutions below:

	x	y	z	
	-12	-5	13	s_{-2}
	-4	3	5	s_{-1}
	-3	4	5	s_0
$-17, 17$	5	12	13	s_1
$-3/2, 3/2$	8	15	17	s_2
$-17, 17$	48	55	73	s_3
$-3/2, 3/2$	65	72	97	s_4
$-17, 17$	297	304	425	s_5
$-3/2, 3/2$	396	403	565	s_6
$-17, 17$	1748	1755	2477	s_7
$-3/2, 3/2$	2325	2332	3293	s_8.
		etc.		

The division into two groups is indicated below:

GROUP I.

	-15	-8	17	s_{-3}
	-4	3	5	s_{-1}
	5	12	13	s_1
$-7, 7$	48	55	73	s_3
$-7, 7$	297	304	425	s_5
$-7, 7$	1748	1755	2477	s_7
		etc.		

GROUP II.

	-12	-5	13	s_{-2}
	-3	4	5	s_0
	8	15	17	s_2
$-7, 7$	65	72	97	s_4
$-7, 7$	396	403	565	s_6
$-7, 7$	2325	2332	3293	s_8
		etc.		

EXAMPLE 3. If we start with any other solution, as s_1' (say s_4), *we will get all the solutions*, but the two sets in example 2 will be in different relative positions. $s_1' = s_4 = (65, 72, 97)$ gives $s_0' = (-15, -8, 17)$, after we have first found $d = 80$. $p' = -137/23$; $q' = -3/20$. The expansion can readily be formed by the reader. He will find the solutions appear in the following order, $s_{-5}, s_2, s_{-3}, s_4, s_{-1}, s_6, s_1, s_8$, etc.

[1] This may be found conveniently by the use of the following lemma: If the principal root of (a, b, c) be expanded into a continued fraction, and if $\dfrac{\sqrt{D} + P_n}{Q_n}$ and $\dfrac{A_n}{B_n}$ represent the nth complete quotient and convergent sets, then

$$(a, b, c) \begin{pmatrix} A_n(-1)^n A_{n+1} \\ B_n(-1)^n B_{n+1} \end{pmatrix} ((-1)^n Q_{n+1}, P_{n+2}, (-1)^{n+1} Q_{n+2}).$$

Application of this lemma to the roots of $(2, 0, -1)$ and $(49, -10, 2)$ gives the desired substitution, which might have been written by inspection in this simple example.

ON THE THEORY OF EQUATIONS FROM THE STANDPOINT OF VECTOR ANALYSIS.

By PETER FIELD, University of Michigan.

The problem of solving an algebraic equation requires that a number which satisfies a certain condition shall be determined. If a graphical solution is attempted, the case of real roots requires a different construction from that for imaginary roots. The advantage of viewing the problem from the standpoint of vector analysis lies in the fact that it gives a geometrical point of view which holds equally well for real or imaginary roots and for real or imaginary coefficients. The idea involves simply the elementary properties of complex numbers.

Let $\mathbf{t}$ be a given unit vector, let a_j be a known operator which when operating on any vector multiplies the modulus of the vector by a positive number r_j and turns the vector through an angle φ_j in a given plane, and let x be an unknown operator of the same nature as a_j, the corresponding multiplier and angle being r and φ. In the notation of Burali-Forti and Marcolongo, *Éléments de Calcul Vectoriel*, $a_j = r_j e^{i\varphi_j}$, $x = re^{i\varphi}$. The equation

$$(a_n x^n + a_{n-1} x^{n-1} + \cdots a_0)\mathbf{t} = 0, \tag{1}$$

expresses the condition that x shall be so chosen that the polygon formed by the $n + 1$ vectors $a_j x^j \mathbf{t}$ shall close. If φ_j is restricted to the values 0 and π, the polygon will have the angle between its consecutive sides φ or $\pi - \varphi$. This corresponds to the equation with real coefficients. In this case if $x = re^{i\varphi}$ is a solution of (1), so is $x = re^{-i\varphi}$. The second polygon differs from the first only in the sense of rotation. In other words, in an equation with real coefficients the imaginary roots enter in pairs. Corresponding to the case of an equation with real coefficients and a real root, there results a polygon whose vertices all lie on the same line.

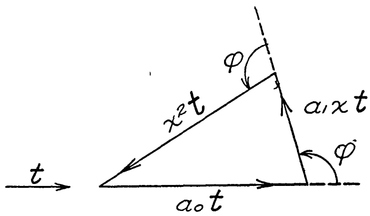

FIG. 1.

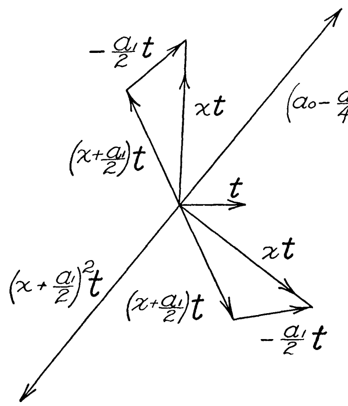

FIG. 2.

Consider in particular the case corresponding to a quadratic equation with

positive real coefficients and imaginary roots. The vector triangle $(x^2+a_1x+a_0)\mathbf{t}$ is isosceles (Fig. 1). Therefore $x = re^{i\varphi}$ where $r = \sqrt{a_0}$, $\varphi = \arccos(-a_1/2\sqrt{a_0})$ and the vector $x\mathbf{t}$ is readily constructed. The most simple construction which is applicable in all cases appears to result from considering the equation in the form

$$\left[\left(x+\frac{a_1}{2}\right)^2 + a_0 - \frac{a_1^2}{4}\right]\mathbf{t} = 0.$$

The two values of $x\mathbf{t}$ are indicated in Fig. 2.

A SUGGESTED LIST OF MATHEMATICAL BOOKS FOR JUNIOR COLLEGE LIBRARIES.[1]

I. ELEMENTARY BOOKS.[2]

Algebra.

DICKSON, L. E. First Course in the Theory of Equations. 1922. \$1.75. Wiley.

FINE, H. B. College Algebra. \$2.40. Ginn.

HALL, H. S. and KNIGHT, S. R. Higher Algebra (Revised and enlarged by F. L. Sevenoak). \$1.60. Macmillan.

LIPKA, J. Graphical and Mechanical Computation. 1921. \$4.00. Wiley.

SMITH, C. A Treatise on Algebra. \$3.00. Macmillan.

Geometry.

ALTSHILLER-COURT, N. College Geometry. \$4.00. Johnson Pub. Co.

CARSLAW, H. S. Elements of Non-Euclidean Plane Geometry and Trigonometry. \$1.80. Longmans, Green & Co.

GODFREY, C. and SIDDONS, A. W. Modern Geometry. \$2.00. Macmillan.

HALSTED, G. B. Rational Geometry (2d edition). \$1.75. Wiley.

KENISON, E. and BRADLEY, H. C. Descriptive Geometry. \$2.25. Macmillan.

LEHMER, D. N. Synthetic Projective Geometry. \$1.28. Ginn.

SMITH, C. and BRYANT, S. Euclid's Elements of Geometry. London, 1904. Bks. I–IV. London. 1904. 3 s. Macmillan.

SMITH, C. Geometrical Conics. 1894. 7 s. Macmillan.

Analytic Geometry.

OSGOOD, W. F. and W. C. GRAUSTEIN. Plane and Solid Analytic Geometry. 1921. \$3.00. Macmillan.

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[1] This list was prepared by Professors B. A. BERNSTEIN, F. CAJORI, E. R. HEDRICK, C. A. NOBLE and T. M. PUTNAM, of the University of California.

A similar list was prepared by a committee of the Association and was printed in the MONTHLY (*1917*, 368–376). EDITOR.

[2] The classification "elementary" and "advanced" is made on the supposition that an "advanced" book is one that is designed for readers having considerable mathematical equipment or considerable maturity of mind.

SMITH, C. An Elementary Treatise on Conic Sections. Revised and enlarged ed., 1910. $2.75. Macmillan.

SMITH, C. An Elementary Treatise on Solid Geometry. 1907. $3.50. Macmillan.

Calculus.

GIBSON, G. A. Elementary Treatise on the Calculus. 1906. $2.90. Macmillan.

OSGOOD, W. F. Introduction to the Calculus. 1922. $2.90. Macmillan.

PEIRCE, B. O. A Short Table of Integrals. (2d revised edition $1.48, Abridged $.40.) Ginn.

PHILLIPS, H. B. Differential Equations. $1.50. Wiley.

THOMPSON, S. P. Calculus made easy. 2d edition. 1916. $1.00. Macmillan.

History.

BALL, W. W. R. A short History of Mathematics. $4.00. Macmillan.

CAJORI, F. History of Elementary Mathematics. 2d edition, 1917. $2.25. Macmillan.

CAJORI, F. History of Mathematics. 2d edition, 1919. $4.00. Macmillan.

HEATH, T. L. History of Greek Mathematics. 2 Vols. $16.70. Clarendon Press, Oxford.

KARPINSKI, L. C. History of Arithmetic. 1925. $2.00. Rand McNally.

SMITH, D. E. History of Mathematics. Vol. I, 1923. $4.00. Vol. II, 1925. $4.40. Ginn.

SMITH, D. E. and KARPINSKI, L. C. The Hindu-Arabic Numerals. 1911. $1.60. Ginn.

TROPFKE, J. Geschichte der Elementar-Mathematik. Parts I–VII, 1921–1924. Berlin. $8.50. W. deGruyter & Co.

Miscellaneous.

(ABBOTT, E. A.). Flatland. By A. Square. $1.25. Little, Brown & Co.

BALL, W. W. R. Mathematical Recreations. 7th edition. $3.50. Macmillan.

BALL, W. W. R. Introduction to String Figures. 1924. 2 s. 6 p. Heffer & Sons.

BOON, F. C. Companion to Elementary School Mathematics. 1924. $4.50. Longmans, Green & Co.

BRUHNS, C. C. A new Manual of Logarithms to Seven Places of Decimals. 1870. $2.25. Lemcke & Buechner, N. Y. (Stechert, N. Y., dealer.)

CARSON, G. ST. L. Essays on Mathematical Education. $1.20. Ginn & Co.

DAVISON, C. Subjects for Mathematical Essays. $1.60. Macmillan.

DEMORGAN, A. A Budget of Paradoxes. 2 Vols. $5.00. Open Court Publishing Co.

DUDENEY, H. E. The Canterbury Puzzles. 1908. $2.00. E. P. Dutton.

FORSYTH, C. H. Mathematical Theory of Life Insurance. $1.25. Wiley.

FORSYTH, C. H. Mathematical Analysis of Statistics. \$2.25. Wiley.
HART, W. L. Mathematics of Investment. 1924. \$2.50. With tables \$3.40. Heath.
HOBSON, E. W. Treatise on Plane Trigonometry. 4th Edition. \$5.00. Macmillan.
KARSTEN, K. G. Charts and Graphs. 1923. \$6.00. Prentice-Hall.
KELLEY, T. J. Statistical Method. 1923. \$4.00. Macmillan.
LENNES, N. J. The Teaching of Arithmetic. 1923. \$2.00. Macmillan.
MACFARLANE, A. Ten British Mathematicians of the Nineteenth Century. \$1.50. Wiley.
MANNING, H. P. Fourth Dimension Simply Explained. 1922. \$1.50. Scientific American Pub. Co.
MOULTON, F. R. Introduction to Astronomy. \$3.25. Macmillan.
PUTNAM, T. M. Mathematical Theory of Finance. 1924. \$1.75. Wiley.
RIETZ, H. L., CRATHORNE, A. R. and RIETZ, J. C. Mathematics of Finance. 1921. \$3.00. Holt.
ROW, T. S. Geometric Paper Folding. Edition by W. W. Beman and D. E. Smith. \$1.00. Open Court Pub. Co.
SCHUBERT, H. Mathematical Recreations. \$.75. Open Court Pub. Co.
SCHULTZE, A. The Teaching of Mathematics in Secondary Schools. \$1.80. Macmillan.
SKINNER, E. B. Mathematical Theory of Investment. Revised edition, 1925. \$3.40. Ginn.
SMITH, D. E. The Teaching of Geometry. \$1.40. Ginn.
SMITH, D. E. Number Stories of Long Ago. 1919. \$.60. Ginn.
VON SANDEN, H. Practical Mathematical Analysis. Translation by H. Levy. 1923. 12 s. 6 d. Methuen.
WELD, LEROY D. Theory of Errors and Least Squares. 1916. \$2.00. Macmillan.
WELD, L. G. Determinants. 4th edition, monograph series no. 3, 1906. \$1.25. Wiley.
WHITTAKER, E. T., and ROBINSON, G. A Short Course in Interpolation. 5 s. Blackie & Sons.

II. ADVANCED BOOKS.

Algebra.

BÔCHER, M. Introduction to Higher Algebra. 1907. \$3.00. Macmillan.
BURNSIDE, W. S., and PANTON, A. W. Theory of Equations. Vol. I, \$5.00; Vol. II, \$5.00. Longmans, Green & Co.
FINE, H. B. The Number System of Algebra. \$1.40. Heath.
GLENN, O. E. Theory of Invariants. 1915. \$3.25. Ginn.
MURNAGHAN, F. D. Vector Analysis and Relativity. 1922. \$2.75. Johns Hopkins Press.
RUNNING, T. R. Empirical Formulas. 1917. \$2.00. Wiley.

WEBER, H. Lehrbuch der Algebra (small edition). 1912. About 15 gold marks. Vieweg und Sohn.

Geometry.

BELL, R. J. T. Coordinate Geometry of Three Dimensions. 2d Edition, 1923. $4.25. Macmillan.

COOLIDGE, J. L. Geometry of the Complex Domain. $6.00. Oxford Univ. Press.

COOLIDGE, J. L. The Elements of Non-euclidean Geometry. $5.35. Oxford Univ. Press.

CREMONA, L. Elements of Projective Geometry. $5.00. Clarendon Press.

DOWLING, L. W. Projective Geometry. 1917. $2.00. McGraw-Hill.

DURELL, C. V. Modern Geometry. 1920. $2.00. Macmillan.

FRANKLAND, W. B. Theories of Parallelism. $2.25. Cambridge University Press (Macmillan).

HEATH, T. L. The Thirteen Books of Euclid's Elements. 3 Vols. Set: $21.00. Macmillan.

HILBERT, D. The Foundations of Geometry. Translation by Townsend. $1.00 cloth. Open Court Pub. Co.

HILTON, H. Plane Algebraic Curves. 1920. $9.35. Clarendon Press.

LOBACHEVSKI, N. I. Theory of Parallels. Translation by Halsted. $1.25. Open Court Pub. Co.

MANNING, H. P. Geometry of Four Dimensions. 1914. $3.00. Macmillan.

MANNING, H. P. Non-Euclidean Geometry. $1.36. Ginn.

MATHEWS, G. B. Projective Geometry. $1.80. Longmans, Green & Co.

NEVILLE, E. H. Prolegomena to Analytical Geometry. 1922. $10.50. Cambridge Univ. Press.

NEVILLE, E. H. The Fourth Dimension. $1.60. Cambridge Univ. Press.

SOMERVILLE, D. M. Y. Elements of Non-Euclidean Geometry. 7 s. 6 d. Harcourt Brace & Co.

SNYDER, V. and SISAM, C. R. Analytic Geometry of Space. $3.00. Holt.

WINGER, R. M. Introduction to Projective Geometry. $4.40. Heath.

WOODS, F. S. Higher Geometry. $5.00. Ginn.

Calculus and Differential Equations.

BATEMAN, H., Differential Equations. 1918. $3.25. Longmans, Green & Co.

COHEN, A. Differential Equations. $2.40. Heath.

FORSYTH, A. R. Treatise on Differential Equations. 4th Edition. $6.00. Macmillan.

HEDRICK, E. R. and KELLOGG, O. D. Applications of the Calculus to Mechanics. $1.36. Ginn.

JOHNSON, W. W. A Treatise on Ordinary and Partial Differential Equations. $3.50. Wiley.

MURRAY, D. A. Differential Equations. $2.25. Longmans, Green & Co.

WILSON, E. B. Advanced Calculus. 1912. $5.00. Ginn.

Miscellaneous.

BIRKHOFF, G. D. Lowell Lectures on Relativity. Macmillan.

BIRKHOFF, G. D. Relativity and Modern Physics. 1923. $4.00. Harvard Univ. Press.

BLISS, G. A. Calculus of Variations. 1925. $2.00. Open Court Pub. Co.

BURKHARDT, H. Theory of Analytic Functions. Translation by Rasor. $4.40. Heath.

BYERLY, W. E. Fourier Series. $3.00. Ginn.

BYERLY, W. E. Harmonic Functions. 4th Edition, Monograph Series No. 5. $1.25. Wiley.

CAJORI, F. Theory of Equations. 1904. $2.00. Macmillan.

CANTOR, G. Contributions to the Founding of the Theory of Transfinite Numbers. Translation by Jourdain. $1.25. Open Court Pub. Co.

CARMICHAEL, R. D. Diophantine Analysis. Monograph No. 16. $1.50. Wiley.

CARMICHAEL, R. D. Theory of Numbers. $1.25. Wiley.

Chambers' Mathematical Tables. Edited by J. Pryde. $2.50. Van Nostrand.

COFFIN, J. G. Vector Analysis. 2d Edition. $2.50. Wiley.

COHEN, A. The Lie Theory of One-parameter Groups. $2.40. Heath.

COOLIDGE, J. L. Treatise on the Circle and the Sphere. 1916. $8.35. Clarendon Press.

COUTURAT, L. Algebra of Logic. Translation by Robinson. $1.50. Open Court Pub. Co.

CURTISS, D. R. Theory of Functions. $2.00. Open Court Pub. Co.

DICKSON, L. E. Algebras and their Arithmetics. 1923. $2.25. Univ. of Chicago, Science Series.

DICKSON, L. E. Algebraic Invariants. Monograph Series No. 14. $1.50. Wiley.

DIXON, A. C. Elementary Properties of Elliptic Functions. London. 1894. 5 shillings. Macmillan.

EISENHART, L. P. A Treatise on the Differential Geometry of Curves and Surfaces. 1909. $5.00. Ginn.

FISKE, T. S. Functions of a Complex Variable. Monograph Series No. 11. $1.25. Wiley.

GOURSAT, E. Course in Mathematical Analysis. Translation by E. R. Hedrick. Vol. I, $5.00. Vol. II (Hedrick and Dunkel), Part I, $3.50; Part II, $3.50. Ginn.

HARDY, G. H. A Course in Pure Mathematics. 3d Edition. 1921. 15 s. Cambridge Univ. Press.

HOBSON, E. W. The Theory of Functions of a Real Variable. 2d Edition. Vol. I. $15.00. Macmillan.

HUNTINGTON, E. V. The Continuum and Other Types of Serial Order. $1.25. Harvard Univ. Press.

JEANS, J. H. Theoretical Mechanics. 1907. $3.50. Ginn.

KEYSER, C. J. Mathematical Philosophy. 1922. $4.70. Dutton.

KEYSER, C. J. Human Worth of Rigorous Thinking. $3.00. Columbia Univ. Press.

LONEY, S. L. An Elementary Treatise on the Dynamics of a Particle and of Rigid Bodies. 2d Edition. $4.75. Macmillan.

MACMAHON, J., Hyperbolic Functions. Monograph Series, No. 4. $1.25. Wiley.

MERRIMAN, M. and WOODWARD, R. S. Higher Mathematics. 2d Edition revised. $5.00. Wiley.

MILLER, G. A., BLICHFELDT, H. F. and DICKSON, L. E. Theory and Applications of Finite Groups. $4.00. Wiley.

NUNN, T. P. Relativity. 1923. $2.40. Dutton.

NUNN, T. P. The Teaching of Algebra, including Trigonometry. $2.75. Longmans, Green & Co.

PEARSON, K. Grammar of Science. Third edition. Part I: Physical, Part II: Biological. Macmillan.

PICARD, C. E. Traité d'Analyse. 3 Vols., 1901–1906. 2d editions: Vol. I, 1901, 16 francs; Vol. II, 1905, 18 francs; Vol. III, 1908, 18 francs. Gauthier-Villars.

PIERPONT, J. Theory of Functions of Real Variables. Two volumes. $5.00 each. Ginn.

POINCARÉ, H. The Foundations of Science. 1912–1921. Translated by Halsted. $5.00. The Science Press, New York.

RIETZ, H. L. (and many others). Handbook of Mathematical Statistics. $4.00. Houghton Mifflin.

RUSSELL, B. Introduction to Mathematical Philosophy. $4.00. Macmillan.

SHAW, J. B. Lectures on the Philosophy of Mathematics. 1918. $1.50. Open Court Pub. Co.

SMAIL, L. L. Elements of the Theory of Infinite Processes. 1923. $3.50. McGraw-Hill Book Co.

TANNERY, J. Introduction a la Théorie des Fonctions d'une Variable. Vol. I, 2d Ed., 1904, 14 francs. Vol. II, 2d Ed., 1910, 15 francs. Hermann in Paris.

TOWNSEND, E. J. Functions of a Complex Variable. 1915. $4.00. Holt.

VEBLEN, O. and LENNES, N. J. Introduction to Infinitesimal Analysis. $2.00. Wiley.

VEBLEN, O. and YOUNG, J. W. Projective Geometry. Vol. I. $4.80. Ginn.

WEBSTER, A. G. Dynamics of Particles and of Rigid, Elastic and Fluid Bodies. 1904. Teubner.

WHITEHEAD, A. N. Introduction to Mathematics. $1.00. Holt.

WHITEHEAD, A. N. Concept of Nature. $5.25. Macmillan.

WHITTAKER, E. T. and ROBINSON, G. The Calculus of Observations. 1924. 18 s. Blackie & Sons.

WILLIAMS, K. P. Dynamics of the Airplane. 1921. $2.50. Wiley.

WYNNE, W. E. and SPRARAGEN, W. Engineering Mathematics. \$2.50. Van Nostrand.

YOUNG, J. W. A. Monographs on Topics of Modern Mathematics. \$4.00. Longmans, Green & Co.

YOUNG, J. W. A. Teaching of Mathematics. Revised edition. 1924. \$2.20. Longmans, Green & Co.

YOUNG, J. W. Fundamental Concepts of Algebra and Geometry. 1911. Macmillan. \$2.00.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

54. The search for three-dimensional analogues of Rolle's Theorem suggests the following question: C is a simple closed curve in three dimensions, having at every point a tangent which turns continuously with respect to the arc. O is a fixed point on C, at which curvature and torsion exist and do not vanish. At no point on C is the tangent parallel to the tangent at O. Under these conditions, do there always exist points P and Q on C, such that OP is parallel to the tangent at Q? Possibly some modification of the assumptions would lead to an interesting decision.[1]

REPLIES TO QUESTIONS.

52. Is there any simple treatment of the regular pentagon constructions the proof of which involves neither medial section nor trigonometric functions?

REPLY BY H. C. BRADLEY, Massachusetts Institute of Technology.[2]

A proof, fulfilling the requirements except perhaps as to simplicity, can be obtained by representing the radius of the circle by the quantity $\sqrt{5}+1$. Only the Pythagorean theorem is necessary in the demonstration.

See Fig. 1. Let O be the center of a circle, OA any radius. Let B be the point where $OB : BA = 1 : \sqrt{5}$. Draw BC perpendicular to OA, meeting the circumference at C. Then AC is 1/5 of the circumference.

Proof. Produce the radius, and obtain the symmetrical points A', B', and C'. Then $CC' = BB'$. If $OB = 1$, then $CC' = 2$.

Bisect arc AC at D. Now, if we can show that $AD = DC = 2$, that is, $AD = CC'$, the semicircle is divided into fifths, and AC is one-fifth of the circumference.

Draw OC, OD, and AC, the latter two intersecting at E.

In $\triangle OBC$, $OB = 1$, $OC = \sqrt{5}+1$. Hence $CB = \sqrt{5+2\sqrt{5}}$.

[1] If C contains two points at which the tangents are parallel but opposite, P and Q may be so determined in at least two ways. EDITOR.

[2] Other replies have already appeared by Norman Anning (*1925*, 75) and C. H. Chepmell (*1925*, 76). See also the discussion by H. C. Bradley (*1924*, 342). EDITOR.

In $\triangle\, ABC$, $AB = \sqrt{5}$, $CB = \sqrt{5 + 2\sqrt{5}}$. Hence $AC = \sqrt{10 + 2\sqrt{5}}$, $AE = \frac{1}{2}AC = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$.

In $\triangle\, OAE$, $OA = \sqrt{5} + 1$, $AE = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$. Hence $OE = \frac{1}{2}\sqrt{14 + 6\sqrt{5}}$, $= \frac{1}{2}(\sqrt{5} + 3)$.

$DE = OD - OE = \sqrt{5} + 1 - \frac{1}{2}(\sqrt{5} + 3) = \frac{1}{2}(\sqrt{5} - 1)$.

In $\triangle\, AED$, $AE = \frac{1}{2}\sqrt{10 + 2\sqrt{5}}$, $DE = \frac{1}{2}(\sqrt{5} - 1)$. Hence $AD = 2$, and the proposition is proved.

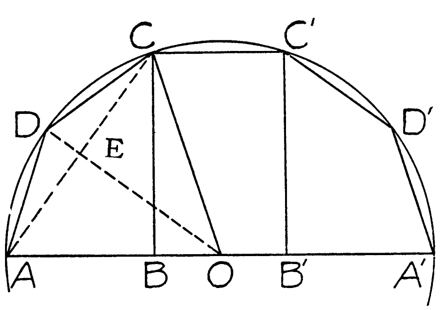

FIG. 1.

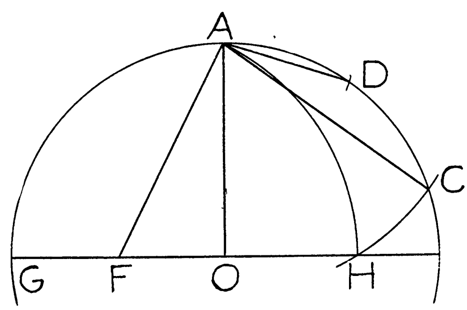

FIG. 2.

We can now prove a construction given in text-books for the sides of the decagon and pentagon. See Fig. 2. The radius OG is perpendicular to OA, and F is its mid-point.

$$OA = \sqrt{5} + 1. \qquad OF = \tfrac{1}{2}(\sqrt{5} + 1).$$

Hence

$$AF = \tfrac{1}{2}\sqrt{30 + 10\sqrt{5}} = \tfrac{1}{2}(\sqrt{5} + 5). \qquad FH = AF = \tfrac{1}{2}(\sqrt{5} + 5).$$

$OH = FH - OF = 2 =$ side of decagon. Then in $\triangle\, AOH$, $AH = \sqrt{10 + 2\sqrt{5}}$ $=$ side of pentagon.

DISCUSSIONS.

I. On a Method of Approximating the Real Roots of a Polynomial.

By G. W. Spenceley, Miami University.

The word "equation" throughout this paper will refer only to the integral algebraic equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots p_n = 0 \tag{1}$$

in which the coefficients are real and rational, and one root at least is real.

In such an equation, if R be a real positive root which is large in comparison

with s, the sum of all the other roots, we shall have

$$R + s = -\frac{p_1}{p_0},$$

and approximately,

$$s = -\frac{p_2}{p_1}.$$

If the approximation to s be accurate to the nth digit in the decimal, then R can be calculated from the above, correct to the same degree.

It is only one step further to the approximating of any real root of a given equation by transforming that equation into another in such a way as to make the desired root go over into a positive real root, large in comparison with the sum of all the other roots of the transformed equation. The purpose of this paper is to describe such a transformation and the method of applying it to the evaluation of any real root of a given equation.

First, we shall find the general transformation which is equivalent geometrically to reflecting all the roots, real and imaginary, in a circle with center a and radius r,—the origin being moved in the process first to the point a and then arbitrarily back a distance b, that is, equivalent to moving the origin to the point a–b. We may do this in three transformatious, which combine into

$$x = \frac{r^2}{y - b} + a. \tag{2}$$

In what follows a, b, and r are real.

First Step. Let x_1 be the real root of equation (1) which it is desired to approximate; and let the other $n - 1$ roots be represented by $x_2, x_3, x_4 \cdots x_n$. Let us further assume that a first approximation a_1 for the root x_1 is known, and that $a_1 < x_1$.

Now if equation (1) be subjected to the transformation (2), the center of the circle of reflection being a_1, and the radius r_1 being chosen sufficiently small to exclude from the circle all the roots, real and imaginary, except x_1, we shall obtain a new equation in y

$$q_0y^n + q_1y^{n-1} + q_2y^{n-2} + \cdots + q_n = 0, \tag{3}$$

whose roots $y_1, y_2, y_3 \cdots y_n$ are the transformed values of $x_1, x_2, x_3 \cdots x_n$ respectively, and such that all the roots of (3) except y_1 will be included in the circle of reflection, y_1 being outside and to the right.

If the roots of the original equation (1) be so separated as to enable us to take $r_1 = 1$, and if at this step we take $b_1 = 0$, we obtain Lagrange's familiar transformation

$$x = (1/y) + a_1. \tag{4}$$

The transformation (2) serves the purpose of separating y_1 from the other

roots of (3), and confining the latter roots to the small region within the circle of reflection.

A first approximation to y_1 is $-q_1/q_0$. Guided by this value we can find by substitution in equation (3) an approximate value a_2 which will be less than y_1 but differ from it by less than unity.

Second Step. To increase further the value of the root y_1, and at the same time diminish the value of the sum of the remaining roots, we shall reflect the y-roots in a circle with center a_2, taking $r_2 = \sqrt{a_2}$, $b_2 = -1$, which gives us for our equation of transformation

$$y = a_2 z/(z - 1). \tag{5}$$

If the resulting equation in z be represented by

$$t_0 z^n + t_1 z^{n-1} + t_2 z^{n-2} + \cdots + t_n = 0,$$

then the sum s of the small roots $z_2, z_3, z_4 \cdots z_n$ is approximately $-t_2/t_1$, and the large root z_1 is obtained from $z_1 + s = -t_1/t_0$.

We have thus far:

$$x_1 = a_1 + \frac{1}{a_2} - \frac{1}{a_2 z_1}.$$

By continuing the process we may obtain the value of x_1 to any degree of accuracy desired.

In practice it sometimes happens that a_2 may be taken more conveniently to the right of y_1, in which case we may reflect in the circle as before, the z_1 now going far to the left and becoming negative. By setting $z = -z_1$ we can change z_1 to a positive root. The resulting transformation is given by

$$y = a_2 z/(z + 1), \tag{6}$$

a very useful addition to equation (5). In this case we should have

$$x_1 = a_1 + \frac{1}{a_2} + \frac{1}{a_2 z_1}.$$

It remains to calculate the large root in the final equation and estimate the error.

Assume the final equation to be

$$c_0 w^n + c_1 w^{n-1} + c_2 w^{n-2} + \cdots + c_n = 0, \tag{7}$$

whose large root w_1 we shall now call R, and the sum of whose small roots w_2, w_3, $w_4 \cdots w_n$ we shall write as s.

By substituting R for w in (7) and transposing terms, we get

$$c_0 R^n = -c_1 R^{n-1} - c_2 R^{n-2} - c_3 R^{n-3} - \cdots - c_n,$$

whence

$$R = -\frac{c_1}{c_0} - \frac{c_2}{c_0R} - \frac{c_3}{c_0R^2} - \cdots - \frac{c_n}{c_0R^{n-1}}. \tag{8}$$

Again,

$$R + s = -c_1/c_0. \tag{9}$$

From (8) and (9) we get

$$s = \frac{c_2}{c_0R} + \frac{c_3}{c_0R^2} + \frac{c_4}{c_0R^3} + \cdots + \frac{c_n}{c_0R^{n-1}}.$$

In this equation substitute

$$R = -(c_1 + c_0s)/c_0,$$

obtained from (9), to get finally

$$s = -\frac{c_2}{c_1\left(1 + \frac{c_0s}{c_1}\right)} + \frac{c_0c_3}{c_1^2\left(1 + \frac{c_0s}{c_1}\right)^2} - \frac{c_0^2c_4}{c_1^3\left(1 + \frac{c_0s}{c_1}\right)^3} + \cdots.$$

On expanding this becomes

$$\begin{aligned} s = -\frac{c_2}{c_1} + \frac{c_0c_2s}{c_1^2} - \frac{c_0^2c_2s^2}{c_1^3} &+ \frac{c_0^3c_2s^3}{c_1^4} - \frac{c_0^4c_2s^4}{c_1^5} + \cdots \\ + \frac{c_0c_3}{c_1^2} - \frac{2c_0^2c_3s}{c_1^3} &+ \frac{3c_0^3c_3s^2}{c_1^4} - \frac{4c_0^4c_3s^3}{c_1^5} + \cdots \\ - \frac{c_0^2c_4}{c_1^3} &+ \frac{3c_0^3c_4s}{c_1^4} - \frac{6c_0^4c_4s^2}{c_1^5} + \cdots \\ &+ \frac{c_0^3c_5}{c_1^4} - \frac{4c_0^4c_5s}{c_1^5} + \cdots \\ &\qquad - \frac{c_0^4c_4}{c_1^5} + \cdots \\ &\qquad\qquad + \cdots. \end{aligned} \tag{10}$$

Terms in columns are of about the same order of magnitude. It is to be observed that s, which is to be computed, appears on both sides of the equation; and that the series on the right are infinite. Excepting the numerical coefficients in the second and following series, any term after the first term in each series may be obtained from the preceding term of the same series by multiplying it by $-c_0s/c_1$. This factor is of course equal to $s/(R + s)$, and hence a small fraction, so that these series in practice are rapidly convergent and s may be calculated by means of the first few terms, those neglected giving limits to the error in s. But to show the convergence of these series, without begging the question, it is

necessary to obtain, in an independent manner,[1] an upper limit to $|s|$ and hence to $|c_0s/c_1|$. In practice this may be omitted and the approximate value $-c_2/c_1$ for s may be used in estimating the error due to the neglected terms. With regard to the numerical coefficients it is to be observed that they are the numbers of Pascal's triangle

$$\begin{array}{ccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & u_n = 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdot & \cdot & \cdot & u_n = n \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & \cdot & \cdot & \cdot & \cdot & \cdot & u_n = n(n+1)/2! \\
1 & 4 & 10 & 20 & 35 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_n = n(n+1)(n+2)/3! \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

These numbers increase from left to right in any row, but the ratio u_n/u_{n-1} decreases monotonically to the limit 1 as n increases and becomes infinite. Hence the convergence of the series when $|c_0s/c_1| < 1$. If the series are alternating, a limit to the error in each is given by the first neglected term in each case. If the signs are alike, a limit to the error in each series is gotten by summing the neglected terms in each series as a geometric series, the numerical coefficient (say N) in the first neglected term of each series being included with $-c_0s/c_1$ to form the geometric factor $-Nc_0s/c_1$.

An upper limit to the error in s being known we can calculate R from equation (9) and set an upper limit to its error. The error in x_1 is then determined from the equation

$$x_1 = a_1 + \frac{1}{a_2} - \frac{1}{a_2R}$$

by means of

$$\Delta x_1 = \frac{dx_1}{dR} \cdot \Delta R = \frac{\Delta R}{a_2R^2},$$

where Δx_1 represents the error in x_1 and ΔR the error in R.

[1] An independent upper limit to $|s|$ may be obtained thus. Let the two points $(a_1 + r_1)$ and $(a_1 - r_1)$ on the x-axis be transformed finally into the points w' and w'' respectively. Let the equations of transformation be

$$x = \frac{r_1^2}{y} + a_1; \quad y = \frac{a_2z}{z \pm 1}; \quad z = \frac{a_3u}{u \pm 1}; \quad u = \frac{a_4v}{v \pm 1}; \quad v = \frac{a_5w}{w \pm 1}.$$

Then none of the small roots $w_2, w_3, w_4 \cdots w_n$ lie outside of the region bounded by w' and w''. Also the origin lies within this same region, and hence, if d represent the distance between w' and w'' taken positively, we shall have

$$d = |w' - w''|.$$

Now there are $n - 1$ such small roots in this region; hence

$$|s| \leq (n-1)d.$$

This distance d will vary according as the signs in the denominators of the equations of transformation are positive or negative; but the maximum d will be gotten by taking them all positive. In this case

$$d = \frac{2r_1}{a_2a_3a_4a_5\cdots}\left[\frac{1}{1 - r_1^2\left(\dfrac{1}{a_2} + \dfrac{1}{a_2a_3} + \dfrac{1}{a_2a_3a_4} + \dfrac{1}{a_2a_3a_4a_5} + \cdots\right)^2}\right].$$

A convenient approximation to s may be obtained from the first three terms of (10); whence

$$s = \frac{c_0c_3 - c_1c_2}{c_1^2 - c_0c_2}. \tag{11}$$

Example. Take

$$x^3 + 4x^2 - 7 = 0 \qquad (\text{root } x_1 = 1 +).$$

Step 1. Set

$$x = \frac{1}{y} + 1$$

to get

$$2y^3 - 11y^2 - 7y - 1 = 0 \qquad (\text{root } y_1 = 6 +).$$

Step 2. Set

$$y = \frac{6z}{z - 1}$$

to get

$$7z^3 - 483z^2 + 45z - 1 = 0.$$

Here $R + s = 483/7 = 69$, and approximately $s = 45/483 = 1/10$ nearly; and hence R is a little less than 69, and we shall consequently use equation (6) in the next transformation.

Step 3. Set

$$z = \frac{69w}{w + 1}$$

to get

$$3104w^3 - 2293356w^2 + 3102w - 1 = 0.$$

Stopping at this point we obtain from equation (10) or (11)

$$s = .00135,\ 26047,\ 964,$$

whence

$$R = 738.83756,\ 49209,\ 768.$$

This gives

$$\begin{aligned} x_1 &= 1 + \frac{1}{6} - \frac{1}{6 \times 69} - \frac{1}{6 \times 69 \times 738.83756,\ 49209,\ 768} \\ &= 1.16424,\ 79384,\ 60211,\ 21305,\ 3 \cdots. \end{aligned}$$

As a check, Horner's method gives

$$x_1 = 1.16424,\ 79384,\ 60211,\ 21305,\ 38421,\ \cdots.$$

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Vorlesungen über die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen. By R. FUETER, Part I (Teubner's Sammlung, Band XLI, 1). Leipzig, Teubner, 1924. 142 pages. Price 7 marks.

The subject with which Professor Fueter's book deals is of central significance in the theories of elliptic functions, of quadratic number fields and of automorphic functions, and may therefore be approached from many different sides. The author has chosen to develop it on the basis of a study of the group of linear substitutions $w = (az + b)/(cz + d)$, in which a, b, c and d are integers. It is

to this subject and to the necessary preliminaries concerning quadratic fields and elliptic functions that the first part of his book is devoted.

It is well known from elementary analysis that the values of a trigonometric function for an integral multiple of θ can be expressed rationally in terms of the values of that function and of its first derivative quâ θ. The theory of elliptic functions develops the fact that, if $\phi(u)$ has the independent periods ω_1 and ω_2, *i.e.*, $\phi(u + \omega_1) = \phi(u + \omega_2) = \phi(u)$, and if k is an integer, then $\phi(ku)$ can be expressed rationally in terms of $\phi(u)$ and $\phi'(u)$. It is shown moreover that the equation determining $\phi(u/k)$ is an Abelian equation (that is, an equation whose roots are all expressible rationally in terms of any one of them), and hence solvable by radicals; and these results hold for arbitrary values of ω_1 and ω_2. When we turn to the case in which k is not rational, we have the remarkable result obtained by Abel (*Collected Works*, vol. I, p. 377) according to which an algebraic relation exists between $\phi(ku)$, $\phi(u)$ and its derivatives only in case k belongs to the field of a quadratic imaginary, and in this case only for special values of the ratio ω_2/ω_1.

The set of problems suggested by this state of affairs is commonly referred to as the problem of "complex multiplication." The moduli of the elliptic functions for which the ratio of the periods is such as to admit a "multiplication theorem" are called "singular moduli." Abel treated the problem for the case in which k belonged to the domain of $\sqrt{-3}$ or $\sqrt{-5}$, and discovered the fact that the singular moduli were expressible in terms of the ratio of the period by means of radicals. Further progress was made by Kronecker, Hermite and others, whose work brought out the strategic position of the problem. The determination of the singular moduli, the setting up and the characterization of the equations for $\phi(ku)$ have given rise to much work in algebra and the theory of numbers. So much in explanation of the title of the book under review.

To say much about its content would require a lengthier exposition of details than can well be justified in a review. For these reasons I shall limit myself to a few indications of the literature and to a brief account of the material found in this volume.

An excellent general statement of the problem is found in Weber's article in volume I of the *Encyklopaedie der math. Wissenschaften,* while a detailed study of the theory and of the many topics related to it can perhaps best be made by the aid of the third volume of Weber's *Lehrbuch der Algebra.* Although apparently conceived on a less ample basis than this classic, Professor Fueter's book may well be looked upon as continuing its tradition; it is intended to include, particularly in the second part, the important advances of recent years.

In the first chapter, on elliptic modular functions, we get an introduction to the theory of the modular group (*i.e.*, the group of linear fractional substitutions $w = (az + b)/(cz + d)$ in which a, b, c and d are integers such that $ad - bc = 1$); the elliptic modular function $j(z)$ is determined in connection with the fundamental region for the modular group. Chapter II introduces modular functions of the nth order in connection with the subgroups of the modular group and

determines the important modular equation of order n, $\Phi_n(t, j(z)) = 0$, whose roots are these modular functions.

The third chapter begins with a brief account of the fundamental properties of the ideals of a quadratic imaginary field, the notion of class number, etc. It brings the proof that the invariant $j(\omega_2/\omega_1)$ has the same value for all ideals belonging to a class; and that the class equation $H_m(t) = 0$ of the field $k(\sqrt{m})$ whose roots are the moduli $j(k_r)$, $r = 1 \cdots m$, defined for the classes $k_1 \cdots k_m$ of equivalent ideals, is an Abelian equation with respect to the field $k(\sqrt{m})$.

In chapter IV the multiplication formulæ for elliptic functions, for even and odd integral values of the multiplier, are obtained in terms of the functions $T(z)$ of Jacobi, and its derivative $T_1(z)$; these functions are related in a simple way to the Weierstrass y-function. In the last chapter it is proved that if (ω_1, ω_2) is the basis of an ideal of the field $k(\sqrt{m})$, ν an integer of this field and x_1 and x_2 integers, then $T((x_1\omega_1 + x_2\omega_2)/\nu)$ is an algebraic number. This result naturally leads then to the problem of determining the equation of lowest degree which this number satisfies and of finding the Galois group of this equation. This in turn necessitates a more detailed study of the properties of a field and the introduction of a variety of new concepts.

The book gives evidence of having been written with great care. One might wish for some variation of emphasis which would facilitate the reading and particularly the assimilation of its contents. For those who have a good knowledge of the classical theory of analytic functions and of elliptic function theory, it furnishes a very satisfactory approach to a fascinating domain of mathematics. Previous knowledge of the theory of algebraic numbers, although not presupposed, would be of material assistance. In execution the book has all the excellence which we have grown accustomed to associate with the name of Teubner.

ARNOLD DRESDEN.

The Mathematical Theory of Investment. By E. B. SKINNER. Revised edition. New York, Ginn and Company, 1924. xi + 269 pages. Price $3.40.

The first edition of Professor Skinner's book, which was published in 1913, was a pioneer American textbook in the field of mathematics of investment. It was a good book and stood alone for some years. Recently, however, a number of other textbooks of a similar nature have appeared, and it is perhaps natural that Professor Skinner should effect a revision at this time, although these newer books showed no conspicuous superiority to his original volume.

The revised edition follows the plan of the first edition, the principal changes being (1) the emphasis placed upon interpolation by the inclusion of a short chapter on that subject, and by a more extensive use of interpolation in the solution of problems involving the finding of the interest rate, and (2) the addition of a considerable amount of material on the subject of depreciation.

The feature—a desirable one—of beginning with a review of those topics of algebra which are essential in the mathematics of investment, has been retained.

It is gratifying to see included in this algebraic introduction the above-mentioned chapter on the important subject of interpolation, which for many years has unfortunately been neglected.

The chapter on graphic representation which was in the first edition has been omitted.

The equation of value and the equation of payments, formerly discussed in the chapter on interest, have been stressed by devoting to them a separate, though short, chapter. The subjects of perpetuities, capitalization, capitalized cost, continuous annuities, and annuities with fractional terms have also been placed in a separate chapter entitled "Some special types of annuities."

The author apparently neglects to state in the revised edition that the formulas for computing the premium that must be paid on a bond are equally applicable to the computation of the discount. The use of bond tables is explained, a decided improvement upon the old edition. Included in the new edition is a discussion of the determination of the rate of yield for a bond bought at a given price. The author explains three ways in which the problem may be solved: by means of bond tables, using interpolation by first and by second differences; by means of annuity tables; and by algebraic approximation. A section on the rate of interest paid by the issuer of the bond and a section on serial bonds have been added.

The chapter called "Sinking funds and depreciation" in the original edition has been broken up into three chapters in the new book: "Sinking funds," "Depreciation," and "The theory of mine valuation."

The chapter on depreciation is one of the outstanding features of the book. The author takes up the straight-line method, the sinking-fund method, the interest-on-investment method—commonly called the compound-interest method—and a modification of it, and the unit-cost method. In his discussion of the sinking-fund method, he distinguishes carefully and clearly between a *replacement* charge and a *depreciation* charge.

The discussion of the valuation of mining properties has been slightly expanded.

The chapter on building and loan associations has been somewhat simplified by removing, or putting into more compact form, certain equations.

Part III, dealing with probability and its applications to financial problems,—notably life annuities and life insurance—is practically unchanged.

New problems have been added in various places, and there are some rearrangements of material. The tables at the end of the volume are identical with those published before.

The revision, which is not at all radical, has improved the book, which seems better adapted to a fairly extensive course in the mathematics of investment than to a brief course in the subject. In addition to its value as a textbook, it is an excellent book of reference.

P. R. Rider.

Politische Arithmetik (Zinseszinsen-, Renten- und Anleiherechnung). By EMIL FOERSTER. Berlin and Leipzig. Walter de Gruyter and Co., 1924. 155 pp.

This book considers the theory of simple interest, compound interest and annuities, and the applications of this theory to the usual problems of the mathematics of finance. A knowledge of college algebra is sufficient preparation for reading practically all of the book. The results and methods of development offer little that is novel, as compared with the subject matter of American books of similar content. Extensive use is made of the more refined methods of interpolation, in certain of the problems where it is more usual to rest content with the results obtained by linear interpolation. In the treatment of annuities and their applications, very little space is devoted to the cases where the payment interval of the annuity is not the same as the conversion period of the interest rate. There are numerous illustrative examples solved in the text, but there are no sets of problems for the reader to solve.

W. L. HART.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

ANNALS OF MATHEMATICS, volume 26, no. 3, March, 1925: "Non-monoidal involutions which contain a web of invariant monoids" by V. Snyder, 165–172; "Definite integrals containing a parameter" by R. L. Jeffery, 173–180; "Integro-differential expressions invariant under Volterra's group of transformations" by A. D. Michal, 181–201; "New proofs of two well known theorems on quadratic forms" by J. F. Ritt, 202–204; "Some relations between compound determinants" by W. H. Metzler, 205–211; "A contribution to the theory of interpolation" by N. Wiener, 212–216; "A property of cyclotomic integers and its relation to Fermat's last theorem (second paper)" by H. S. Vandiver, 217–232; "On an infinite system of non-abelian groups of order nm^n" by W. E. Edington, 233–238; "Conformal and geodesic mapping" by A. Bramley, 239–246.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 31, nos. 5–6, May–June, 1925: "Life insurance as a social service and as a mathematical problem" by R. Henderson, 227–252; "Simplifications relating to Sylow's theorem" by G. A. Miller, 253–256; "Reduction of Euler's equations to a canonical form" by J. H. Taylor, 257–262; "Integro-differential equations" by I. A. Barnett, 263, 265; "A new type of double sextette closed under a binary (3, 3) correspondence" by L. D. Cummings, 266–274.

ISIS, volume 7, no. 1, November, 1924: "Drei planimetrische Aufgaben des arabischen Mathematikers Abul-jud Muhammed Ibn Al-lith" by C. Schoy, 5–8.

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉS, volume 4, no. 2, 1925: "On the formal modular invariants of binary forms" by W. L. G. Williams, 169–192.

SCIENTIFIC MONTHLY, volume 21, no. 2, August, 1925: "Fundamental facts in the history of mathematics" by G. A. Miller, 150–156.

TOHOKU MATH. SOCIETY, volume 25, nos. 1, 2, May, 1925: "On a series of rational functions formally analogous to Fourier's series" by W. P. Udinski, 1–23; "Determination of plane algebraic curves which are invariant under involutory Cremona transformations" by A. Emch, 63–76.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

Pi Mu Epsilon, University of Missouri, Columbia, Mo.

Pi Mu Epsilon, honorary mathematical fraternity, was founded at the University of Syracuse and incorporated under the laws of the State of New York in 1914. There are now nine chapters, the University of Missouri Chapter being the fourth.

The chapter officers for the year are:

Director: Verne T. Bickel '23.
Vice-director: W. E. Wirtel '26.
Secretary: Selma Gartman '26.
Treasurer: Richard Shewmaker '26.
Corresponding Secretary: Kathryn Wyant, Instructor in Mathematics.

The program for the year has consisted of two series of talks, one by the professors of departments related to mathematics and the other by members of the fraternity.

October 14: "The spirit of mathematics" by Professor W. D. A. Westfall.
October 28: "Mathematics of early times" by C. G. Jaeger; "Differential equations of motion" by Harlan Hibbard '25.
November 11: "Requirements for degrees at Cambridge and the mathematical tripos" by Professor L. M. Defoe.
December 9: "A problem in statistical mechanics" by Professor H. M. Reese.
December 11: Initiation.
December 29: National convention in Washington, D. C. Three delegates attended.
January 13: "Elliptic functions" by L. H. McFarlan; "Hyperbolic functions," Kathryn Wyant.
January 27: "An application of arithmetic to industry" by Professor H. Schlundt.
February 10: Report of Washington convention, by Verne Bickel.
February 29: "Historical development of laws of universal gravitation" by Professor E. S. Hayne.
March 10: "Slide rule" by R. R. Stokes '25; "Adding machine" by H. W. Wood, Jr. '25; "Planimeter" by W. E. Hoeflin '25.
March 28: Social meeting.
April 7: "The future of mathematics in high school" by M. G. Neale, Dean of School of Education.
April 28: "Laplace's equation" by J. P. Foltz '24; "Gauss's theorem" by Hilbert Moore '23; "Green's theorem" by Selma Gartman '26.
May 12: Professor G. E. Wahlin.

(Report by Miss Kathryn Wyant.)

The Pascal Circle, Trinity College, Washington, D. C. [*1924*, 399.]

The officers for the year 1924–1925 were:

Honorary president, Professor Marie Cecilia Mangold;
President, Anne Foley, '25;
Vice-president, Helena Crowley, '25;
Secretary, Anne McLarney, '26;
Treasurer, Merie Swiney, '27.

The program for the year 1924–1925 was the following:

November 4, 1924. Report of the meetings of the past year was read by the president. It was decided to spend the time of the meetings in the study of tests in secondary schools, and in relay work in mathematical calculation by class teams.
November 18. An article on "Tests in arithmetic" was read by Miss Josephine Gillis, '24, from the report on the reorganization of mathematics in secondary schools. A discussion followed. The team work was introduced.

February 3, 1925. Margaret Harz, '27, read an article on "Algebra in the ninth grade." A discussion followed. Team work in calculation was taken up for the remainder of the period.
February 18. Ruth Eileen Lynch, '26, gave an outline of "Algebraic tests" from the report on the reorganization of mathematics in secondary schools. Relay team work.
March 3. Ruth Eileen Lynch, '26, continued her outline covering "Algebraic tests." Discussion. Relay work.
April 1. "The calculation of the date of Easter" was given by Miss Lillian Seracci, '26. Discussion.

(Report by the Secretary.)

Napierian Club of DePauw University, Greencastle, Indiana.

The Club was organized Dec. 6, 1924. On Jan. 8, 1925, the following officers were elected: President, Quinton Stone, '25; Vice-President, John Thompson, '25; Secretary, Fern Schuette, '25; Treasurer, Charlotte Stafford, '25.

The program for this year was as follows:

February 5, 1925. "Eclipse as seen from New York," Professor W. V. Brown.
February 12, 1925. "The life and work of Napier," Miss Davison, '25. "History of the decimal point," Mr. Tokey, '25.
March 19, 1925. "The squared circle," Mr. Heinzmann, '25. "Curve of pursuit," Professor W. V. Brown.
April 2, 1925. Evening spent at the observatory.
April 30, 1925. "The scale of notation," Miss Stafford, '25. "Magic squares," Mr. Arnold, Mr. Pickering, '25.

On May 14, the following officers were elected for next year:

President, James Brown, '26;
Vice-President, Ruth Bickel, '26;
Secretary, Gertrude Hendrix, '26;
Treasurer, Orin Sykes, '26.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3148. Proposed by EUGENE M. BERRY, Purdue University.

Let

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \neq 0$$

and X_i be the cofactor of x_i in Δ and Y_i the cofactor of y_i, etc. Prove that if

$$\begin{vmatrix} 1/x_1 & 1/x_2 & 1/x_3 \\ 1/y_1 & 1/y_2 & 1/y_3 \\ 1/z_1 & 1/z_2 & 1/z_3 \end{vmatrix} = 0$$

2841 [1920, 274]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

The square number,

$$9\frac{49}{64} = 9 + \frac{49}{64} = 3^2 + \frac{7^2}{8^2}$$

is of the type form

$$k^2 + \frac{(2k+1)^2}{(2k+2)^2};$$

how may the *forms* of the terms of the fractional part be determined *deductively?*

Generally, required that

$$k^2 + \frac{\{\varphi_1(k)\}^2}{\{\varphi_2(k)\}^2}$$

be a perfect square, show how $\varphi_1(k)$ and $\varphi_2(k)$ may be found.

SOLUTIONS.

2706 [1918, 216]. Proposed by H. F. MACNEISH, New York City.

Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the given point and the given line may be equal to the part within the circle.

SOLUTION BY LOUIS WEISNER, University of Rochester.

The following analysis shows that in general the construction is impossible with straight edge and compass alone.

Take the given point O as origin, and as x-axis the line joining O and the center C of the circle. Let the equation of the given line be $y = mx + b$, and the equation of the circle

$$(x - c)^2 + y^2 = a^2.$$

The equation of the required line is of the form $y = kx$, where k is to be determined; let it cut the given line in A and the circle in $B(x_1, y_1)$ and $D(x_2, y_2)$. We readily find that the coördinates of A are $\left(\frac{b}{k-m}, \frac{kb}{k-m}\right)$ and hence $\overline{OA}^2 = \frac{(1+k^2)b^2}{(k-m)^2}$.

Substituting $y = kx$ in the equation of the circle, we find that x_1 and x_2 satisfy the equation,

$$(1 + k^2)x^2 - 2cx + c^2 - a^2 = 0.$$

Without solving this quadratic equation, we find that

$$(x_1 - x_2)^2 = \frac{4[(a^2 - c^2)k^2 + a^2]}{(1 + k^2)^2}.$$

Now, as $y_1 = kx_1$ and $y_2 = kx_2$,

$$\overline{BD}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (1 + k^2)(x_1 - x_2)^2 = \frac{4[(a^2 - c^2)k^2 + a^2]}{1 + k^2}.$$

Since $\overline{OA} = \overline{BD}$, we have the following equation to solve for k:

$$4(k - m)^2([a^2 - c^2]k^2 + a^2) = b^2(1 + k^2)^2.$$

Remembering that a, b, c, m are independent given quantities, it is quite clear that, when this equation is simplified, the terms being arranged according to powers of k, we shall have a *general* equation of the fourth degree. Now the solution of a general biquadratic equation always involves an irreducible cube root. The construction is therefore, in general, impossible with straight edge and compass alone.[1]

3093 [1924, 353; 1925, 267]. Proposed by FRANK MORLEY, Johns Hopkins University.

A solution by the PROPOSER was overlooked when the solution of this problem was printed (*1925*, 267). EDITOR.

[1] *Cf.* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 322.

3114 [1925, 94]. Proposed by B. R. HEADSTROM, Boston, Mass.

What is the per cent. error in the following supposed method of trisecting an angle? Given the angle ABC with its vertex at B, lay off on its sides the equal lengths BD and BE and draw the line DE. With this latter length as diameter draw a semi-circle on the side opposite to B; also draw arcs of circles with D and E as centers and radii equal to that of the semi-circle, cutting the latter in M and N, respectively. Then the lines BM and BN trisect the given angle.

SOLUTION BY B. F. FINKEL, Springfield, Mo., and OTTO DUNKEL, Washington University.

Let O be the middle point of DE, and draw the straight line BO producing it until it cuts NM in P. Since $\angle PON = 30°$ it will be convenient to take $OE = 2$. Let the angle to be trisected $\angle DBE = 2\theta$ and set $\angle OBN = \alpha$. Then $\cot\alpha = BP/PN = \sqrt{3} + 2\cot\theta$. The error in taking $\angle NBE$ as $2\theta/3$ is $\epsilon = 2\theta/3 - (\theta - \alpha) = \alpha - \theta/3$. Hence the per cent. of error is found from

$$\frac{\epsilon}{2\theta} = \frac{\cot^{-1}[\sqrt{3} + 2\cot\theta]}{2\theta} - \frac{1}{6},$$
$$= \frac{1}{2\theta}\tan^{-1}\left[\frac{\tan\theta}{2 + \sqrt{3}\tan\theta}\right] - \frac{1}{6}.$$

If θ is very small, then, approximately,

$$\frac{\epsilon}{2\theta} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Setting $\epsilon = 0$, it is found that $\tan\theta/3 = 0, \ \pm 2 - \sqrt{3}, \ 1/\sqrt{3}$; hence the construction is exact for the angles $2\theta = 0, 90°, 180°, \cdots$.

The maximum error may be found from

$$\frac{3}{2}[\tan^2\theta + (\sqrt{3}\tan\theta + 2)^2]\frac{d\epsilon}{d\theta} = (\tan\theta - \sqrt{3} + \sqrt{2})(\tan\theta - \sqrt{3} - \sqrt{2}).$$

Hence $d\epsilon/d\theta$ is zero for

$$\theta_1 = 17° \, 37' \, 55.9'', \qquad \theta_2 = 72° \, 22' \, 4.1'',$$
$$\epsilon_1 = 1° \, 13' \, 33.7'', \qquad \epsilon_2 = -\epsilon_1.$$

Thus as the angle 2θ increases from 0 to $2\theta_1$ the error increases from 0 to ϵ_1, and as the angle continues to increase to 90° the error decreases from ϵ_1 to 0. As the angle increases from 90° to $180° - 2\theta_1$ the error decreases from 0 to $-\epsilon_1$, and after this point the error increases until 2θ is 180° when the error is again zero.

For $2\theta = 45°$, $\epsilon = 1° \, 10'$; for $2\theta = 20°$, $\epsilon = 1° \, 2.42'$. A better approximation is given in the problem and its solution **2972 [1925, 95]**.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. KUHN, Ohio State University, Columbus, Ohio.

Dr. IRVING LANGMUIR, of the General Electric Company, Schenectady, has been awarded the Cannizzaro prize of the Reale Accademia dei Lincei.

Trinity College, Dublin, has conferred the honorary degree of doctor of science on Professor R. A. MILLIKAN, of the California Institute of Technology.

Miss ALICE A. GRANT has been appointed instructor of mathematics at State College of Washington.

Mr. W. H. McEwen has been appointed instructor of mathematics at Regina College, Regina, Canada.

Mr. E. L. Mickelson has been appointed instructor of mathematics at the University of Wisconsin.

Miss Marcia Latham, of Hunter College, died May 9, 1925. Her translation of Descartes' *Geometry*, prepared in collaboration with Professor D. E. Smith, recently appeared through the Open Court.

The new Distinguished Service professorships at the University of Chicago are a part of the $17,500,000 development program that is now under way. Each one carries a salary of $10,000 and is subject to assignment, not to any particular department, but to whatever individual is deemed most worthy on the basis of his record. The selection of Professor Michelson as the holder of the first of these professorships was determined by selective ballot of the members of the University Senate consisting of all full professors. By oversight this appointment was called a "fellowship" instead of a professorship in the August–September issue of the Monthly.

The ninth Summer meeting of the Association at Ithaca was unusually successful both in the large number attending and in the spirit of the occasion. No little interest was added on account of the Colloquium lectures of the American Mathematical Society which were held during the week. The lectures were given by Professor L. P. Eisenhart on "The New Differential Geometry" and Professor Dunham Jackson on "The Theory of Approximation." The attendance was considerably larger than on any previous colloquium. The report of these lectures and of the meetings of the Society will appear in the Bulletin of the Society. The report of the Association meetings will appear in full in an early issue of the Monthly. Numerous matters of interest and importance occupied the attention of the Trustees during two long sessions.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

DESIRED COPIES OF THE MONTHLY

THROUGH an oversight the Association has not reserved a sufficient number of copies of the MONTHLY for October, 1920, and for August, September, and October, 1921, also May, September, and October, 1924. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary,

W. D. Cairns, Oberlin, Ohio

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By H. S. EVERETT, Bucknell University.

Misprinted names in the text are corrected in this index.

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P. 71, l. 9, *for* "$+\cdots)^2\}^{1/2}$" *read* "$+\cdots)^2)\}^{1/2}$."
P. 71, last line, *for* "μ_s/σ_x^5" *read* "μ_5/σ_x^5."
P. 72, lines 3 and 5, *for* "$(\beta_4 - \beta_2 - \beta_1)$" *read* "$(\beta_4 - \beta_2^2 - \beta_1)$."
P. 72, line 10, numerator, for "$+\frac{2}{3}$" read "$+\frac{1}{3}$."
P. 162, **Theorem,** *delete* "*such that* $0 < f(x_1)/[f(x_1) - f(x_n)] \lesseqgtr 1$."
P. 388, above third display, *for* "transformed determinant" *read* "original determinant."
P. 388, below third display, *for* "*Journal* $\cdots\cdots$, p. 47." *read* "*Transactions of the American Mathematical Society*, vol. 26, 1924, p. 113."

THE AMERICAN MATHEMATICAL MONTHLY

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SECOND ANNUAL MEETING OF NEBRASKA SECTION.

The second annual meeting of the Nebraska Section was held at the University of Nebraska, at Lincoln, on May 9, 1925, in affiliation with the Nebraska Academy of Science.

The following papers were presented:

(1) "Factorization of large numbers" by Mr. J. E. OPP, University of Nebraska.

(2) "Constructions of the triangle" by Mr. E. B. OGDEN, University of Nebraska.

(3) "A class of summable trigonometric series" by Mr. G. D. NICHOLS, University of Nebraska.

(4) "Symbolic forms for derivatives of products and determinants" by Professor W. C. BRENKE, University of Nebraska.

(5) "Are the standards and requirements in mathematics being lowered—

(*a*) in the high school?" by Miss ELLEN H. FRANKISH, North High School, Omaha.

(*b*) in the first year courses in college?" by Professor J. M. HOWIE, Nebraska Wesleyan University.

Abstracts of the papers:

(1) Mr. Opp dealt with the factorization of large numbers by the use of a quadratic irrationality and the construction of a table of linear forms of numbers which must contain one of the required factors.

(2) Mr. Ogden gave an outline of methods used in solving, by ruler and compass construction, about one thousand plane triangles. About two hundred and fifty of these problems are either solved or suggested in various texts on geometry; the others are original.

(3) Mr. Nichols established the following theorem on the sum of a non-convergent series, using the Cesaro mean value definition of "sum." "The series $\sum n^p \sin nx$ is summable of order k, if $k - 1 < p < k$."

(4) Professor Brenke showed how the formula

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q}(uvw) = (u + v + w)_x{}^p (u + v + w)_y{}^q$$

can be regarded as giving the indicated derivative of the product uvw, or of the determinant whose elements by columns are u_i, v_i, w_i, $i = 1, 2, 3$, with extension to the general case.

(5*a*) Miss Frankish expressed the opinion that the standards and requirements in mathematics have not been lowered, during the last ten years, in the Omaha Central High School, which is a large school with about 80 per cent. of its graduates going to college. Increasing difficulty arises in keeping up to college entrance

standards because the rate of increase in high school attendance is greater than the rate of increase of population. There seem to be three ways of meeting the situation. First, the standards might be lowered. Second, the protests of school officers and tax payers might be ignored. Third, two courses in mathematics might be offered, one with a higher standard than that of the other. The last plan seems best in spite of some difficulties of administration.

(5*b*) Professor Howie secured opinions on this subject from nearly one hundred colleges and universities. These will be published as a part of the proceedings of the Nebraska Academy of Science.

At the business session the report of the nominating committee was adopted, to the effect that the present officers be continued for another year, namely, W. C. Brenke, chairman, Emma E. Hanthorn of the Kearney State Teachers College, secretary, and R. M. McDill of Hastings College, member of the executive committee.

Ellen H. Frankish, *Secretary pro tem.*

PROJECTIVE GEOMETRY—*FIELDS OF RESEARCH.*[1]

By L. W. DOWLING, University of Wisconsin.

Projective geometry, like other growing things, may be compared to a tree. Its roots reach back to ancient Greek civilization and its branches reach upward and outward toward the unknown.

Obviously it is not possible, in a short paper, to trace many of these branches to the buds and leaves whence new growth may be expected, nor to trace minutely any one of them.

To drop the metaphor, I shall speak of a few lines of development that may be characterized as programs of thought, or as fields of research, and which seem to promise further development.

1. The Poncelet Program. Modern projective geometry may be said to date from the publication in 1822 of Poncelet's great work on the projective properties of figures. This work bears the touch of romance as it was conceived in the Russian prison at Saratoff some ten years before its publication. Poncelet was an officer in Napoleon's army during the disastrous campaign against Russia.

The guiding principle in Poncelet's work is that of central projection and section and the main results are descriptions of those properties of figures which remain unchanged under the transformations of projection and section.

This guiding principle is necessary in the synthetic development of projective geometry and great use has been made of it, particularly by Italian geometers, in exploring the geometry of hyperspace. But we do not yet know all about the geometry of hyperspace. In the light of recent geometrical development, excited

[1] Presented at the Ithaca meeting of the Association, September, 1925.

by Einstein's theory of relativity, further research on the synthetic geometry of hyperspace may be expected. Doubtless central projection and section will still be a guiding principle or, at least, a very useful concept.

2. The Steiner Program. The Steiner program appeared in the *Systematischen Entwickelung* ten years after the publication of Poncelet's great work. Its basic ideas are:

1. Definition of primitive forms (point-row, sheaf of rays or pencil of rays, sheaf of planes, etc.).
2. Postulation of ideal elements and the notion of the perspective position of two primitive forms.
3. Definition of projectively related primitive forms by means of chains of perspectivity (followed by Cremona 40 years later).
4. Generation of new forms by means of projectively related primitive forms.

In this program, the invariance of the cross-ratio of corresponding quadruples of elements in projectively related primitive forms appears as a theorem. The use of this invariant as a definition of projective relationship is due to Chasles (*Aperçu Historique*, 1837).

By means of this program Steiner and his co-workers and followers added a vast number of geometric forms to those previously known and derived many of the properties of these forms. The whole field thus opened has been explored both by synthetic and by analytic methods. But it would be wrong to suppose that all the details of this very general program have been filled in. One may suggest, perhaps, as an example a discussion of rational curves. Thus two cubic equations:

$$\lambda^3 U + \lambda^2 V + \lambda W + Z = 0,$$
$$\lambda^3 A + \lambda^2 B + \lambda C + D = 0,$$

where the coefficients of the powers of λ are linear functions of the coördinates of a point in a plane, represent two projectively related envelopes of rays of the third class. These envelopes generate a rational sextic curve in the plane. That is, the two equations have at least one root in common provided their eliminant vanishes. This eliminant is of sixth degree in the coördinates of the point and thus represents a sextic curve. This curve is rational since the coördinates of any point on it are immediately expressible as rational functions of λ. The two equations will have two roots in common, and thus the curve have a double point, provided the coördinates of the point cause the determinants of the matrix

$$\left\| \begin{matrix} U & V & W & Z & 0 \\ 0 & U & V & W & Z \\ A & B & C & D & 0 \\ 0 & A & B & C & D \end{matrix} \right\|$$

to vanish simultaneously. The number of common solutions is 10 which may, of

course, be combined in various ways according to the assumptions made about the coefficients of λ in the original equations. An analytic discussion would involve the theory of restricted equations. A synthetic discussion would involve a study of certain subsidiary loci whose common points are singular points on the rational curve generated by the projectively related envelopes. We touch closely here the work initiated by Hesse (1866) and carried forward by Klein, Franz Meyer, Haase and others.

We know some things about the location of singular points on a proper rational curve but much remains unknown. It would seem feasible, in the way suggested, to add to our knowledge of configurations of points in a plane that can be multiple points on proper rational curves of orders higher than 5. The method, too, is capable of extension to higher space. Thus, two projectively related sheaves of planes of second class generate a ruled surface of fourth order etc.

One may mention also the application of the Steiner program to the study of non-euclidean geometry, in particular to those representations of non-euclidean geometry in Euclidean space. No difficulty would be encountered in carrying out the Steiner program on a sphere which, with proper restrictions, can be taken as a Euclidean picture of the elliptic plane, but I know of no analogous study of the geometry on a surface of constant negative curvature as the Euclidean picture of geometry in the hyperbolic plane. And I feel very sure that much remains to be done in projective non-euclidean geometry of hyperspace which might very well be accomplished along the lines of the Steiner program.

3. The Von Staudt Program. Von Staudt added three general features to the Steiner program (*Beiträge,* 1847):

1. The definition of projective relationship by means of harmonic division, thus removing the cumbersome chain of perspectivity as a definition from the Steiner program and measurement as a fundamental notion from the Chasles definition.

2. The introduction of the notion of the polarity in the plane and in space and defining conics and quadric surfaces as loci of self conjugate elements in polarities. Imaginary conics and quadric surfaces appear for the first time defined by means of uniform polarities in the plane and in space.

3. A geometric definition of imaginary points, lines, and planes by means of elliptic involutions on the sustaining form.

In regard to this program, the consideration of double polarities in the plane and in space seems capable of further extension. Thus, synthetically, two polarities in a bundle of rays, one of which is the orthogonal polarity, lead most easily to the properties of cones of the second order. Two polarities in a plane lead to quadric transformations and hence, by repetition, to Cremona transformations. Analytically this amounts to a study of simultaneous symmetric bilinear forms. In space of an odd number of dimensions, skew-symmetric bilinear forms would have to be considered.

But Von Staudt's great achievement was the geometric definition of imaginary

elements. The necessity for such a definition had been recognized by Poncelet, Steiner, Chasles and others but no satisfactory synthetic definition existed before 1847. In spite of this, Von Staudt's work bore little fruit for many years probably because of the formal style in which it appeared. It remained for Reye (*Geometrie der Lage,* 1866) to introduce Von Staudt and his work to mathematicians in general and to geometers in particular.

In 1872, Klein proposed a simplification of Von Staudt's definition of imaginary elements. Two conjugate imaginary points on a straight line, for example, are represented by a cyclic involution of order three on the line. Thus, synthetically,

$$ABC \barwedge BCA \barwedge CAB$$

represents one imaginary point on the line ABC, and

$$CBA \barwedge BAC \barwedge ACB$$

represents the conjugate imaginary point on the same line.

Analytically, if λ and μ are the coördinates of two real points on a line, the projectivity

$$\frac{\mu - z}{\mu - \bar{z}} = \omega\frac{\lambda - z}{\lambda - \bar{z}}; \qquad \omega^3 = 1, \qquad \omega \neq 1, \tag{1}$$

is cyclic and of order three and has the conjugate imaginary points z and $\bar{z}$ as its double points. The projectivity *defines* these imaginary points.

For $z,\ \bar{z} = x \pm iy$, the projectivity reduces to

$$\mu = \frac{(\sqrt{3}x - y)\lambda - \sqrt{3}(x^2 + y^2)}{\sqrt{3}\lambda - (\sqrt{3}x + y)} \tag{2}$$

and furnishes the bond between the Klein representation of imaginary points and the familiar representation by means of the Wessel-Argand diagram or the Gauss-plane.

The simpler involution

$$\mu = \frac{\lambda x - (x^2 + y^2)}{\lambda - x} \tag{3}$$

has exactly the same double points (Doppelpunktsinvolution; involutione unita) and furnishes the bond between the Von Staudt elliptic involution representation and the Gauss-plane.

Ramorino has given a complete account of theories about imaginary numbers and their geometric representations (*Giorn. di Mat.,* 1897–98).

Von Staudt also arrived at the notion of chains of imaginary points. Given any three points, real or imaginary, the locus of a fourth point such that the cross-ratio of the four points is a given real number is a chain. When imaginary points on a line are pictured as real points in a Gauss-plane, chains are represented

as straight lines or circles in the Gauss-plane. While much was previously known about the geometry of lines and circles in a plane, the Von Staudt chain was a strange new thing in geometry.

4. The Segre Program. Segre did much for geometry in several directions. Had he been asked, he would doubtless have called himself an algebraic geometrician. The program I am attributing to him deals with imaginary geometry, or more properly, the geometry of imaginary elements. This work was initiated by him in papers appearing in *Atti di Torino*, 1889–90–91, and in *Math. Ann.*, 1890. Here we meet the hyperalgebraic forms; *i.e.*, algebraic forms in the real components of an imaginary element. Or, what comes to the same thing, algebraic forms in the coördinates of conjugate imaginary elements. Thus forms in z, $\bar{z}$ (Eq. 1) or in x, y (Eqs. 2 or 3) are hyperalgebraic forms. Hermitian forms are examples. The geometric representation of such forms leads to the study of hyperalgebraic manifolds. Consider imaginary points in a plane. Each point has four real components and four-dimensioned space is required to form real pictures of geometric relations in the plane. Three equations connecting the real components of an imaginary point are the equations of a *thread.* Von Staudt's chains are particular threads. All imaginary points whose real components satisfy two equations are said to lie on a membrane or tissue. A single equation in the real components is the equation of a tri-dimensioned manifold. Segre's article in the *Math. Ann.* is concerned in the main with the real representations of hyperalgebraic manifolds in hyperspace.

Hyperalgebraic manifolds of an odd number of dimensions are entirely new entities in geometry. Segre also studied the linear transformations amongst the elements of hyperalgebraic manifolds and so was lead to anti-projectivities, anti-collineations, anti-polarities, etc. In an anti-projectivity, for example, the cross-ratios of corresponding quadruples of elements have conjugate imaginary values. In the study of linear transformation in the hyperalgebraic domain, Segre was anticipated somewhat, although unknown to him, by the Danish geometer C. S. Juel. Juel's thesis, published at Copenhagen 1885, deals with certain linear transformations in the hyperalgebraic domain which he called "symmetralities." This thesis was later published in *Acta Mathematica,* 1890.

Analysis in the hyperalgebraic domain leads to the definition and study of hypercomplex numbers and thus touches closely the work of Weierstrass, *Gött. Nach.*, 1884, Study in various papers, and others.

Thus is revealed an astonishing wealth of detail—a wealth beyond the dreams of the most avaricious geometer. The postulation of ideal points, the introduction of irrational points as limiting points in infinite series constructable by harmonic division (Dedekind postulate), and the geometric definition of imaginary points make synthetic geometry co-extensive with the algebra of n-ary forms; but it is doubtful if any of the great spirits who created and perfected the algebra of quantics ever dreamed that the prosaic symbols with which they wrought could hide such a wealth of geometric forms as has been uncovered along the lines

suggested by Segre. In witness to this remark one has only to read Coolidge's *Complex Geometry* (1924) and the papers by Coolidge, Young, Graustein, and others in this country. No one can doubt that there is much yet to be discovered in this field.

5. The Klein Program. Perhaps the most far-reaching and fruitful program for geometry was that of Felix Klein (*Erlanger Program,* 1872). In this program, Klein proposed to classify geometries by means of groups of transformations. Real projective geometry appears as the study of the invariant properties of geometric forms under the group of real linear point transformations. By extending the group to include reciprocal transformations—*i.e.*, point to line, point to plane, or point to hyperplane—projective geometry is made to include dual configurations and their properties.

The general group of projective transformations contains various subgroups. Those projective transformations which leave unchanged the ideal line in a plane (the ideal R_{n-1} in n-space) form a group to which belongs affine geometry. Affine geometry has come into prominence recently through the writings of Weyl and others in connection with the theory of relativity. Those projective transformations which leave invariant the circular points in any plane form a group to which belongs elementary geometry including the modern geometry of the triangle and its related circles.

Other subgroups have been considered which leave invariant certain loci. For example, those projective transformations which leave unchanged a proper conic (real or imaginary) form a group to which belongs plane non-euclidean geometry.

The projective group is capable of further extension by including within it anti-projectivities, anti-collineations, etc., thus including the Segre geometry of imaginary elements within the projective fold. To connect this field with that opened up by Study in *Theorie der Dynamen* and in various papers appears to me to be a problem of considerable attractiveness. And the synthetic study of continuous groups of projective transformations, begun by Newson in 1895 and carried on by Emch and others, cannot be regarded as complete.

The work of Enriques-Fano (*Ann. di Mat.,* 1897) shows that finite continuous groups of Cremona transformations in the plane and in three-space are equivalent to projective transformations in hyperspace. Thus, in space of a sufficiently great number of dimensions, all geometries admitting finite continuous groups of transformations may be classed as projective. Unquestionably many details remain to be filled-in in this comprehensive program: details concerning the inter-relations, isomorphisms, etc., among the various finite continuous groups of transformations contained within the $(n^2 - 1)$-parametered group of projective transformations in n-space.

What has been said thus far applies to entire space (Gesammtraum) and to finite continuous groups of transformations amongst the elements of this space. But there exist infinite continuous groups of point-transformations in this space;

e.g., the group of all analytic point-transformations. This group has no proper invariant other than the dimensions of the space. If, however, we consider only the infinitesimal neighborhood of a point, the differentials of the coördinates of the point undergo a projective transformation by virtue of the group of analytic transformations we are considering. Here we enter the domain of projective differential geometry and become interested in projective differential invariants—a field so well cultivated by Wilczynski, Green, Lane and others. Here we meet the Christoffel-Lipschitz theory of quadratic differential forms which forms the mathematical basis for the modern theory of relativity. A promising recent addition to this field is the geometry of paths initiated by Veblen and Thomas.

And what can be said about the geometry of imaginary points in the neighborhood of any given point, real or imaginary? Is there a complex projective differential geometry, or a projective differential geometry of imaginary elements? A few scattered theorems and notions, perhaps, and little more.

In conclusion, one may say that what has been called the "Segre program" together with the theory of groups of transformations is the most promising field for further research. This field is broad, is relatively new, and almost every issue of a mathematical magazine bears witness to its fecundity.

It has been said that projective geometry is a finished subject—all very pretty and beautiful but lacking in inspiration because completed. Nothing could be further from the truth. No young enthusiast seeking adventure in mathematical fields need avoid projective geometry because of fancied sterility. If he has the strength of purpose to persevere until he can read understandingly the various articles on geometry in the *Encyklopœdia*, he must feel as every traveler feels when, after struggling up the Alps, he sees all Italy lie before him.

THE USE OF COMPUTING RODS IN CHINA.

By DAVID CHIN-TE CHENG, Hinghwa City, Fukien, China.

The use of computing rods was a matter of daily importance in early times in China. We have no means of knowing just who invented the system nor at what date, but in early Chinese literature we find it stated that computers were trained in the art during a period of ten years; and that the use of computing rods was looked upon as fundamental in the solving of problems. For instance, it was asserted that the *K'iu-ch'ang Suan-shu* [1] (Arithmetic in Nine Sections), which probably was written by Li Shou [2] during the reign of Huang-ti, the Yellow Emperor, who began his reign in the year 2704 B.C.,[3] could be easily understood if

[1] *K'iu-ch'ang Suan-shu* or *Chiu-chang Suan-shu*, Smith, *History of Mathematics*, vol. I, pp. 31–33; Smith and Mikami, *History of Japanese Mathematics*, pp. 11–13; Cajori, *History of Mathematics*, p. 71.

[2] Wu Ch'eng Ch'uan, *Kang Chien I Chih Lu* (Brief History of China, monograph 1, p. 4; Yuan Yuan, *Ch'ou Jen Chuan* (Biography of Chinese Mathematicians), monograph 1, p. 1.

[3] Smith, *History of Mathematics*, vol. I, p. 24.

the system was known in which purple rods were used for positive numbers, and black for negative. Some writers claim that this system was used even as early as 2704 B.C. In the *Chou-pei Suan-king*,[4] probably written about 1105 B.C., Shang Kao states that "the process of numbers came from circles and squares; circle came from square, square came from squaring rule, and squaring rule came from the numbers nine times nine, which is equal to eighty-one." [5] "During the time of Huan Kung of Ch'i in the Chou dynasty" (about 680 B.C.), so Mei Fu, a writer of the same period asserts, "the numbers nine and nine were used." [6] The numbers "nine and nine" meant the numbers from one to nine, and a recognition of the product nine times nine equals eighty-one. The fact that the computing rods were based upon the numbers from one to nine led certain writers to assert definitely that the people during that time used them in solving problems. The *Sun Tze Suan Ching*, written by Sun Tze about 350 B.C., used this system in multiplication, division, and squaring. The computing rods were known c. 542 B.C.[7] and are referred to as counting stalks in a statement of Hiao-tze, the ruler of Ts'in from 361 to 337 B.C. They are mentioned again about 215 B.C., and some specimens of this period were displayed in a museum of the Emperor Ngan (397–419). These were known as arrow rods and were eighty in number. They were about 18 inches long, some made of bone and others of horn. The reason for using such long rods was that the early inhabitants of China performed their computations on the ground. In the reign of Wu-ti (140–37 B.C.) of the Han dynasty, it is related that an astronomer Sang Hung (about 113 B.C.) was very skillful in his use of the rods. In the third century of our era it is recorded that Wang Jung, a minister of state, spent his nights in reckoning his income with ivory calculating rods, and the expression "to reckon with ivory rods" is still used as an allusion to wealth. In the time of the Emperor Ch'eng (326–343) the counting rods were made of wood, ivory, iron, bamboo, or paper, and two centuries later the Emperor Siuen Wu (500–516) had counting rods cast in iron for the use of the people.[8] After the Han dynasty it became the custom to compute on a table, and shorter rods were therefore used; these being about four American inches long and one tenth of an inch thick. The fact is that the early mathematics in China, until the Sung and Yuan dynasties (before 1368), was based entirely upon the use of computing rods; but during the Ming dynasty (after 1368) the system lost its significance entirely on account of the use of the abacus (*suan pan* [9]) and the introduction of European methods.

The historian Mei Wen Ting (1633–1721), in his work on *Li Suan Ch'uan Shu* (the chapter on *Ku Suan Ch'i K'ao*, a study of the ancient calculating instru-

[4] Smith, *History of Mathematics*, vol. I, pp. 29–31; Smith and Mikami, *History of Japanese Mathematics*, p. 9; Mikami, *The Development of Mathematics in China and Japan*, p. 4; Cajori, *History of Mathematics*, p. 71.

[5] Tai Chen, *Tse Suan* (Computing Rods), Introduction.

[6] Tai Chen, *Tse Suan* (Computing Rods), Introduction.

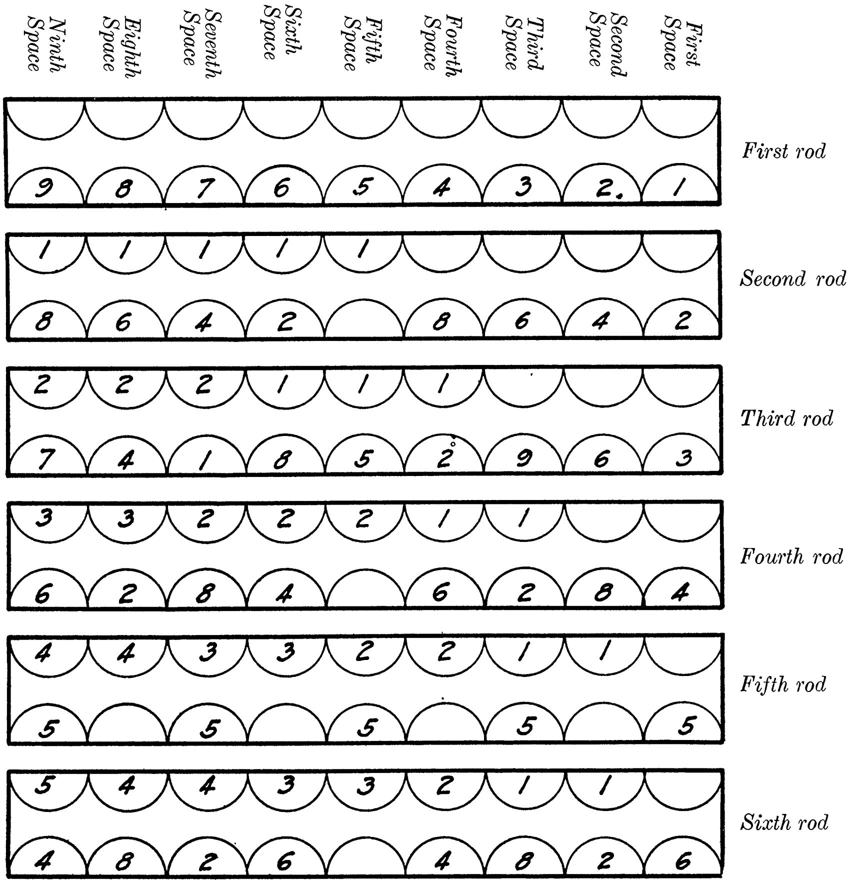
[7] Smith, *History of Mathematics*, vol. I, p. 96.

[8] Smith, *History of Mathematics*, vol. II, pp. 169–170.

[9] Smith, *History of Mathematics*, vol. II, p. 168.

ments), states that about the beginning of the Christian era 271 rods constituted a set, or handful, and that they formed a hexagon that had nine rods on a side. This means that they were arranged in six groups of which the ends of each formed a triangular number of $1 + 2 + \cdots + 9$ units, or 45 in all. Six of these make 6×45, or 270, and these six were grouped about one central rod, making 271, thus affording an illustration of the use of figurate numbers in the East.[10]

We are not certain as to the nature of the very early rods, but we have accurate information as to the later ones. This comes to us from various standard works, a list of which is given in the bibliography at the end of this article. In the case of the later system, beginning about the thirteenth century, each rod was divided into nine spaces, and in each space there were drawn two semicircles, one above and one below. In the nine spaces on the rods the numbers are definitely arranged in the following way:



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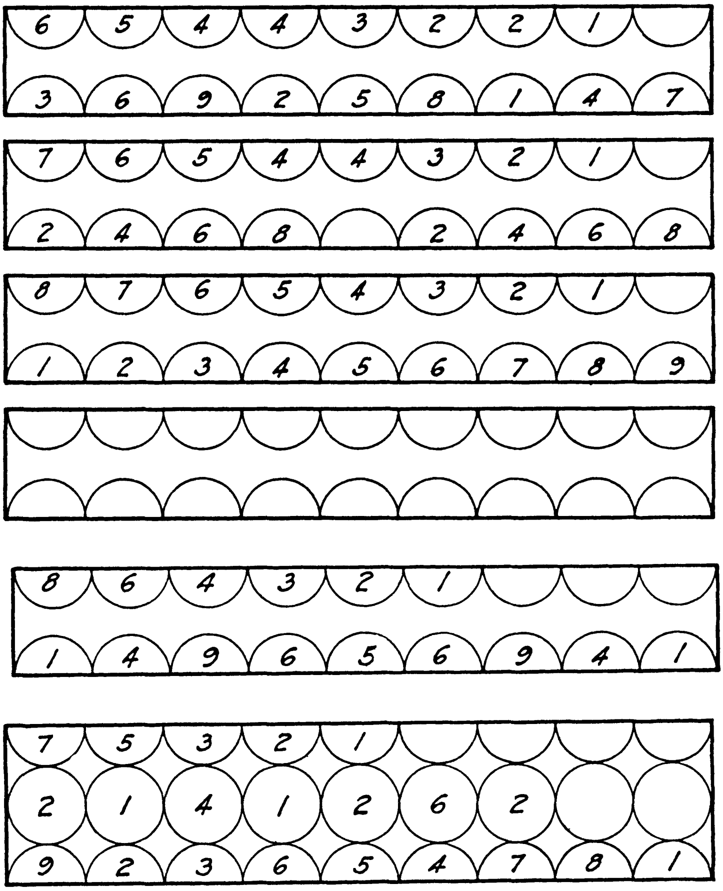
[10] Mei Wen Ting, *Ku Suan Ch'i K'ao* (Ancient calculating instrument); Smith, *History of Mathematics*, vol. II, p. 170.



Seventh rod

Eighth rod

Ninth rod

Zero rod

Square root rod

Cube root rod

Spaces on any rod are numbered from the right-hand end. When in the following discussion reference is made to the number of a rod, the number in the lower semicircle of the first space is meant, zero rod being the one in which that space is vacant. The square-root and cube-root rods are not numbered. To find the number in any space of the regular computing rods: multiply the number of the space by the number of the rod.

The numbers in the upper semicircles are considered as belonging to the tens' place, while those in the lower semicircles are considered as units. When two rods are used, the lower semicircle of the upper rod and the upper semicircle of the lower rod are taken to form one whole circle, and the numbers within each circle are added together so as to form the digit of that place. When two rods are used, three digits are thus formed; when three rods are used, four digits are formed, and so on. This will be clearly understood by reading problems numbered 1 and 2, which follow.

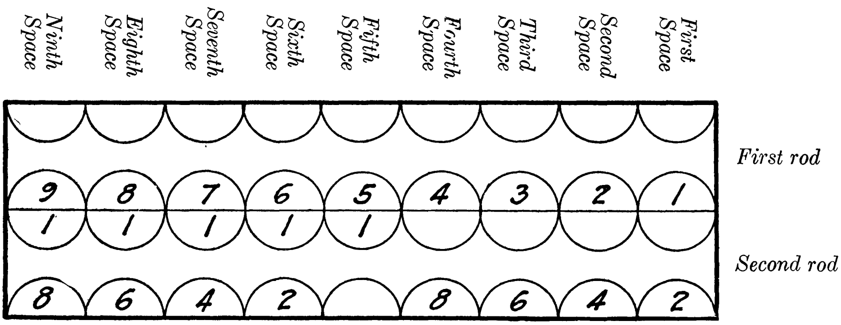
Computing rods are used in multiplication, division and squaring. Addition

and subtraction are the only prerequisites to the working with them. The selection of the rods to be used depends upon the figures to be employed in the operation.

Multiplication on computing rods is somewhat similar to the process in modern arithmetic, as is evident from the following examples:

Problem 1. Twelve tailors who are in the army each get 360 *chin* of rice. What is the total amount of rice given?

Solution: To multiply 360 by 12, the first and second rods are used.

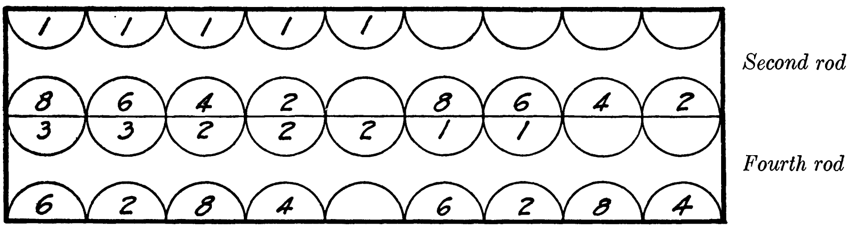


Assuming the operation which involves the 0, we first multiply the 6 in the 360 by 12. This is done by reading to the sixth space and obtaining 072. We next multiply the 3 by reading to the third space and getting as a result 036. Adding these two results together, we have 4320 as the final answer.

$$\begin{array}{r|l} 0 & 000 \\ 6 & 072 \\ 3 & 036 \\ \hline & 4320 \end{array}$$

Problem 2. In *Fang t'ien* (squaring the farm) 1 *mu* is equal to 240 *pu*. How many *pu* are in 125 *mu?*

Solution: Using 240 as the multiplier, the second and fourth rods are used and an extra 0 is annexed to the result.



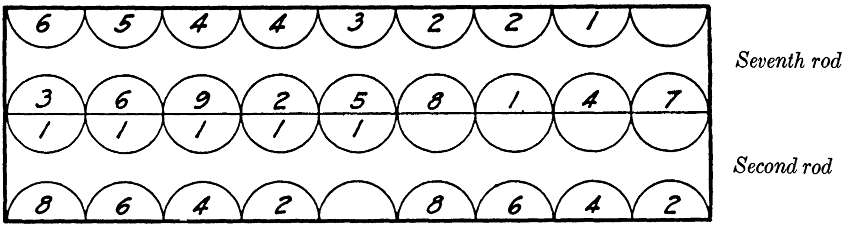
First multiply the 5 of 125, by reading to the fifth space and getting 120; next to multiply by 2, read to the second space and get 048; and finally to multiply by 1, read the first space, 024. Adding the three results, the answer is 30,000.

$$\begin{array}{r|l} & 000 \\ 5 & 120 \\ 2 & 048 \\ 1 & 024 \\ \hline & 300000 \end{array}$$

The process of division by computing rods is just the opposite of that of multiplication, the quotient being read from the number of the space in which the figure most nearly approximates to the figure to be divided.

Problem 1. The sun moves 360 degrees in the year, which is divided into 72 equal parts. How many degrees are there in each part?

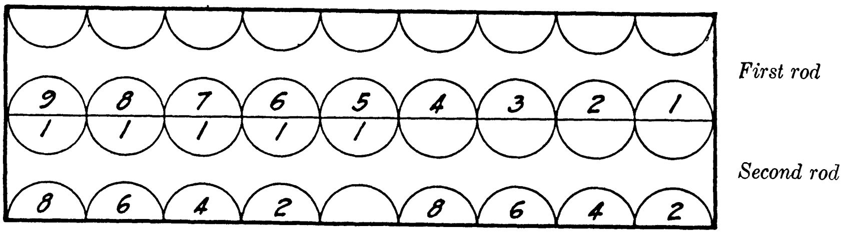
Solution: Since 360 is divided by 72, the seventh and second rods are used.



Read to the fifth space, the number is 360; subtracting, 360 − 360 = 0, and so 5 is the quotient.

Problem 2. 129,600 years is divided into 12 parts. How many years are there in each part?

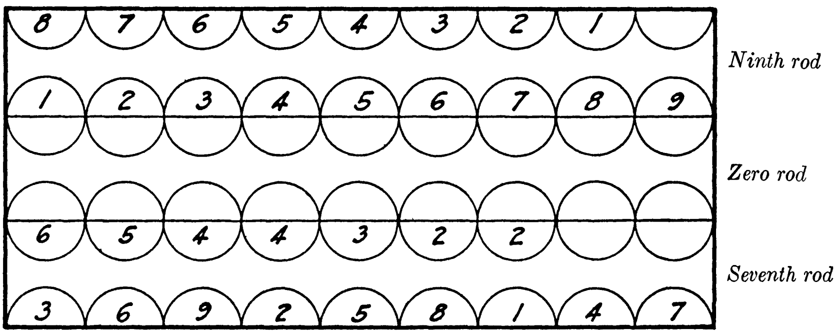
Solution: Since 129,600 is divided by 12, the first and second rods are used.



Read 012 from the first space on the rod; subtracting, 129,600 − 120,000 = 9600, and so the first figure of the quotient is 1. Next read 096 from the eighth space; subtracting, 9600 − 9600 = 0, and so the second figure of the quotient is 8. Between the first and the second figure of the quotient there is one place vacant, which is filled with 0, and similarly with the last two places. The answer is 10,800.

Problem 3. 21,768 yards of cloth are divided among 907 people. How many yards of cloth does each person get?

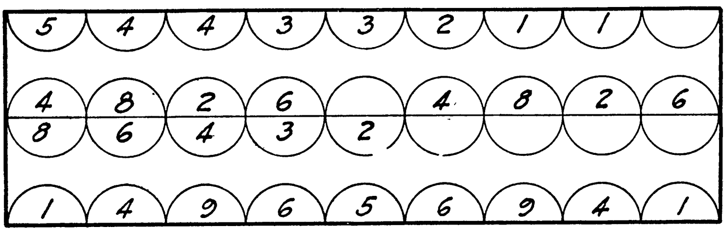
Solution: Since 21,768 is divided by 907, the ninth, zero, and seventh rods are used.



Read 1814 from the second space, which is nearest to 2126; subtracting, 21,768 – 18,140 = 3628, and the first figure of the sum becomes 2; next read 3628 from the fourth space; subtracting, 3628 – 3628 = 0, and so the second figure of the quotient is 4. The answer is 24 yards.

According to Mei Wen Ting "the method of finding square root was used in the *Chou-peï Suan-king*, in which Shang Kao told Prince Chóu-kung that the use of the squaring rule was for the earth while the use of the circle was for the heavens." The use of the computing rod in finding square roots was a common practice in this period, and resembled closely our present system.

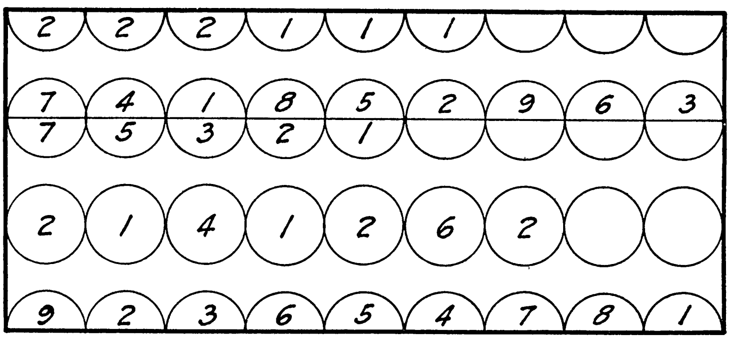
Problem: Find the square root of 129,600.
Solution: Mark off the number by placing a dot over every other place, beginning with units. Thus 12 is in the first left-hand period. Reading to the third space on the square-root rod we have 9, which is nearest to 12, and so the first figure of the root is 3, with a remainder of 3. Multiply the 3 in the root by 2, which gives 6, which shows that we are to use the sixth of the general computing rods together with the square-root rod.



Sixth rod

Square root rod

Read to the sixth space on both of the rods, and the number is 396, which is exactly the next part in the number of which the square root is desired. Therefore the second figure of the root is 6, and, annexing zero, the final answer becomes 360.
Problem: Find the cube-root of 1331.
Solution: We first use the special cube-root rod, as follows:



Third rod

Cube root rod

In cube root mark off the periods by a dot over every third figure. Since 1 is the first figure of the number, the first space on the cube-root rod is read, giving 1. Subtracting this from the first period in the dividend there is no remainder. The first figure of the root is therefore 1. Now take 1 times 10, which is 10, and 10 times 10 times 3, which is 300; then using the third rod together with the cube-root rod, read from the first space the number is 301. Then 1 (first space) times 10 equals 30, and 301 + 30 = 331. Subtracting, 331 – 331 = 0, and thus the second figure in the root is 1. Therefore the final answer is 11.

Having now shown the nature of the work with this kind of computing rods, the question naturally arises as to the antiquity of the device. Evidently the

ancient rods referred to by Mei Wen Ting (1633–1721) were mere computing sticks such as were used to represent the coefficients of an equation. The statement that the later kind came into use about the thirteenth century is also due to Mei Wen Ting. He also says that the squaring rule was used in the *Chou-pei Suan-king,* and that the computing rod was used in finding the square root. That it was this later kind of rod is, however, very improbable, since no certain mention is made of this type until modern times. That these later rods were first used about the thirteenth century may be true, but they are described by no writers before Mei Wen Ting. This puts the description about the close of the seventeenth century, a hundred years after the Jesuit influence began in China, and more than fifty years after Napier's rods became generally known in the West. While the Chinese rods are by no means identical with those of Napier, there is a resemblance. It therefore becomes an interesting problem for us in China to ascertain (1) whether our rods were unquestionably used as early as the thirteenth century, as our later historians assert; and (2), if not, whether they were suggested by the Napier rods, which had no doubt reached our country through the Jesuit missionaries. In fact, Mei Wen Ting suggests this very possibility.

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The above works are all in the library of Professor David Eugene Smith.

ON THE MATHEMATICAL SIGNIFICANCE OF THE CHINESE HO T'U AND LO SHU.

By DAVID CHIN-TE CHENG, Hinghwa City, Fukien, China.

The *K'iu-ch'ang Suan-shu,* or Arithmetic in Nine Sections,[1] which probably was written by Li Shou [2] during the reign of Huang ti, the Yellow Emperor, who

[1] Smith, *History of Mathematics,* vol. I, pp. 31–33; Cajori, *History of Mathematics,* p. 71; Smith and Mikami, *History of Japanese Mathematics,* pp. 11–13.

[2] Or Li Shu. See Wu Sheng Chuen, *Kang Chien I Chih Lu* (Briefer History of China), monograph 1, p. 4; Yüan Yüan, *Chou Jen Chuan* (Biography of Chinese Mathematicians), monograph 1, p. 1.

began his reign in the year 2704 B.C.,[3] and the *Chóu-peï Suan-king,*[4] which related to number mysticism, mensuration, and astronomy, in the dialogues between the Prince Chóu-kung and his minister Shang Kao in the Chow dynasty, about 1105 B.C., have been regarded as the oldest books on mathematics in China and perhaps the world. According to tradition, the origin of Chinese mathematics can be traced back to the *Ho-t'u,*[5] a figure appearing on the back of a dragon horse during the reign of Fuh-hi (2852–2738 B.C.[6]). Basing his work upon this figure and after careful observations of the appearance on the heavens and the earth Fuh-hi drew the *Pa-kua,*[7] or the eight trigrams, so familiar in Chinese mysticism. Tradition also states that the *Lo-shu,* the world's oldest specimen of a magic square,[8] was first found upon the back of a tortoise which appeared to the Emperor Yu (*c*.2200 B.C.), when he was embarking on the Yellow River. Hsia Hou Yang (*c*.550) repeats the legend, saying that "Mathematics (Chinese) began with the Emperor Fuh-hi, and Li Shou wrote the *Nine Sections* in the Yellow Emperor's time";[9] and Mo Jo (*c*.1303) refers to the same period and to the two mystic symbols, as follows: "The number 1 is the beginning of all things and from 1, doubling continuously, we have 2, 4, 8, etc. Is this not a natural mathematical process? This process was derived from the *Ho-t'u* and the *Lo-shu.*" [10] Mei Wen Ting, the celebrated historian (1693), remarks that "since the appearance of the *Ho-t'u* and the *Lo-shu* we have the odd and even numbers" [11]; and Wu Shih Hsien (1750) says that "Mathematics does not begin with Sun Tze[12]; with the *Ho-t'u* and the *Lo-shu* we have both odd and even numbers." [13]

Of course these traditions are not considered as historical records by Chinese scholars any more than they are by those of the West. They are looked upon simply as part of the poetry of my people, just as stories of King Arthur and his Table Round are looked upon as part of the folklore of England, and as scholars generally look upon the stories of Wilhelm Tell or the tales of Captain John Smith and of Captain Kidd. The *Ho-t'u* and the *Lo-shu* exist, we do not know their origin; but there have grown up, in relation to them, certain mathematical theories which do not seem to have been made known as yet to European and American students, and which it is the purpose of this paper to present.

There are two theories relating the origin of the Chinese mathematics among

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[3] Smith, vol. I, p. 24.

[4] Smith, vol. I, pp. 29–31; Cajori, p. 71; Smith and Mikami, p. 9; Mikami, *The Development of Mathematics in China and Japan,* p. 4.

[5] Smith, vol. I, p. 29.

[6] Smith, vol. I, p. 23.

[7] Smith, vol. I, pp. 25–27.

[8] Smith, vol. I, p. 28; Cajori, pp. 76–77.

[9] Hsia Hou Yang, *Hsia Hou Yang Suan Ching,* introduction.

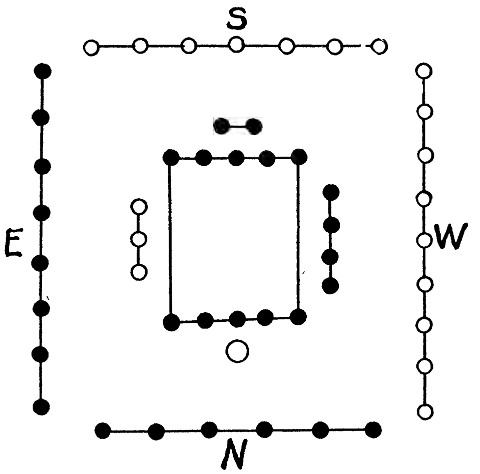
[10] Chu Shih Chieh, *Szu Yuan Yu Chien* (on Chinese algebra). Introduction by Mo Jo.

[11] Mei Wen Ting, Arithmetic. Introduction.

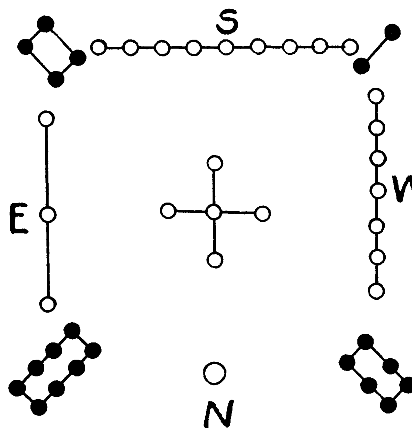
[12] The date of Sun Tze who wrote *Sun Tze Suan Ching* may be about 350 B.C., or possibly as late as the first century. It is very uncertain.

[13] Tan Wen, *Shu Hsueh Hsun Yuan* (To find the Origin of Mathematics). Introduction by Wu Shih Hsien.

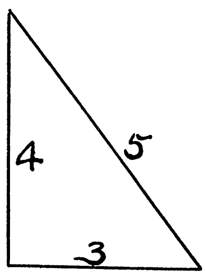
some of the Chinese writers, both of which are based on the figures of the *Ho-t'u* and the *Lo-shu*, and these will now be set forth.



Ho-t'u



Lo-Shu



4	9	2
3	5	7
8	1	6

I. Emperor K'ang Hsi of the Ch'ing dynasty, in 1713 A.D., published two series of his celebrated books containing logarithmic tables. These were issued in fifty-three monographs entitled *Shu Li Ching Yun* (*Principles of Mathematics*) and treating of arithmetic, geometry, and mensuration. In his first monograph of the first series he traced the beginnings of Chinese mathematics. He stated that "addition and subtraction were derived from the *Ho-t'u,* and that multiplication and division were derived from the *Lo-shu*." [14] The explanation is substantially as follows:

a. The origin of addition and subtraction. According to Emperor K'ang Hsi, the numbers of the *Ho t'u* begin with one, the center is five, and they end with ten. The numbers 1, 3, 5, 7, and 9 are odd or masculine numbers, and 2, 4, 6, 8, and 10 are even or feminine numbers.[15] As to the position of the numbers

[14] Emperor K'ang Hsi, *Shu Li Ching Yun,* Chapter I.

[15] In the *Ho t'u* and *Lo shu* the masculine numbers are represented by white (celestial) circles, and the feminine numbers by black (terrestrial) ones.

in the *Ho-t'u*, 1 is the beginning of numbers, it represents the masculine principle and it is placed on the colder north; 2, which represents the feminine principle, is placed on the warmer south; 3, the masculine, is placed east, the position of the rising sun; 4, the feminine, is placed on the west; 5, the masculine, is placed in the center,—the only position left vacant, the other four directions having been taken; 6 is the sum of $5 + 1$, whence $6 - 5 = 1$, and therefore it has the same position as the number 1; 7, masculine, is the sum of $5 + 2$, whence $7 - 5 = 2$, and therefore it has the same position as the number 2; 8, feminine, is the sum of $5 + 3$, whence $8 - 5 = 3$, and therefore it has the same position as the number 3; 9, masculine, is the sum of $5 + 4$, whence $9 - 5 = 4$, and therefore it has the same position as the number 4; 10, feminine, is the sum of $5 + 5$, whence $10 - 5 = 5$, and therefore it has the same position as the number 5. This, according to K'ang Hsi, was the beginning of addition and subtraction in China.

b. The origin of multiplication and division. According to the same writer, the explanation of the positions of the numbers in the *Lo-shu* is as follows: 5 is placed in the center as a person wearing the numeral 9 as his hat; the numbers 3 and 7 are placed on the right and left, the numbers 2 and 4 occupy the position of the shoulders, and the numbers 8 and 6 represent the legs and stand on the number 1. There are five odd or masculine numbers and four even or feminine numbers. Number 1 is the beginning of the numbers, and hence it does not move. The numbers begin with 2 and 3. Beginning with 3 and proceeding clockwise with the odd numbers, we have 3, 9, 7, and 1. Beginning with 2 and proceeding counter-clockwise, we have the following even numbers: 2, 4, 8, and 6. The masculine number 3 is placed on the east, meaning that all things grow from the orient. Beginning at the east and moving to south, we have $3 \times 3 = 9$; moving to the west, we have $3 \times 9 = 27$, and taking away the complete number 20 we have the remainder 7; moving to the north, we have $3 \times 7 = 21$, and taking away the complete number 20 we have the remainder 1. Beginning with the feminine number 2, located in the southwest and representing the right shoulder, and moving to the southeast, we have $2 \times 2 = 4$, which is located on the left shoulder; moving to the northeast we have $4 \times 2 = 8$, which is located on the right leg; and moving to the northwest we have $8 \times 2 = 16$, and taking away the complete number 10 we have the remainder 6, located on the left leg. If we add the numbers horizontally, vertically, or diagonally, the sum is always fifteen, and so we have in the *Lo-shu* what is known as the magic square. The total is 45 which is the product of 9×5. This is therefore asserted to be the beginning of multiplication and division in China.

II. The second theory, held by some of the writers, is that the *Ho-t'u* was the origin of right triangle or the Pythagorean Theorem, and the *Lo-shu* was the origin of addition, subtraction, multiplication, and division. Li Shih Hsiung (1673) stated "the right triangle was derived from the *Ho-t'u*, and addition, subtraction, multiplication, and division were derived from the *Lo-shu;* the whole of the *Nine Sections* was related to the right triangles." [16] Tai Ti Yuan (1750)

[16] Fang Chung T'ung, *Shu Tu Yen*, Introduction by Li Shih Hsiung.

stated that "the *Nine Sections* were derived from the *Ho-t'u* and the *Lo-shu*." [17] Tung Yung (1750) stated that "the *Nine Sections* treated of the right triangle, which in turn was derived from the *Ho-t'u*, and that addition, subtraction, multiplication, and division were derived from the *Lo-shu*. Therefore the *Ho-t'u* and the *Lo-shu* were the origins of mathematics." [18] Tan Wen (1750), in the first chapter of his *Shu Hsueh Hsun Yuan* (*To Find the Origin of Mathematics*), stated that "the theory of numbers began with the *Ho-t'u* and the *Lo-shu*."

According to Tan Wen, Fuh-hi derived the *Pa-kua* from the *Ho-t'u*, and the direction of movement proceeded from north to east, thence to south, thence to the center, thence to west, and finally to the north. In the *Ho-t'u* the sum of the odd numbers 1, 3, 5, 7, and 9 is 25, and the sum of the even numbers 2, 4, 6, 8, and 10 is 30, the total number being 55. Adding 1 and 10 we have 11, and this multiplied by 10 and divided by 2 gives 55. Tan Wen's explanation as to the position and arrangement of numbers in the *Lo-shu* are the same as that given by Emperor K'ang Hsi.

Fang Chung T'ung (1687) in his *Shu Tu Yen*, a book treating of geometry, the abacus, arithmetic, the computation rods, and the *Nine Sections*, asserts that "The *Nine Sections* treats of the right triangle, which in turn was derived from the *Ho-t'u*, and hence the *Ho-t'u* was the origin of mathematics." [19] The numbers in the figure of the *Ho-t'u*, according to Fang Chung T'ung, are based upon the right triangle, having the hypotenuse 5, and the sides 3 and 4. The number 1 is the difference between the hypotenuse 5 and the side 4, or the side 4 minus the side 3; the number 2 is the difference between the hypotenuse 5 and the side 3; the number 3 is the side 3; the number 4 is the side 4; the number 5 is the hypotenuse; the number 6 is the difference between the hypotenuse 5 and the side 4, plus the hypotenuse 5; the number 7 is the difference between the hypotenuse 5 and the side 3, plus the hypotenuse 5, or it is the sum of sides 4 and 3; the number 8 is the sum of the hypotenuse 5 and the side 3; the number 9 is the sum of the hypotenuse 5 and the side 4. The *Chóu-peï Suan-king* treats of the right triangle having the sides 3 and 4 and the hypotenuse 5, 25 being the square of the hypotenuse. The sum of the numbers in the *Ho-t'u* is 55; the middle number 5 is of no consequence, and 50 is the sum of the squares of 3, 4, and 5. The number 3, which is masculine and is located at the left, added to the hypotenuse 5, equals 8, and so the numbers 3 and 8 have the same position at the left of the figure (east); the number 4, which is feminine and at the right, added to the hypotenuse 5, equals 9, and so the numbers 9 and 4 have the same position at the right (west); and the number 5, which is masculine and is the hypotenuse, is equal to $\sqrt{3^2 + 4^2}$ and stands in the center. The hypotenuse 5 minus the side 3 equals 2, which is located above; and the hypotenuse 5 minus the side 4 equals 1, which is located below. The hypotenuse 5 minus the side 3

[17] Tan Wen, *Shu Hsueh Hsun Yuan* (*To Find the Origin of Mathematics*). Introduction by Tai Ti Yuan.

[18] Tan Wen, *Shu Hsueh Hsun Yuan* (*To Find the Origin of Mathematics*). Introduction by Tung Yung.

[19] Fang Chung T'ung, *Shu Tu Yen*, Chapter I.

equals 2, and this plus the hypotenuse 5 equals 7; therefore the numbers 7 and 2 have the same position; the hypotenuse 5 minus the side 4 equals 1, and this plus hypotenuse 5 equals 6; therefore the numbers 6 and 1 have the same position. This is the origin of the right triangle.

As to the *Lo-shu,* Fang T'ung stated that "addition, subtraction, multiplication, and division are all derived from the *Lo-shu,*" and that "the addition of a masculine number and a feminine number gives a masculine number";[20] for example, $1 + 6 = 7$, $7 + 2 = 9$, $9 + 4 = 13$, and taking away the complete number 10 we have 3; and $3 + 8 = 11$, from which, by taking away the complete number 10, we have 1. Numbers begin with the masculine ones, and these govern the feminine ones. This was the beginning of addition. Subtracting masculine numbers from feminine ones we have $6 - 1 = 5$, $8 - 3 = 5$; and subtracting feminine numbers from masculine ones we have $9 - 4 = 5$, $7 - 2 = 5$. Taking away the side numbers, and retaining the middle number 5, was the beginning of subtraction. In the *Lo-shu* the opposite numbers 1 and 9 have the following relations: $9 \times 1 = 9$, $1 \times 9 = 9$, $9 \div 1 = 9$, $9 \div 9 = 1$; the opposite numbers 2 and 8 give $2 \times 8 = 16$, $8 \times 2 = 16$, $16 \div 8 = 2$, and $16 \div 2 = 8$; the opposite numbers 3 and 7 give $3 \times 7 = 21$, $7 \times 3 = 21$, $21 \div 3 = 7$, and $21 \div 7 = 3$; the opposite numbers 4 and 6 give $4 \times 6 = 24$, $6 \times 4 = 24$, $24 \div 4 = 6$, and $24 \div 6 = 4$. This was the beginning of multiplication and division.

Such are some of the fanciful theories concerning the significance of the two most ancient pieces of Chinese mathematics that we have. They show the play of fancy and imagination in the work of the scholars of my country in the period before the modern introduction of European science. They lack the mathematical insight of the West, but the West lacks the poetic insight of the East. If one were to be deprived of either, with which would it be better for him to dispense? Perhaps the best answer is to say that it is not necessary to lack both. If so, which will teach the other this fact?

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPRINT OF OLD QUESTIONS.

It is thought that some of the questions of this department which have hitherto remained unanswered should be reprinted for the attention of new readers and others who had previously overlooked them.

[20] Fang Chung T'ung, *Shu Tu Yen,* Chapter I.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19, 125; *1922*, 159]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental value as it has to do with the most general solid whose volume is given by the prismatoid formula.[1]

Previous contributors have shown that $f(x)$ may have *one*, but not *six*, continuous derivatives.

39 [*1900*, 256; *1921*, 125; *1922*, 160].[2] There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion it should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line.[3] In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an inscribed polygon of greatest area R. For a fixed value of the integer n and for all convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$ the upper limit of Q/P is 2.

3. Let the area of a simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \cdots = a?$$

I have proved the statement to be true in the special case where A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see part 1 of this question), always remaining inscribed to K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

43 [*1921*, 260; *1922*, 161]. Is any rapid method known for the evaluation of the Sylvester determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape and of its frequent occurrence.

45 [*1921*, 305; *1922*, 161].

See the reply appearing in this number, which disposes of one part of the question, but leaves something to be done with regard to the second part.

46 [*1922*, 210]. A geometrical construction will often become impracticable in special cases where some of the construction points are imaginary, although the final result is real. Is there any system according to which a construction failing in this way may be replaced by one that will work?

For example, let a line l_1 cut a circle c_1 in P_1 and Q_1, and let a line l_2 cut a circle c_2 in P_2 and Q_2. If the four points P_1, Q_1, P_2, Q_2 are real, the intersections of P_1P_2 with Q_1Q_2 and P_1Q_2 with P_2Q_1 may be found directly. If the four points are imaginary, the intersections named will still

[1] A summary of results so far obtained in this problem is given in the last reprint of questions (*1922*, 159). These results leave the following question to be answered:

How many continuous derivatives may a real function $f(x)$ (*not a cubic or lower polynomial*) *possess while satisfying the given equation for every* h *in a certain real interval* $0 < h < H$?

[2] As contributed by Professor S. Kakeya.

[3] For a varied form of this problem see W. B. Ford, "On Kakeya's Minimum Problem," *Bulletin of the Amer. Math. Soc.*, vol. 28, 1922, pp. 45–53. See also Question **42** in this department (*1921*, 65, 126; *1922*, 161).—Editor.

be real, and there ought to be a simple way of getting them as the points common to a line and a circle.[1]

51 [*1924*, 85]. Can any reader supply approximate formulas for the problem of a cable suspended from two points at different levels?

NEW QUESTION.

55. Is it possible by ruler and compasses to construct an angle equal to one radian? [2]

REPLIES TO QUESTIONS.

45 [*1921*, 305; *1922*, 161]. Is every non-trivial solution in integers of the equation $t^3 = x^3 + y^3 + 1$ expressible in the form $x = 9r^4 - 3r$, $y = 9r^3 - 1$, $t = 9r^4$? If there are non-trivial solutions not expressible in this form, can a general solution be found?

REPLY BY N. B. MITRA, Ewing Christian College, Allahabad, India.

The answer to the first part of the question is in the negative, as the following investigation will show.[3]

The problem is to solve in integers the equation

$$x^3 + y^3 + z^3 = 1. \tag{1}$$

We have

$$x^3 + y^3 = 1 - z^3. \tag{2}$$

If $x + y = 1 - z$, then from (2), $x^2 - xy + y^2 = 1 + z + z^2$. Eliminating z we get $xy - (x + y) + 1 = 0$ which gives the trivial solutions, $x = 1$, $y = -z$ and $y = 1$, $x = -z$. If $x + y \neq 1 - z$, then integers p and q, prime to each other (one of them may be unity), exist such that

$$\frac{1 - z}{x + y} = \frac{p}{q}. \tag{3}$$

We may without loss of generality suppose p to be positive and this we do. We shall show (see below) that in this case q cannot be negative.

From (2) and (3),

$$q(x^2 - xy + y^2) = p(1 + z + z^2). \tag{4}$$

Eliminating z from (3) and (4),

$$x^2(p^3 - q^3) + xy(2p^3 + q^3) + y^2(p^3 - q^3) - 3p^2q(x + y) + 3pq^2 = 0. \tag{5}$$

In order that integral solutions of (1) other than the two trivial ones found above may exist, equation (5) must be capable of affording integral values of x and y.

[1] Perhaps it would be better to omit the last ten words of the question. It should be possible for some reader to solve at least the particular problem here proposed. The more general question, though probably admitting an affirmative answer, calls for a well-planned representation of imaginary points.

[2] Proposed in the course of Discussion I appearing below.

[3] The solution given by Professor Bradley (*1921*, 307) was arrived at by me by quite a different method and was published in the *Journal of the Indian Mathematical Society*, vol. XIII (Feb. 1921), p. 17.

Since p and q are prime to each other, equation (4) cannot subsist unless p is a factor of $x^2 - xy + y^2$ and q of $1 + z + z^2$. Now $x^2 - xy + y^2$ and $1 + z + z^2$ are both of the form $a^2 + 3b^2$, and factors of numbers of this form are always of this form also. Hence p and q must both be of the same form. In other words p and q are both of the form $3^m f^2 N$, where $m = 0$ or 1, f is any integer and N is composed entirely of factors which are primes (including 1) of the form

$$6n + 1. \tag{6}$$

By giving any suitable values to p and q consistent with (6) we can get all the possible integral solutions of (5) or else satisfy ourselves that there are none.

Equation (5) may be thrown into the form (always, since $p \neq q$)

$$X^2 - DY^2 = H, \tag{7}$$

where

$$X = 3q^2\{x(4p^3 - q^3) - 3p^2q\}, \tag{8.1}$$

$$Y = (2p^3 + q^3)x + 2(p^3 - q^3)y - 3p^2q, \tag{8.2}$$

$$D = 3q(4p^3 - q^3), \tag{8.3}$$

$$H = 36pq^3(p^3 - q^3)^2. \tag{8.4}$$

Equations (8.1) and (8.2) show that X and Y are integral. Hence equation (7) must be capable of affording integral values of X and Y, if (1) is possible.

If q is negative, D and H are both negative and (7) has no real solutions. Hence for the possibility of (1), q must be positive.

If $q > p\sqrt[3]{4}$, D is negative and H is positive; therefore the number of integral solutions of (7) in this case is limited and there may be none.

If $q < p\sqrt[3]{4}$, D and H are both positive; in this case, if D is not a perfect square, (7) will admit of an infinite number of solutions, provided H, if $< \sqrt{D}$, occurs among the divisors belonging to the development of $\sqrt{D}$ as a S.C.F. and all the solutions may be obtained in the usual way. For other values of H there are either no solutions or an infinite number. If however D is a perfect square, the number of solutions is limited and there may be none.

From the values of X and Y thus obtained we are to pick out those which give integral values of x and y from (8.1) and (8.2); and from the values of x and y we are to reject those which do not give integral values of z from (3). In the particular case $q = 1$, every integral value of x and y gives an integral value of z.

The values of X and Y obtained as above form a periodic series to any given modulus; thus for any given values of p and q, a finite number of trials will determine whether they give integral values of x and y and of z.

The following numerical examples will illustrate the method.

(*a*) Take $p = 3$, $q = 1$. Then we get $X^2 - 321Y^2 = 3\cdot 156^2$, where $X = 321x - 81$ and $Y = 55x + 52y - 27$.

Putting $X = \pm 156X_1$, $Y = \pm 156Y_1$, we get $X_1^2 - 321Y_1^2 = 3$, which gives $X_1 = 18r \pm 321s$, $Y_1 = 18s \pm r$, where $r^2 - 321s^2 = 1$, so that

$$r = \frac{1}{2}\{(215 + 12\sqrt{321})^n + (215 - 12\sqrt{321})^n\},$$

$$s = \frac{1}{2\sqrt{321}}\{(215 + 12\sqrt{321})^n - (215 - 12\sqrt{321})^n\},$$

where n is any integer. Putting $n = 0$ and 1, we get $r = 1$ and 215, $s = 0$ and 12; $X_1 = 18$ and 7722, $Y_1 = 1$ and 431: whence $x = 9$ and 3753, $y = -6, -12$ and $-2676, -5262$, and $z = -8$, 10 and -3230, 4528.

Hence we get the following solutions:

$$-6^3 + 9^3 - 8^3 = 1, \qquad (a, 1)$$

$$-12^3 + 9^3 + 10^3 = 1, \qquad (a, 2)$$

$$-2676^3 + 3753^3 - 3230^3 = 1, \qquad (a, 3)$$

$$-5262^3 + 3753^3 + 4528^3 = 1. \qquad (a, 4)$$

Giving other values to n, we may get as many solutions as we like.

Again, putting $X = \pm 156X_1/f$, $Y = \pm 156Y_1/f$ and solving the equation $X_1^2 - 321Y_1^2 = 3f^2$, where f is any integral factor of 156, we may get as many other solutions as we like.

(b) Take $p = 4$, $q = 1$. Then we get $X^2 - 85Y^2 = 84^2$, where

$$X = 85x - 16, \qquad Y = 43x + 42y - 16.$$

All the integral solutions of $X^2 - 85Y^2 = 84^2$ are obtained by putting $X = \pm 84r/f$, $Y = \pm 84s/f$ and solving in integers $r^2 - 85s^2 = f^2$, where f is any factor of 84.

Thus, taking $f = 1$, we have $X = \pm 84r$, $Y = \pm 84s$, $r^2 - 85s^2 = 1$. Hence in this case,

$$\left.\begin{aligned} r &= \frac{1}{2}\{(285769 + 30996\sqrt{85})^n + (285769 - 30996\sqrt{85})^n\}, \\ s &= \frac{1}{2\sqrt{85}}\{(285769 + 30996\sqrt{85})^n - (285769 - 30996\sqrt{85})^n\}, \end{aligned}\right\} \qquad (9)$$

where n is any integer.

The residues, modulo 85, of the successive values of r form a periodic series of which the period is (84, 1). None of these make x integral.

Taking $f = 2$, we again find no integral values for x.

Taking $f = 3$, we have $X = \pm 28r_2$, $Y = \pm 28s_2$, where $r_2^2 - 85s_2^2 = 9$. Hence we get $r_2 = 37r \pm 85 \cdot 4s$; $s_2 = 37s \pm 4r$, r and s having the values given by (9). Here we get $X = -12r \pm 112s - 16(r-1)/85$ or $12r \pm 112s + 16(r+1)/85$. The former of these make X integral if $r \equiv 1 \pmod{85}$ and the latter if $r \equiv 84 \pmod{85}$; and this is the case according as n is even or odd in (9).

Thus, taking $n = 0$, we get the solution

$$-12^3 + 10^3 + 9^3 = 1, \qquad (b, 1)$$

and, taking $n = 1$, the two solutions

$$11468^3 - 14258^3 + 11161^3 = 1, \qquad (b, 2)$$

$$6954572^3 - 5593538^3 - 5444135^3 = 1. \qquad (b, 3)$$

If we take $f = 7$, we get the solutions

$$172^3 - 138^3 - 135^3 = 1, \qquad (b, 4)$$

$$-812^3 + 1010^3 - 791^3 = 1, \qquad (b, 5)$$

$$464196268^3 - 577145658^3 + 451797561^3 = 1, \qquad (b, 6)$$

$$-98196140^3 + 78978818^3 + 76869289^3 = 1. \qquad (b, 7)$$

Putting $f = 14$, we obtain

$$67402^3 - 83802^3 + 65601^3 = 1. \qquad (b, 8)$$

And so on.

Equation (5) may also be thrown into the form

$$(p^3 - q^3)u^2 + (2p^3 + q^3)uv + (p^3 - q^3)v^2 = m,$$

where

$$u = (4p^3 - q^3)x - 3p^2q, \qquad v = (4p^3 - q^3)y - 3p^2q, \qquad m = 3pq^2(4p^3 - q^3)(2q^3 - p^3).$$

Thus the problem of obtaining integral solutions of (5) is identical with that of the representation of the integer $2m$ in the binary quadratic form $(2p^3 - 2q^3, 2p^3 + q^3, 2p^3 - 2q^3)(u, v)$.

Of the values of u and v thus found we are to pick out those which satisfy $u \equiv v \equiv -3p^2q \pmod{4p^3 - q^3}$. From these values of x and y we are to choose those which make z integral.

If $u = v$, then $x = y$, and (2) reduces to $1 - z^3 = 2x^3$, which has no other integral solution except the trivial ones $x = 0$, $z = 1$ and $x = 1$, $z = -1$.

N.B. The six results $(a, 1)$, $(a, 2)$, $(b, 4)$, $(b, 5)$, $(b, 2)$ and $(b, 8)$ were given by the late Mr. Ramanujan in the form of a question (without demonstration) in the *Journal of the Indian Mathematical Society*, vol. VII (1915), Question No. 681.

I. The Definition of Radian.

By A. A. Bennett, Lehigh University.

It is customary to define the word "radian," as used in the circular measure of angles, by some such statement as the following: "A radian is the angle subtended at the center of a circle by an arc equal in length to the radius." Probably few teachers of trigonometry would care to be quizzed on the exact meaning that they wish to convey by the words, "an arc equal in length to the radius." This is a notoriously deep hole which most of us are relieved to find the students unquestioningly swimming over without attempting to touch bottom. It would seem inappropriate with only such a harmless application in mind to make vague mention of the difficulties in the limit notions of real variables. Were this the only possible method of definition, the current practice could only deserve regretful condonement, but this is not the case. Of course essentially the same

situation has arisen in the previous training of the student where the notion of π is first introduced. It is this fact that makes it possible to avoid the difficulty here. The student can be made to understand that the symbol π is used to denote a certain definite real number approximately equal to $3\frac{1}{7}$. This number, π, is definable by series in many ways and does not require for its definition any comparison of lengths of arc. As a consequence of various theorems involving limiting processes, it is *convenient* to *define* the ratio of the circumference to the diameter as a number and in particular as this number, π. Throughout trigonometry it must be assumed that for each positive number, a, there is a unique angle, which is the $(1/a)$th part of a straight angle. The "construction" may be regarded as involving at worst the limit of an infinite sequence of bisections completely determined by the number, a. This is part of the continuity assumed in the topic and is not subject to ruler and compasses restrictions such as appear in the question of the trisection of an angle. One might remark, parenthetically, that even the angle of one degree cannot be constructed with ruler and compasses. In particular the angle which is the $(1/\pi)$th part of a straight angle is *defined* as a radian. In the same sense that one may assume that the diameter goes into the circumference exactly π times, one may say that a central angle of one radian subtends an arc equal to the radius. The same difficulties attend both statements, and neither is logically necessary in trigonometry, although the origin of the term "radian" is not without interest.

The title of this Department suggests that a question not unrelated to the brief discussion made above may be appropriate here. The question I here propose will probably remain long unanswered. "Is it possible by ruler and compasses to construct an angle equal to one radian?" Presumably not, but it is not clear that the transcendence of π implies this conclusion.

II. Note on the Continuity of a Function Defined by a Definite Lebesgue Integral.

By H. J. Ettlinger, University of Texas.

In the recent June-July number of the Monthly, R. L. Jeffery proved the following theorem.[1]

Hypothesis: 1. *$f(x, y)$ is a real bounded function of (x, y) on the square $a \leq x \leq b$, $a \leq y \leq b$.*

2. *$f(x, y)$ is a summable function of x on (a, b) for each y in (a, b).*

3. *$f(x, y)$ is a continuous function of y at $\bar{y}$, for every x in (a, b), except a null set M_0.*

Conclusion: *$F(y) = L\int_a^b f(x, y)dx$ is continuous at every point $\bar{y}$ in (a, b).*

The proof of this theorem may be obtained as a direct result of the Duhamel-Moore theorem as follows.[2]

[1] The continuity of a function defined by a definite integral, this Monthly (*1925*, 297–299).

[2] R. L. Moore, On Duhamel's theorem, *Annals of Mathematics*, second series, vol. 16 (1911), pp. 45–49. See also H. J. Ettlinger, A simple form of Duhamel's theorem and some new applications, this Monthly (*1922*, 241).

Let $y_1, y_2, \cdots, y_n, \cdots$ be any infinite sequence of values in (a, b) such that $\lim_{n\to\infty} y_n = \bar{y}$. Then for any fixed x in (a, b) not in M_0, $\lim_{n\to\infty} f(x, y_n) = f(x, \bar{y})$. Consider $|F(\bar{y}) - F(y_n)| = \left| L\int_a^b [f(x, \bar{y}) - f(x, y_n)]dx \right| \leq \sum_{i=1}^{n} h_{in}e_{in}$, where, following Lebesgue's definition of integrability, h_{in} represents the points of division of the range of the function $|f(x, \bar{y}) - f(x, y_n)|$, and e_{in} is the measure of the set of points $E[h_{i-1,\, n} \leq |f(x, \bar{y}) - f(x, y_n)| < h_{i,\, n}]$.[1] Now h_{in} is bounded for all values of i ($\leq n$) and n, since $f(x, \bar{y})$ and $f(x, y_n)$ are bounded for all values of (x, y) in the square. If $\bar{x}$ is not in M_0, $\lim_{n\to\infty} h_{i_{P_n}n} = 0$, where P is the point in (a, b) corresponding to $x = \bar{x}$. Hence by the Duhamel-Moore theorem, $\lim_{n\to\infty} \sum_{i=1}^{n} h_{in}e_{in} = 0$. Hence $\lim_{n\to\infty} F(y_n) = F(\bar{y})$, *i.e.*, $F(y)$ is continuous at $y = \bar{y}$.

The above process is the equivalent of proving a generalization of a theorem due to Lebesgue[2] which may be stated as follows: *If a bounded sequence of measurable functions $f_n(x)$ converges for each x in (a, b) except a null set M_0 to a limit function $f(x)$, then* $\lim_{n\to\infty} L\int_a^b f_n(x)dx = L\int_a^b f(x)dx$.

It can be readily shown that if hypothesis 3 be modified to require integrability in y, for every x in (a, b), except a null set M_0, then $F(y)$ is an integrable function of y on (a, b).

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

History of Mathematics. By DAVID EUGENE SMITH. Ginn and Company, 1923, 1925. Volume I, *General Survey of the History of Elementary Mathematics*. xxii + 596 pages. Price $4.00. Volume II, *Special Topics of Elementary Mathematics*. xii + 725 pages. Price $4.40.

The excellence of this history of elementary mathematics is due primarily to the many years which Professor Smith has devoted to a first hand study of original sources. He had long collected early and rare books not only for himself, but more extensively for Mr. George A. Plimpton, also of New York, whose remarkable library contains the most important and extensive collection of early arithmetics in the world, besides many rare books and manuscripts on other fields of elementary mathematics. The constant enthusiastic study of these and other sources during so many years has made Smith an outstanding authority on the history of elementary mathematics. Fortunately he writes in a very clear and pleasing style. In these two volumes he has presented an authoritative and

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[1] *Leçons sur l'intégration*, Gauthier-Villars, Paris (1904), p. 112.

[2] *L. c.*, p. 114.

entertaining history of elementary mathematics, which should be accessible to every one who teaches or is otherwise interested in mathematics.

This work has been written for the purpose of supplying teachers and students with a usable text book on the history of elementary mathematics, that is, of mathematics through the first steps in the calculus. . . . The general plan adopted in the preparation of this work is that of presenting the subject from two distinct standpoints, the first, as in Volume I, leading to a survey of the growth of mathematics by chronological periods, with due consideration to racial achievements; and the second, as in Volume II, leading to a discussion of the evolution of certain important topics. . . . A general historical presentation is desirable for the purpose of relating the development of mathematics to the development of the race, of revealing the science as a great stream rather than a static mass, and of emphasizing the human element, but . . . this ought to lead to a topical presentation by which the student may understand something of the life history of the special subject which he may be studying. (*Quoted from author's preface.*)

Each chapter of Volume I treats a certain period; these periods are separated by the following dates: 1000 B.C., 300 B.C., 500 A.D., 1000, 1500, 1600, 1700. The subdivision of a chapter is usually by countries. For such a subdivision there is discussed in chronological order the contributions made by influential mathematicians of a given country in a given period.

Volume II treats the history of arithmetic, geometry, algebra, trigonometry, and calculus. The 68 pages of Chapter VII deal with mathematical recreations, history of commercial problems, puzzle problems leading to indeterminate equations in integers, and magic squares. The 42 pages of Chapter IX treat the history of measures of weight, length, metric system, and time (calendars and timepieces).

An important feature throughout is the abundance of cuts, diagrams, maps, reproductions of engravings, paintings, lithographs, and photographs, as well as facsimiles of pages of rare books and manuscripts.

What Smith has done so ably for elementary mathematics needs to be done for the various fields of advanced mathematics. A dozen men of promise in America should be encouraged to fit themselves to become our future historians of advanced mathematics.

L. E. DICKSON.

The History of Mathematics in Europe. By J. W. N. SULLIVAN. London, Oxford University Press, 1925. 110 pages. Price $1.00.

To outline the history of mathematics within the narrow compass of about a hundred pages, and to do this in a manner attractive to the general reader, is a difficult task which Sullivan has accomplished with remarkable success. The booklet contains pictures of fourteen mathematicians, Nicolas of Cusa, Pacioli, Cardan, Galileo, Napier, Kepler, Descartes, Fermat, Pascal, Wallis, Huygens, Newton, Laplace, Lagrange, and reproductions of portions of two medieval manuscripts. The history is carried down to the close of the eighteenth century. The human interest is enhanced by the insertion of biographical detail. Of Stifel we read,

His conversion [to Protestantism] was brought about by his discovery that the number 666, the number of the beast in the Book of Revelation, really referred to Pope Leo X, ... If Leo

DeCIMVs be written, the letters MDCLVI make up the number 1656 . . . the name should be followed by the symbol X. The result is now too great by 1000. But the 1000 springs from the letter M which, in this case, must stand for *Mysterium* and not for a number. Thus the required result is obtained.

Doubtless the conversion was well under way before Stifel gave his interpretation of the significance of 666.

In some cases the author missed opportunities to point out the evolution of ideas or notations, as when he failed to connect Descartes' exponential notation with the earlier notations of Chuquet, Bombelli, Stevin, Hérigone and Hume. In a few instances the author overlooked the latest researches, as when he spoke of Egyptian geometry as chiefly connected with land-surveying problems, and made no reference to the remarkable approximation to the area of a circle found in the Ahmes papyrus, nor to the exact computation of the volume of the frustrum of a quadrangular pyramid recently found in the Egyptian "Moscow papyrus." An imperfect characterization of Descartes' great work, *La Géométrie*, lies in the statement, "The wholly new contribution made by Descartes was in importing the idea of motion into geometry." In reality the important new contribution was the geometric interpretation of an algebraic equation containing two variable coördinates. There are no facts to support Sullivan's statement that Napier "had a precursor in Joost Bürgi." Other minor defects, here and there, might be pointed out, but they do not seriously lessen the value of the work as a whole. Students desiring a rapid general survey of the growth of mathematics and caring more for readability than for extreme accuracy of detail will peruse Sullivan's book with pleasure and profit.

FLORIAN CAJORI.

An Introductory Account of Certain Modern Ideas and Methods in Plane Analytic Geometry. By CHARLOTTE ANGAS SCOTT. Second edition with notes and corrections. New York, G. E. Stechert & Co., 1924. xx + 288 pages. Price $4.00.

The first edition of this book was published in 1894 and immediately met with hearty approval, as it furnished about the only account in English of many of the "modern ideas and methods in plane analytic geometry." But the book has long been out of print, and of the recent works which have appeared in the meantime, none in English occupies itself with just this field. In this interval some of the fundamental concepts have undergone extensive changes, and now some of the aspects are rather old-fashioned, but the student of geometry will still find all of it profitable reading, and some of the chapters are as useful today as they ever were. The only change in the new edition is the addition of six pages of notes and corrections, most of which are simpler proofs than those given in the earlier text.

We welcome the reappearance of this instructive and interesting volume.

VIRGIL SNYDER.

North Star Navigation. By L. M. BERKELEY. New York, White Book and Supply Co., 1924. 86 pages. Price $3.75.

This little pamphlet gives what is probably the simplest, most practical and valuable method yet suggested for finding the observer's position on the earth, at the same time giving as a valuable by-product the azimuth of Polaris. It seems strange that so simple a method should not have been developed soon after the invention of reasonably accurate chronometers. High praise is due the author for the elegance and simplicity with which he has presented the method and especially for the clearness of presentation of the mathematical theory and the adaptation to the practical needs of the navigator.

It seems to the reviewer that the method of obtaining a ship's position, which is so ably and clearly presented by Mr. Berkeley, can hardly fail to replace the Sumner or St. Hilaire methods, as those have largely displaced the older methods. One of the strongest points in favor of Mr. Berkeley's method is that the position may be found without any knowledge of the approximate position by dead reckoning or even without a knowledge of change of position by dead reckoning.

The only criticism which should be offered is on the notation and the use of terms. Where a notation has been long in current use and so become "classic," it is much better to use the "classic" notation than to introduce a new notation. Also, it is unfortunate to confuse meridians and hour circles as is done on page 26, where, in the first paragraph, the meridians of Greenwich and the north star are spoken of where the hour circles through the zenith of Greenwich and through the north star are intended.

It may further be remarked that the methods developed by Mr. Berkeley are useful to the surveyor in getting azimuth and latitude and will no doubt be taught in many of our engineering colleges provided the tables are made easily available.

S. L. BOOTHROYD.

Στοιχεῖα Αναλυτικῆς Γεωμετρίας: μερος πρωτον. By NEILOU SAKELLARIOU, with an introduction by CONSTANTINE CARATHÉODORY. Athens, Press of E. and I. Mplazoudaky, 1924. 288 + x pages.

It's no fun being an heir of all the ages. If you are one and write a book, your readers may expect too much of you. Not every Greek mathematician can be a Euclid, an Apollonius or a Carathéodory, and Professor Sakellariou has not set out to produce a *κτῆμα ἐς ἀεί*, but only to write an introduction to analytic geometry suited to the needs of students in the University of Athens and other institutions of higher learning, and he has, *exceptis excipiendis,* succeeded admirably in this first part, to which Professor Carathéodory has contributed an introduction.

The recent writer in the *Times Literary Supplement,* who rather sadly acquiesces in the Americanization of the world, could find further evidence of it were he to examine this book, for it would remind him, in subject matter and lists of problems, not of Salmon or Clebsch but of a score of American text-books for

beginners. In arrangement, the book differs from most American text-books in that solid analytic geometry makes its appearance very soon; the second chapter treats of planes and straight lines in space. It is not until the fourth that the circle is taken up along with the sphere, while the other conic sections have the final fifth chapter reserved for them.

A fuller treatment of directed lines than American books of similar character usually contain is a praiseworthy feature of the work. The proofs of the theorems of Menelaus and Ceva appear very soon and this adds interest to the earlier part of the book, the most difficult and least interesting part of analytic geometry. Furthermore a list of formulæ at the end of the volume is very convenient for reference. But however excellent a book may be in other particulars, we have a right to complain of the author who does not treat clearly and correctly the fundamental questions of tangents and asymptotes. We can forgive seventy times seven divisions by zero in an elementary book; we may even without asperity allow an author to give a definition of circles which applies to the real case only and then say that every equation of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

represents a circle, but the unpardonable sin is to mislead the student on the meaning of tangent, for that is fundamental, and a Greek should set us a good example in this matter, since his fathers have eaten fewer sour grapes than our own mathematical progenitors.

We can be sure that it was of certain passages on tangents in mathematical works that the Red Queen was thinking when she shook her head and said: "You may call it nonsense if you like, but *I've* heard nonsense in comparison with which this would be as sensible as a dictionary." Euclid's definition of a tangent to a circle as a line which cuts it in only one point is perfectly explicit and, with certain obvious exceptions, is valid for all conic sections and determines the same straight line as the more modern definition of tangent. It is the basis of the so-called discriminant method and its greatest disadvantage is that by its use the opportunity of introducing the fundamental concept of limit is lost. But Professor Sakellariou's definition is neither fish nor flesh for he says: "If we consider a right line, which cuts a curve in two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ and itself turns about the point P_1 in such a manner that P_2 continually approaches P_1 until finally P_2 coincides with P_1, we say that the line is tangent to the curve under consideration at P_1." These words miss the point in each definition of tangent and end in saying—nothing at all. If P_2 is to coincide with P_1, we have only one point on the line and one point will not determine a straight line, whether we call it two points or not. If Euclid's definition is the one meant, the point P_2 has nothing to do with the case and the essential fact that the line shall cut the curve at one point only is omitted. Nor does this definition carry more than a hint of the limit definition of tangent, as the slope of the tangent, under the limit definition of tangent, depends on the slope of the secant when P_2 does NOT coincide

with P_1 and what happens when P_2 coincides with P_1 has nothing to do with it.

Such a definition would be expected to lead our author into trouble and it does so straightway. It is true that if P_1 and P_2 are two *different* points on the circle $x^2 + y^2 = r$, not at opposite ends of a diameter,

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}$$

and that the latter ratio is the slope of the line P_1P_2, but when P_2 coincides with P_1, Professor Sakellariou must not claim that this ratio is still the slope of the line P_1P_2, only one point of which is fixed.

We have laid emphasis on this matter, not because Professor Sakellariou is an uncommon sinner, but precisely for the opposite reason; his treatment of tangents is only too typical, and it must be set down to his credit that his treatment of asymptotes is far superior to that of many text-books with which we are familiar.

W. L. G. WILLIAMS.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN DE LA SOCIETÉ MATHÉMATIQUE DE FRANCE, volume 52, nos. 3–4: "Un problème de probabilités dénombrables" by Norbert Weiner, 569–577; "Sur les séries de Fourier restreintes et la convergence presque partout" by W. H. Young, 578–584.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 31, no. 7, July, 1925: "The number of even and odd absolute permutations of n letters" by J. M. Thomas, 303; "The absolute value of the product of two matrices" by J. H. Wedderburn, 304–308; "On the number of representations of an integer as the sum or difference of two cubes" by E. T. Bell, 309–311; "Contact curves of the rational plane cubic" by L. W. Griffiths, 312–317; "Note on the projective geometry of paths" by T. Y. Thomas, 318–322; "The tensor character of the generalized Kronecker symbol" by F. D. Murnaghan, 323–329; "Two general functional equations" by W. H. Wilson, 330–334; "Functional invariants, with continuity of order p, of one parameter Friedholm and Volterra transformation groups" by A. D. Michal, 335–345; "On the distribution of quadratic and higher residues" by H. S. Vandiver, 346–350.

JOURNAL OF MATHEMATICS AND PHYSICS, M. I. T., volume 4, no. 4, July, 1925: "Note on quasi-analytic functions" by Norbert Wiener, 193–199; "File multiplication of ordered determinants" by Lepine Hall Rice, 200–204; "A theory of ordered determinants with application to polyadics" by Frank L. Hitchcock, 205–237; "A new method in the theory of quantics" by Frank L. Hitchcock, 238–256.

MESSENGER OF MATHEMATICS, volume 54, no. 12, April, 1925: "The x functions of Glaisher and class numbers" by E. T. Bell, 186–188.

SCIENCE, volume 62, nos. 1595–1598, July 24 and 31 and August 7 and 14, 1925: "Some mathematical aspects of cosmology" by W. D. MacMillan, 63–72, 96–98, 121–126; "Relation of the restricted to the general theory of relativity" by W. F. G. Swann, 145–147.

THE QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS, volume 50, no. 2: "On surfaces whose asymptotic curves are cubics" by C. H. Sisam, 149–153; "Some two dimensional loci" by J. L. Walsh, 154–164.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 3, July, 1925: "The linear complex of conics" by E. E. Libman, 265–269; "On the Weddle surface and

analogous loci" by Arnold Emch, 270–278; "Polynomials of several variables and their residue systems" by A. J. Kempner, 287–298; "A criterion for the conformal equivalence of a Riemann space to a Euclidean space" by Jesse Douglas, 299–306; "The deflection of a rectangular plate, fixed at the edges" by H. W. March, 207-217; "On irredundant sets of postulates" by Alonzo Church, 318–328; "Three dimensional manifolds of states of motion" by Harold Hotelling, 329–344; "Relations between the critical points of a real function of n independent variables" by Marston Morse, 345–396.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS.

FIEDLER'S CYCLOGRAPHY.

By B. H. BROWN, Dartmouth College.

Fiedler [1] establishes a correspondence between the circles in a plane and the points of space as follows. At the center of any circle we erect a perpendicular to the plane of the circle on a specified side of the plane, which we shall call *above*, equal in length to the radius of the circle. The point at the end of this perpendicular shall "represent" the circle. Conversely, every point above the plane represents a circle. A point of the plane represents itself considered as a point circle.

In two articles E. Müller [2] has expressed his astonishment that so little interest has been shown in this extraordinarily instructive representation. This astonishment we share. Here is a representation which is real, natural, easily visualized, one where important systems of circles are represented by simple space loci and conversely; and yet this representation has been almost totally neglected while artificial, imaginary, but analytically perfect representations have absorbed all the attention of geometers. We shall here indicate some of the simplest applications and theorems; the reader may then turn to Fiedler, and to articles by Müller and Ogura.[3]

THEOREM 1. *The circles tangent to two intersecting half-lines in a plane are represented by the points of a half-line through the point of intersection of the half-lines, making with the plane an angle whose cotangent is the cosecant of half the angle between the half-lines.*

APPLICATION. THEOREM.[4] *The three external centers of similitude of three coplanar circles lie in a straight line.* PROOF. The plane determined by the three representing points of the circles will cut the plane of the circles in the straight line

[1] Fiedler, *Cyclographie*, Leipzig, 1882. *Die darstellende Geometrie.* See indexes, vols. 1 and 2.

[2] Müller, *Jahresbericht*, vol. 14, 1905; vol. 20, 1911.

[3] Ogura, *Tohoku Math. Journal*, vol. 3, 1913.

[4] *Cf.* Archibald, this MONTHLY (*1915*, 6). The method of Fiedler may be at once extended to a correspondence between the hyperspheres of $(n - 1)$-space and the points of n-space; and the theorems in a short note by the author, this MONTHLY (*1916*, 155), follow immediately.

of the theorem. If we represent one of the circles by a point *below* the plane, we can show as easily the THEOREM: *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*

THEOREM 2. *Conversely, the points of every half-line above the plane will be represented by circles tangent to two intersecting or parallel lines which will be real if the angle which the half-line makes with the plane is* $\leq \pi/4$.

COROLLARY 1. *The circles tangent to a line at a point, and which lie on one side of the line, are represented by the points of a half-line through this point, perpendicular to this line, and making an angle of* $\pi/4$ *with the plane.*

COROLLARY 2. *The circles of one system cutting a line under a constant angle* θ *are represented by points of a half-plane through this line.*

COROLLARY 3. *The circles of one system cutting a line under a constant angle* θ *at a given point are represented by the points of a half-line through this point, perpendicular to the common tangent to the circles, and making an angle* $\pi/4$ *with the plane.*

THEOREM 3. *The circles externally tangent to a given circle are represented by the points of a portion of a cone of revolution through the given circle.*

APPLICATION. PROBLEM OF APOLLONIUS: *To construct a circle tangent to three mutually external circles.* Analysis of the problem from Fiedler's point of view leads at once to the well-known construction of Gergonne. The student is referred to Fiedler [1] or to the very concise analysis of Coolidge.[2]

THEOREM 4. *The circles cutting a given circle under a fixed angle are represented by the points of a portion of an equilateral one-sheeted hyperboloid of revolution.*

PROOF: Let the fixed circle be of radius R and consider any circle of radius r cutting it under the fixed angle θ. Then if c is the distance between their centers,

$$c^2 = R^2 + r^2 - 2Rr \cos \theta.$$

Let x, y be a pair of rectangular axes in the plane, intersecting at the center of the given circle. Setting

$$c^2 = x^2 + y^2, \qquad r = z,$$

we have

$$x^2 + y^2 = R^2 + z^2 - 2Rz \cos \theta$$

as the equation of the locus of representing points.

APPLICATION. By considering Corollary 3 to Theorem 2 in connection with Theorem 4 we can discuss the rulings on the hyperboloid.

APPLICATION. STEINER'S PROBLEM:[3] *To construct a circle meeting three given circles at given angles.* The generalization from the solution of the problem of Apollonius is easy and natural.[4]

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[1] Fiedler, *Cyclographie, loc. cit.*, p. 30.

[2] *A Treatise on the Circle and the Sphere*, Oxford, 1916, p. 185.

[3] Steiner, *Crelle*, vol. 1, 1826, p. 162.

[4] Fiedler, *Cyclographie.*

CLUB ACTIVITIES.

Mathematical Club, Tulane University, New Orleans, La. [*1924*, 400.]

The Tulane Mathematical Club held three meetings during the year 1924–1925. The first was in November and the paper of the afternoon was on unified mathematics by Professor R. L. Menuet. The February meeting was addressed by Dr. C. G. Latimer on "The Geometrical Construction of the Complex Roots of a Cubic with Some Historical Remarks." The attendance was about forty at each of these meetings.

For the March meeting we had Professor F. R. Moulton of the University of Chicago who spoke on "Ballistics." This lecture was attended by about 100.

(Report by Professor H. E. Buchanan, Chairman.)

Mathematics Club of the College of the Ozarks, Clarksville, Arkansas. [*1925*, 95.]

The Mathematics Club of the College of the Ozarks was organized for the year of 1924–25 as follows:

Pres., Annie Moon '26.
Vice-Pres., John Neal '27.
Sec.-Treas., Gladys Anderson '27.

Two members of the Club, Annie Moon and Gladys Anderson, were admitted as active members of "Chi Phi Mu" mathematical fraternity, the charter of which was recently granted to our mathematics department.

Meetings were held bi-monthly. Papers on the following topics were presented by members of the Club:

1. Einstein's theory of relativity.
2. Korzybski's concept of man.
3. Pedagogy of mathematics.
4. Mathematical philosophy.
5. Mathematical recreations.
6. History of mathematics.
7. Mathematical fallacies.
8. Geometry and faith.
9. Mathematics and religion.
10. Symmetry (general and applied).

(Report by Professor C. R. Hillard.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3153. Proposed by W. D. CAIRNS, Oberlin College.

Find the laws of motion, energy, etc., when the length of a simple pendulum is gradually shortened say by holding the thread between two fingers and drawing it up between them. Special cases: Draw up the thread (*a*) only at the ends of the swing; (*b*) at a uniform rate.

3154. Proposed by R. H. SCIOBERETI, Berkeley, California.

Find the equation of a spherical curve (Γ) such that if Q, P, N denote the intersections of a fixed great circle (C) with (1) the curve (Γ), (2) the great circle passing through a point M on the curve and orthogonal to the fixed circle (C), (3) the great circle normal to the curve at M respectively, then $QP = PN$, [the distances being measured along (C)].

EDITOR'S NOTE: Since to each point M on Γ correspond two points N and two points P, it follows that Γ must intersect C in two diametrically opposite points.

3155. Proposed by R. E. MORITZ, University of Washington.

Show that

$$\frac{d^n}{dx^n}[x^{m+n}(\log x)^n] = n!\,x^m\left[1 + {}^m_nP_1 \log x + {}^m_nP_2\frac{(\log x)^2}{2!} + \cdots + {}^m_nP_n\frac{(\log x)^n}{n!}\right],$$

where ${}^m_nP_k$ denotes the sum of the products of the numbers $m + 1$, $m + 2$, $\cdots$, $m + n$, taken k at a time.

2676 [1918, 75]. Proposed by E. R. SMITH, State College, Pa.

Find the greatest term of the series

$$\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(-r,\ -sq,\ sp-r+1,\ 1),$$

where s, r, sq and sp are positive integers, $r < s$, p and q are proper fractions such that $p + q = 1$, and $F(-r,\ -qs,\ sp-r+1,\ 1)$ is a hypergeometric series. If s, r and $s - r$ are large, show that the greatest term is approximately equal to

$$\sqrt{\frac{s}{2\pi pqr(s-r)}}.$$

2775 [1919, 213]. Proposed by H. T. BURGESS, University of Wisconsin.

Solve in finite form, if possible, the differential equation

$$\frac{d^2y}{dt^2} + ay\frac{dy}{dt} + by = g,$$

where a and b are arbitrary constants and $g = 32.16$. When $t = 0$, $y = 0$, $dy/dt = 0$.

2784 [1919, 366]. Proposed by T. H. GRONWALL, New York City.

Show that all solutions in integers of $y^2 = 1 + x + x^2 + x^3 + x^4$ are given by

$$\begin{aligned} x &= -1, \quad 0, \quad 3;\\ y &= \pm 1, \quad \pm 1, \quad \pm 11. \end{aligned}$$

2873 [1921, 36]. Proposed by D. H. RICHERT, Bethel College, Newton, Kan.

At B is the enemy's battery. At M_1 a battery is to be placed to silence B. Listening posts are installed at M_1, M_2, M_3, all provided with stop-watches. From the maps at hand, the three sides of the triangle $M_1M_2M_3$ are known. B is not visible from any one of the points M_1, M_2, M_3. The sound of a gun fired at B reaches M_1 at the time T, and M_2 at the time $T + \tau_1$ sec., and it reaches M_3 at the time $T + \tau_2$ sec. How far is B from M_1?

SOLUTIONS.

3115 [1925, 94]. Proposed by H. W. REDDICK, Cooper Union Institute of Technology.

Given a plane triangle and a system of coördinate axes in a 3-space, find the number of ways of placing the triangle so that its vertices lie on the axes.

SOLUTION BY FRED MILLER, Student, Cooper Union Institute of Technology.

Let the vertices of the given triangle be lettered A, B, C and its corresponding opposite sides a, b, c. Consider first the case in which a side lies upon an axis. If the triangle lies in the xy-plane with the side c on the x-axis and the vertex C on the y-axis, there are four possible positions, since C may be on the positive or negative y-axis and $c = AB$ may take the positive or negative direction along the x-axis. Four other positions are obtained by taking C on the x-axis and c on the y-axis. There are then eight positions in which c lies on one of the two axes in the xy-plane. Replacing in this reasoning c by a and b in turn, we find that there are 24 possible positions in the xy-plane. Considering all three planes, the total number of positions is 72, and this is true whether the triangle be acute, right or obtuse angled.

Next consider the case in which no side lies upon an axis, for example A on the x-axis, B on the y-axis, C on the z-axis, and denote the intercepts by x, y, z. Then $x^2 + y^2 = c^2$, $y^2 + z^2 = a^2$, $z^2 + x^2 = b^2$, and from these equations follow

$$x = \pm \sqrt{(b^2 + c^2 - a^2)/2} = \pm \sqrt{bc \cos A},$$

$$y = \pm \sqrt{(c^2 + a^2 - b^2)/2} = \pm \sqrt{ca \cos B},$$

$$z = \pm \sqrt{(a^2 + b^2 - c^2)/2} = \pm \sqrt{ab \cos C}.$$

Now if each angle of the triangle is acute, there are eight solutions for the intercepts considering the various combinations of $+$ and $-$ signs. But a, b, c may be permuted in each solution in $3!$ ways; there are therefore 48 possible positions.

If one angle is a right angle, the corresponding intercept is zero and the triangle must lie in a coördinate plane. This case has already been covered above. If one angle is obtuse, the corresponding intercept is imaginary and this position is impossible.

There are then for an acute angled triangle $72 + 48 = 120$ possible positions; for any other triangle there are 72 positions.

3116 [1925, 95]. Proposed by A. A. BENNETT, University of Texas.

Show that if a number is k times the number obtained by reversing the order of its digits, and neither has a zero for leading digit, then $k = 1$, 4, or 9. Show by example that 4 and 9 are actually possible.

SOLUTION BY HARRY LANGMAN, New York City.

The condition implies $k < 10$ and we shall suppose that $k > 1$. Let the unit's digit of the first number be a and the digit of the highest order, b. Then $kb \equiv a \pmod{10}$ and $ka \leqq b$. Hence if a is odd, neither k nor b can be even. With these restrictions the remaining possibilities are represented in the following table which is explained below:

k	a	b	a	Result
2	2	4 5	8 0	x x
	4	8 9	6 8	x x
3	1	3 5	9 5	x x
	2	6 7 8	8 1 4	x x x
	3	9	7	x
4	2	8 9	2 6	? x
5	1		0 5	x x
6, 8	1		even	x
7	1	7 9	9 3	x x
9	1	9	1	?

Here if $k = 2$, for example, a (second column) is restricted by the condition $2a < 10$. Hence $a = 1$, 2, 3 or 4, but as a must be even, 1 and 3 are omitted. The left-hand digit of the first number, b, is clearly ka plus what may be carried—at most $k - 1$. For $k = 2$, the number carried is either 0 or 1. Hence for $a = 2$ the only possibilities for b are 4 and 5, and for $a = 4$ they are 8 and 9 (third column). But the unit's digit of the first number, a, is obtained from kb. Hence for $k = 2$ and $b = 4$ or 5 the corresponding possibilities for a (fourth column) are 8 and 0, in each case contradicting the value in column 2. This is indicated by the "x" in the last column.

This shows that the only possibilities for k, omitting $k = 1$, are $k = 4$ and $k = 9$. In

the first case $a = 2$ and $b = 8$; in the second, $a = 1$ and $b = 9$. Using the method of skeleton division, it is easily seen that for $k = 4$ the only numbers possible are 8712, 87912, 879912, $\cdots$, $879\cdots912$, $\cdots$, and for $k = 9$ they are 9801, 98901, 989901, $\cdots$, $989\cdots901$, $\cdots$. These numbers are all multiples of 99.

EDITOR'S NOTE: It should be observed that 87128712, 871287128712, 98019801, etc., also are solutions.

Also solved by W. A. REES.

3117 [1925, 95]. Proposed by W. J. SIDIS, New York City.

It is well known that, except for a possible factor 3, all prime factors of $m^2 \pm mn + n^2$ must be of the form $6k + 1$, if m and n are prime to each other. Prove that, if n is an odd prime, all prime factors of

$$x^{n-1} \pm x^{n-2}y + \cdots \pm xy^{n-2} + y^{n-1},$$

excluding a possible factor n, must be of the form $2kn + 1$, provided x and y are prime to each other.

SOLUTION BY HARRY LANGMAN, New York City.

In the given expressions the terms must be either all positive or alternately positive and negative; let us call these cases "first" and "second."

Let p be a prime factor of the first expression, $A(x, y)$. Then both x and y are prime to p. We have

$$(x - y)A = x^n - y^n \equiv 0 \pmod{p}.$$

We have also

$$x^{p-1} \equiv y^{p-1} \equiv 1 \pmod{p}.$$

Suppose $p - 1$ not a multiple of n. Then $p - 1$ is prime to n, and we may write $\alpha(p - 1) - \beta n = 1$. But $x^{\alpha(p-1)} \equiv y^{\alpha(p-1)} \pmod{p}$, from which $x^{\beta n+1} \equiv y^{\beta n+1}$, giving $x \equiv y \pmod{p}$. If we assume $x - y$ prime to p, we must have $p - 1$ a multiple of n, and, as n is odd, and p is odd, this multiple must be even, giving $p = 2kn + 1$. If $x - y$ is a multiple of p, we may put $x = y + mp$, from which

$$A(x, y) = A(y + mp, y) \equiv A(y, y) \equiv ny^{n-1} \equiv n \equiv 0 \pmod{p},$$

requiring $p = n$.

In the second polynomial, $A(x, -y)$, let $-y \equiv y' \pmod{p}$, and put $-y = y' - mp$. Then

$$A(x, -y) = A(x, y' - mp) \equiv A(x, y').$$

This is in the first form and the theorem follows.

3118 [1925, 95]. Proposed by HARRY LANGMAN, New York City.

If $n > 2$ and ϵ is a primitive root of $\epsilon^n = 1$, show that

$$\begin{vmatrix} \epsilon & \epsilon^2 & \epsilon^3 & \cdots & \epsilon^{n-1} \\ \epsilon^2 & \epsilon^4 & \epsilon^6 & \cdots & \epsilon^{2(n-1)} \\ \epsilon^3 & \epsilon^6 & \epsilon^9 & \cdots & \epsilon^{3(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \epsilon^{n-1} & \epsilon^{2(n-1)} & \epsilon^{3(n-1)} & \cdots & \epsilon^{(n-1)(n-1)} \end{vmatrix} = (-1)^{(n-1)(n-2)/4} n^{(n-2)/2}.$$

SOLUTION BY J. J. NASSAU, Case School of Applied Science.

Let us write e instead of ϵ.

It is clear that

$$\begin{vmatrix} e & e^2 & e^3 & \cdots & e^{n-1} \\ e^2 & e^4 & e^6 & \cdots & e^{2(n-1)} \\ e^3 & e^6 & e^9 & \cdots & e^{3(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{n-1} & e^{2(n-1)} & e^{3(n-1)} & \cdots & e^{(n-1)(n-1)} \end{vmatrix}$$

$$= e \cdot e^2 \cdot e^3 \cdots e^{n-1} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ e & e^2 & e^3 & \cdots & e^{n-1} \\ e^2 & e^4 & e^6 & \cdots & e^{2(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{n-2} & e^{2(n-2)} & e^{3(n-2)} & \cdots & e^{(n-2)(n-1)} \end{vmatrix}$$

The coefficient of this determinant being equal to $e^{n(n-1)/2}$ or 1, we have the original determinant equal to the well-known difference-product

$$\zeta^{1/2}(e, e^2, e^3, \cdots, e^{n-1}).$$

But

$$\zeta(e, e^2, e^3, \cdots, e^{n-1}) = \begin{vmatrix} S_0 & S_1 & S_2 & \cdots & S_{n-2} \\ S_1 & S_2 & S_3 & \cdots & S_{n-1} \\ S_2 & S_3 & S_4 & \cdots & S_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ S_{n-2} & S_{n-1} & S_n & \cdots & S_{2(n-2)} \end{vmatrix},$$

where $S_i = (e)^i + (e^2)^i + \cdots + (e^{n-1})^i$ (Scott and Mathews, *Theory of Determinants*, second edition, page 152).

Making use of Newton's formulæ for sums of powers of roots (Burnside and Panton, *Theory of Equations*, second edition, page 290), we have for the equation $x^n - 1 = 0$,

$$\begin{aligned} S_0 &= (e)^0 + (e^2)^0 + \cdots + (e^{n-1})^0 = -(e^n)^0 + n = n - 1, \\ S_1 &= e + e^2 + \cdots + e^{n-1} = -e^n = -1, \\ S_2 &= (e)^2 + (e^2)^2 + \cdots + (e^{n-1})^2 = -(e^n)^2 = -1, \\ &\cdots\cdots\cdots\cdots \\ S_n &= n - 1, \\ S_{n+1} &= -1. \end{aligned}$$

Therefore

$$\zeta(e, e^2, e^3, \cdots, e^{n-1}) = \begin{vmatrix} n-1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & n-1 \\ -1 & -1 & -1 & -1 & \cdots & n-1 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -1 & -1 & -1 & n-1 & \cdots & -1 & -1 \\ -1 & -1 & n-1 & -1 & \cdots & -1 & -1 \end{vmatrix}.$$

Subtracting the second row from the first, the third, fourth, etc., we have

$$\zeta(e, e^2, e^3, \cdots, e^{n-1}) = \begin{vmatrix} n & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & 0 & \cdots & n & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & n & \cdots & 0 & 0 \\ 0 & 0 & n & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{(n-1)(n-2)/2} n^{n-2}.$$

By extracting the square root of both sides the proposition is established.

A more general form of the given determinant appears in the *Annals of Mathematics*, second series, vol. 28, nos. 1 and 2, page 111.

Also solved by THEODORE BENNETT who sent in two solutions, each differing from the other and from the one above, and by W. A. REES.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. KUHN, Ohio State University, Columbus, Ohio.

Professor E. V. HUNTINGTON of Harvard University has recently given a series of twelve lectures at Beloit College on the subject "The Logical Foundations of Elementary Mathematics."

Associate Professor J. W. CAMPBELL of the University of Alberta has been promoted to a full professorship of mathematics.

Professor ETHELWYNN R. BECKWITH, who was acting head of the department of mathematics at the College for Women at Western Reserve University last year, has been appointed head of the department of mathematics at Milwaukee-Downer College.

Mr. A. B. LEWIS of the University of Mississippi has been promoted to an assistant professorship of mathematics.

Professor C. D. SMITH has resumed the duties of head of the department of mathematics at Louisiana College after a year's leave of absence spent at the University of Iowa.

Miss JEANNETTE RYNO has been appointed to an assistant professorship of mathematics at Lander College, Greenwood, S. C.

Mr. F. A. PARISI has been appointed to an assistant professorship of mathematics at the College of St. Teresa.

Mr. D. H. RICHERT of Bethel College has been promoted to a full professorship of mathematics.

Professor H. W. TAYLOR, formerly professor of mathematics at Emporia College, has been appointed professor of mathematics at Southwestern College.

At the University of Oklahoma, Professor S. W. REAVES, who for the past two years has been acting dean of the College of Arts and Sciences, has been appointed dean. Associate Professor E. D. MEACHAM has been promoted to a full professorship of mathematics. Associate Professor NATHAN ALTSHILLER-COURT has resumed his duties after a leave of absence for one year spent in Paris. Miss DORA MCFARLAND has returned to her duties as instructor of mathematics after spending a year in study at the University of Chicago. Dr. ELSIE J. MCFARLAND, formerly of the University of California, Mr. S. B. TOWNES, and Mr. J. C. BRIXEY have been appointed instructors in mathematics.

At the University of Georgia, Professor C. M. SNELLING, Dean and Professor of Mathematics, has been elected Acting Chancellor and Professor R. P. STEPHENS of the same department has been elected Acting Dean. Mr. FORREST CUMMING has been promoted to an adjunct professorship of mathematics and E. M. EVERETT has been appointed to an instructorship. Professor D. F. BARROW of the department of mathematics has been elected Secretary of the Georgia Academy of Science.

At the Mississippi Agricultural and Mechanical College, Dr. B. M. WALKER, formerly head of the department of mathematics, has been elected president of the college. Associate Professor H. FOX has been made professor and head of the department of mathematics. Mr. C. R. STARK has been appointed to an associate professorship of mathematics and Mr. A. EDMONDSON has been appointed to an instructorship.

At Bowdoin College, Professor W. A. MOODY is on leave of absence for the year 1925–1926. Assistant Professor E. S. HAMMOND has been made professor and acting head of the department for the current year. Mr. C. T. HOLMES has been appointed to an assistant professorship of mathematics and Mr. ROY HALE to an instructorship.

At Western College for Women, Dr. HELEN TAPPAN, formerly associate professor of mathematics at Iowa State College, has been appointed professor and head of the department of mathematics. Miss RUTH DEWEY has been appointed instructor.

At the University of Wyoming, Mr. O. H. RECHARD has been promoted to associate professorship of mathematics. Miss GRETA NEWBAUER has been appointed to an instructorship and Mrs. R. H. HARRIS and Miss MARIAN PRATER have been appointed student instructors of mathematics.

At Georgia Wesleyan College, Professor J. C. HINTON has retired as head of the department of mathematics and astronomy after thirty-five years of service. Professor FREDERICK WOOD of Lake Forest College has been elected to succeed him. Miss RUTH LEONARD has been appointed to an instructorship in mathematics.

The following appointments to instructorships of mathematics are announced:

Case School of Applied Science, Mr. P. D. WILKINS, formerly instructor in mathematics at Tufts College.

Goucher College, Dr. MARIAN M. TORREY, formerly instructor in mathematics at the University of Illinois.

Kansas State Agricultural College, Mr. R. C. STALEY.

Otterbein College, Mr. H. E. MENKE.

South Dakota State College, Mr. G. D. GORE.

Toledo University, Mr. J. B. WINSLOW.

Union College, Mr. E. W. POWELL.

Wheaton College, Miss KATHLEEN SEARS.

Atlanta University, Mr. CLEMENT SUTTON.

Beloit College, Mr. L. COTHERN.

Professor J. A. MILLER, Associate Professor R. W. MARRIOTT, and Mr. D. B. McLAUGHLIN, instructor in mathematics, of Swarthmore College have gone to Sumatra in order to observe the total solar eclipse of January 14, 1926.

At the Virginia Polytechnic Institute, Blacksburg, Va., Professor J. E. WILLIAMS, head of the department of mathematics, has been promoted to dean

of the college. He still continues as head of this department. At its June commencement Hampden-Sidney College conferred the degree of LL.D. upon Dr. WILLIAMS.

At the same institution, the following instructors in the department of mathematics have been promoted to assistant professorships: T. W. HATCHER, H. C. AHALT, A. V. MORRIS, and F. S. GLASSETT.

Dr. F. S. NOTESTEIN, for many years Professor of Mathematics and Astronomy at Alma College, died July 22, 1924, after a short illness.

Professor JOHN PHILLIPS of the department of mathematics, Kansas University, died May 25, 1925.

The Mathematical Association of America announces that the second of the Carus Mathematical Monographs is ready for the printer. It is expected that it will be ready for distribution in January, 1926, on the same plan as the first monograph was distributed in 1925. The title is *Functions of a Complex Variable* by D. R. CURTISS, of Northwestern University. It will be a volume of the same style and of approximately the same number of pages as the first monograph the sale of which, both to members of the Association and to the general public, has been gratifying.

At Yale University, the Sterling fellowships have been established by a gift of one million dollars from the trustees of the estate of the late JOHN W. STERLING to stimulate scholarship and advanced research in all fields of knowledge. They are open equally to graduates of Yale University and other approved colleges and universities in the United States and foreign countries, to both men and women, whether graduate students, or instructors or professors when on leave of absence, who desire to carry on studies and investigations under the direction of the graduate faculty of Yale University or in affiliation with that body.

The Sterling fellowships are divided into two general classes: *research or senior fellowships* and *junior fellowships*. Candidates for research or senior fellowships must have the Ph.D. degree, or must have had such training and experience in research as are indicated by this degree. Candidates for junior fellowships must be well advanced in their work towards the Ph.D. degree. In exceptional circumstances, holders of either class of fellowships, who have been in residence at Yale University for a year or more, may be permitted to carry on their investigations in part elsewhere, at home or abroad. The Sterling fellowships are awarded on the understanding that the recipients shall not engage in teaching during the tenure of appointment.

The stipends of the research or senior fellowships range from \$1,000 to \$2,500 or more, dependent upon the character of the proposed investigation. The

stipends of the junior fellowships range from $1,000 to $1,500. For special purposes, such as carrying through to completion a piece of investigation, awards may be made of less than $1,000. Fellows who have not yet obtained the Ph.D. degree are subject to the usual tuition and laboratory fees. All fellows are appointed for a single year, but may be reappointed with or without additional stipend.

Holders of Sterling fellowships are required to submit reports on their work, either at stated intervals or at the expiration of their fellowships; and when the results of their work are published they are expected to give proper credit to the assistance they have received as Sterling fellows.

Applications for these fellowships should be addressed to the Dean of the Graduate School of Yale University, New Haven, Connecticut, on blanks which may be obtained from him. Applications for the junior fellowships must be submitted by March 1, and applications for the senior fellowships by April 1.

PERSONAL MENTION—NECROLOGY.

ADDENDA AND CORRIGENDA.

P. 71, l. 9, *for* "$+\cdots)^2\}^{1/2}$" *read* "$+\cdots)^2)\}^{1/2}$."
P. 71, last line, *for* "μ_s/σ_x^5" *read* "μ_5/σ_x^5."
P. 72, lines 3 and 5, *for* "$(\beta_4 - \beta_2 - \beta_1)$" *read* "$(\beta_4 - \beta_2^2 - \beta_1)$."
P. 72, line 10, numerator, for "$+ \frac{2}{3}$" read "$+ \frac{1}{3}$."
P. 162, **Theorem,** *delete* "*such that* $0 < f(x_1)/[f(x_1) - f(x_n)] \gtreqless 1$."
P. 388, above third display, *for* "transformed determinant" *read* "original determinant."
P. 388, below third display, *for* "*Journal* $\cdots\cdots$, p. 47." *read* "*Transactions of the American Mathematical Society*, vol. 26, 1924, p. 113."

CONTENTS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.